ON THE BIHARMONICITY OF PRODUCT MAPS

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We introduce the warped product of maps defined between Riemannian warped product spaces and we give necessary and sufficient conditions for warped product maps to be (bi)harmonic. We obtain then some characterizations of nontrivial harmonic metrics and nonharmonic biharmonic metrics on warped product spaces.

1. Introduction

Let $f : (M^m, g) \to (N^n, h)$ be a map between the m-dimensional Riemannian manifold (M, g) and the n-dimensional Riemannian manifold (N, h).

The energy of the map f is given by

$$E(f) = \int_M e(f) v_g, \qquad (1.1)$$

where v_g is the volume form on (M, g) and $e(f)(x) := (1/2) \| df(x) \|^2_{T^*M \otimes f^{-1} TN}$ is the energy density of f at the point $x \in M$. In local coordinates $(x^i)_{i=1}^m$ on M and $(y^a)_{a=1}^n$ on N, the energy density is given by $e(f)(x) = (1/2) g^{ij}(x) h_{ab}(f(x))(\partial f^a / \partial x^i)(\partial f^b / \partial x^j)$.

Critical points of the energy functional are called harmonic maps.

The first variational formula of the energy gives the following characterization of harmonic maps: the map f is harmonic if and only if its tension field $\tau(f)$ vanishes identically, where the tension field is given by

$$\tau(f) := \operatorname{trace}_g \nabla df = g^{ij} \left(\frac{\partial^2 f^a}{\partial x^i \partial x^j} - {}^M \Gamma_{ij}^k \frac{\partial f^a}{\partial x^k} + {}^N \Gamma_{bc}^a \frac{\partial f^b}{\partial x^i} \frac{\partial f^c}{\partial x^j} \right) \frac{\partial}{\partial y^a}, \qquad (1.2)$$

with ${}^M \Gamma_{ij}^k$ and ${}^N \Gamma_{bc}^a$ being the Christoffel symbols of the metrics g and h, respectively. The theory of harmonic maps was introduced by Eells and Sampson [7] and has been

developed by many authors. The bienergy $E_2(f)$ of the map f is defined by

$$E_2(f) := \frac{1}{2} \int_M \|\tau(f)\|^2 v_g. \tag{1.3}$$

As it was observed in [1], the bienergy is the first variation of the energy with respect to a particular variation of the map; precisely that with $-\tau(f)$ as variational vector field.

The map f is said to be biharmonic if it is a critical point of the bienergy functional. The Euler-Lagrange equation associated to the bienergy functional is given by

$$\tau_2(f) := -\Delta^f \tau(f) - \operatorname{trace}_g R^N(df, \tau(f)) df = 0, \tag{1.4}$$

where $\Delta^f = -\operatorname{trace}_g(\nabla^f \nabla^f - \nabla^f_\nabla)$ is the Laplacian on the sections of $f^{-1}(TN)$, and R^N is the Riemannian curvature operator of (N, h) (see [8, 9]).

Note that $\tau_2(f) = -J_f(\tau(f))$, where J_f is the Jacobi operator of f, which gives the second variation of the energy functional at its critical (harmonic) points.

Harmonic maps are obviously biharmonic and are absolute minimum of the bienergy. In [9], the author proved that harmonicity and biharmonicity are equivalent if M is compact and $R^N \leq 0$, or if f is a Riemannian immersion with $\|\tau(f)\|$ constant (see [11]) and $R^N \leq 0$.

Examples of nonharmonic biharmonic maps are given in [5] and also in [8], where it is proved that the generalized Clifford torus $\mathbb{S}^p(1/\sqrt{2}) \times \mathbb{S}^q(1/\sqrt{2}) \to \mathbb{S}^{m+1}$ with $p + q = m$, $p \neq q$, is a nonharmonic biharmonic submanifold of $\mathbb{S}^{m+1}$.

A classification of nonharmonic biharmonic submanifolds of $\mathbb{S}^3$ is given in [3, 4].

The method of conformal deformation of metrics has been used by several authors to study the existence or properties of (bi)harmonic maps.

In the case of warped or twisted product spaces, the same techniques can be used, not by deforming conformally the whole metric but by acting only on the warping or twisting functions. In the present paper we consider a particular class of maps defined between warped product spaces, the so-called warped product maps, and we examine (bi)harmonic properties of these maps in relation to that of the component maps. We apply the results obtained to characterize (bi)harmonic metrics on warped product spaces.

Throughout the paper, manifolds, metrics, and maps are assumed to be smooth.

2. Harmonicity of warped product maps, harmonic metrics, and nonharmonic biharmonic metrics on warped product spaces

2.1. Warped product of manifolds. Let (M, g) and (N, h) be two Riemannian manifolds of dimensions m and n, respectively, and $\lambda \in C^\infty(M)$ a strictly positive function on M.

The warped product of (M, g) and (N, h), with warping function λ, is the product manifold $M \times N$ endowed with the metric G_λ defined by

$$G_\lambda =: \pi^* g + (\lambda \circ \pi)^2 \sigma^* h, \tag{2.1}$$

where π and σ are the projections of $M \times N$ on M and N, respectively.

The metric G_λ is also called the warped product of the metrics g and h with warping function λ.

For $X, Y \in \mathscr{C}(TM)$ and $U, V \in \mathscr{C}(TN)$,

$$G_\lambda(X + U, Y + V) = g(X, Y) + (\lambda \circ \pi)^2 h(U, V), \tag{2.2}$$

where we use the same convention as in [10] for the notations of vector fields.

Let $^g\nabla$, $^h\nabla$, and ∇ be the Levi-Civita connections of (M, g), (N, h), and $(M \times N, G_\lambda)$, respectively. We have the following (see [10]).

LEMMA 2.1. *For $X, Y \in \mathscr{C}(TM)$ and $U, V \in \mathscr{C}(TN)$, it holds that*

(i) $\pi_*(\nabla_X Y) = {}^g\nabla_X Y$ *and* $\sigma_*(\nabla_X Y) = 0$,

(ii) $\nabla_X U = \nabla_U X = (d\lambda(X)/\lambda)U$,

(iii) $\pi_*(\nabla_U V) = -(\langle U, V \rangle / \lambda)\operatorname{grad}\lambda = -\lambda h(U, V)\operatorname{grad}\lambda$,

(iv) $\sigma_*(\nabla_U V) = {}^h\nabla_U V$.

Let $(X_1, X_2, \ldots, X_{m+n}) = (x_1, x_2, \ldots, x_m, u_1, u_2, \ldots, u_n)$ be a local coordinates system in $M \times N$, where $(x_1, x_2, \ldots, x_m)$ and $(u_1, u_2, \ldots, u_n)$ are local coordinates systems in M and N, respectively. We use the indices notations $\hat{I}, \hat{J}, \hat{K}, \ldots \in \{1, \ldots, m+n\}$ on $M \times N$, $i, j, k, \ldots \in \{1, \ldots, m\}$ on M, and $\bar{i}, \bar{j}, \bar{k}, \ldots \in \{1, \ldots, n\}$ on N and for $\hat{I}, \hat{J}, \hat{K}, \ldots \in \{1, \ldots, m\}$, we put $\hat{I} = i$, $\hat{J} = j$, $\hat{K} = k, \ldots$; for $\hat{I}, \hat{J}, \hat{K}, \ldots \in \{m, \ldots, m+n\}$, we put $\hat{I} = m + \bar{i} \equiv \bar{i}$, $\hat{J} = m + \bar{j} \equiv \bar{j}$, $\hat{K} = m + \bar{k} \equiv \bar{k}$. We have

$$G_\lambda = (G_{\lambda \hat{I}\hat{J}}) = \begin{pmatrix} g_{ij} & 0 \\ 0 & \lambda^2 h_{\bar{i}\bar{j}} \end{pmatrix}, \qquad G_\lambda^{-1} = \left(G_\lambda^{\hat{I}\hat{J}}\right) = \begin{pmatrix} g^{ij} & 0 \\ 0 & \lambda^{-2} h^{\bar{i}\bar{j}} \end{pmatrix}. \tag{2.3}$$

Let $^1\Gamma_{ij}^k$, $^2\Gamma_{\bar{i}\bar{j}}^{\bar{k}}$, and $\Gamma_{\hat{I}\hat{J}}^{\hat{K}}$ be Christoffel's symbols on (M, g), (N, h), and $(M \times N, G_\lambda)$, respectively.

A direct computation from Lemma 2.1 leads to the following (see, e.g., [12, page 111]).

LEMMA 2.2. (i) $\Gamma_{ij}^k = {}^1\Gamma_{ij}^k$, *for all $i, j, k \in \{1, \ldots, m\}$.*

(ii) $\Gamma_{ij}^{\bar{k}} = \Gamma_{i\bar{j}}^k = 0$, *for all $i, j, k \in \{1, \ldots, m\}$ and for all $\bar{j}, \bar{k} \in \{1, \ldots, n\}$.*

(iii) $\Gamma_{i\bar{j}}^{\bar{k}} = (\partial \log \lambda / \partial x_i)\delta_{\bar{j}}^{\bar{k}}$, *for all $i \in \{1, \ldots, m\}$ and for all $\bar{j}, \bar{k} \in \{1, \ldots, n\}$.*

(iv) $\Gamma_{\bar{i}\bar{j}}^k = -(1/2)(\operatorname{grad}\lambda^2)^k h_{\bar{i}\bar{j}}$, *for all $k \in \{1, \ldots, m\}$ and for all $\bar{i}, \bar{j} \in \{1, \ldots, n\}$.*

(v) $\Gamma_{\bar{i}\bar{j}}^{\bar{k}} = {}^2\Gamma_{\bar{i}\bar{j}}^{\bar{k}}$, *for all $\bar{i}, \bar{j}, \bar{k} \in \{1, \ldots, n\}$.*

The relations (i)–(v) of Lemma 2.2 can be also obtained directly from the formula

$$\Gamma_{\hat{I}\hat{J}}^{\hat{K}} = \frac{1}{2}G_\lambda^{\hat{K}\hat{L}}\left(\frac{\partial G_{\lambda \hat{J}\hat{L}}}{\partial X_{\hat{I}}} + \frac{\partial G_{\lambda \hat{J}\hat{L}}}{\partial X_{\hat{I}}} - \frac{\partial G_{\lambda \hat{I}\hat{J}}}{\partial X_{\hat{L}}}\right). \tag{2.4}$$

2.2. Warped product maps. Let (M, G_λ) be the warped product of two Riemannian manifolds (M_1, g_1) and (M_2, g_2) with warping function $\lambda \in C^\infty(M_1)$, and let (N, H_ϱ) be the warped product of the Riemannian manifolds (N_1, h_1) and (N_2, h_2) with warping function $\varrho \in C^\infty(N_1)$.

Let $\phi_1 : (M_1, g_1) \to (N_1, h_1)$ and $\phi_2 : (M_2, g_2) \to (N_2, h_2)$ be two maps.

Defintion 2.3. The warped product of ϕ_1 and ϕ_2, with warping functions λ and ϱ, is defined as the map $\Phi_{\lambda,\varrho}$ by

$$
\begin{aligned}
\Phi_{\lambda,\varrho} &: (M, G_\lambda) \longrightarrow (N, H_\varrho), \\
(P_1, P_2) &\longmapsto (\phi_1(P_1), \phi_2(P_2)).
\end{aligned}
\tag{2.5}
$$

The maps ϕ_1 and ϕ_2 are then called the components of $\Phi_{\lambda,\varrho}$.

Let $(X_1, X_2, \ldots, X_{m_1+m_2}) = (x_1, x_2, \ldots, x_{m_1}, u_1, u_2, \ldots, u_{m_2})$ and $(Y_1, Y_2, \ldots, Y_{n_1+n_2}) = (y_1, y_2, \ldots, y_{n_1}, v_1, v_2, \ldots, v_{n_2})$ be local coordinates systems in $M_1 \times M_2$ and $N_1 \times N_2$, respectively, defined as above with the same notations' conventions for the indices for which we use $\hat{I}, \hat{J}, \hat{K}$ on $M_1 \times M_2$; i, j, k on M_1; $\bar{i}, \bar{j}, \bar{k}$ on M_2; $\hat{A}, \hat{B}, \hat{C}$ on $N_1 \times N_2$; a, b, c on N_1, and $\bar{a}, \bar{b}, \bar{c}$ on N_2.

The second fundamental form of the warped product map $\Phi_{\lambda,\varrho}$ of ϕ_1 and ϕ_2 is given by the following.

LEMMA 2.4. (i) *For $i, j \in \{1, \ldots, m_1\}$ and $a \in \{1, \ldots, n_1\}$,*

$$
(\nabla d\Phi_{\lambda,\varrho})_{ij}^a = ({}^1\nabla d\phi_1)_{ij}^a.
\tag{2.6}
$$

(ii) *For $i, j \in \{1, \ldots, m_1\}$, $a \in \{1, \ldots, n_1\}$, $\bar{j} \in \{1, \ldots, m_2\}$, and $\bar{a} \in \{1, \ldots, n_2\}$,*

$$
(\nabla d\Phi_{\lambda,\varrho})_{ij}^{\bar{a}} = (\nabla d\Phi_{\lambda,\varrho})_{i\bar{j}}^a = 0.
\tag{2.7}
$$

(iii) *For $i \in \{1, \ldots, m_1\}$, $\bar{j} \in \{1, \ldots, m_2\}$, and $\bar{a} \in \{1, \ldots, n_2\}$,*

$$
(\nabla d\Phi_{\lambda,\varrho})_{i\bar{j}}^{\bar{a}} = -\frac{\partial\log\lambda}{\partial x_i} \cdot \frac{\partial\phi_2^{\bar{a}}}{\partial u_{\bar{j}}} + \frac{\partial\log\varrho}{\partial y_b} \cdot \frac{\partial\phi_1^b}{\partial x_i} \cdot \frac{\partial\phi_2^{\bar{a}}}{\partial u_{\bar{j}}}.
\tag{2.8}
$$

(iv) *For $\bar{i}, \bar{j} \in \{1, \ldots, m_2\}$ and $a \in \{1, \ldots, n_1\}$,*

$$
(\nabla d\Phi_{\lambda,\varrho})_{\bar{i}\bar{j}}^a = \frac{1}{2}(\operatorname{grad}\lambda^2)^k \frac{\partial\phi_1^a}{\partial x_k} g_{2\bar{i}\bar{j}} - \frac{1}{2}(\operatorname{grad}\varrho^2)^a \frac{\partial\phi_2^{\bar{b}}}{\partial u_{\bar{i}}} \cdot \frac{\partial\phi_2^{\bar{c}}}{\partial u_{\bar{j}}} h_{2\bar{b}\bar{c}}.
\tag{2.9}
$$

(v) *For $\bar{i}, \bar{j} \in \{1, \ldots, m_2\}$ and $\bar{a} \in \{1, \ldots, n_2\}$,*

$$
(\nabla d\Phi_{\lambda,\varrho})_{\bar{i}\bar{j}}^{\bar{a}} = ({}^2\nabla d\phi_2)_{\bar{i}\bar{j}}^{\bar{a}}.
\tag{2.10}
$$

Let $\tau(\Phi_{\lambda,\varrho})$, $\tau(\phi_1)$, and $\tau(\phi_2)$ be the tension fields of the warped product map $\Phi_{\lambda,\varrho}$ and of its components ϕ_1 and ϕ_2. From Lemma 2.4 we have the following proposition.

PROPOSITION 2.5.

$$
\tau(\Phi_{\lambda,\varrho}) = \tau(\phi_1) + \lambda^{-2}\tau(\phi_2) + \lambda^{-2}\left[\frac{m_2}{2}d\phi_1(\operatorname{grad}_{M_1}\lambda^2) - \mathrm{e}(\phi_2)(\operatorname{grad}_{N_1}\varrho^2) \circ \phi_1\right],
\tag{2.11}
$$

with $d\phi_1(\operatorname{grad}_{M_1}\lambda^2) = [d\phi_1^a(\operatorname{grad}_{M_1}\lambda^2)](\partial/\partial y^a)$, where $\mathrm{e}(\phi_2)$ is the energy density of ϕ_2, and $\operatorname{grad}_{M_1}$ and $\operatorname{grad}_{N_1}$ denote the gradient operators on M_1 and N_1, respectively.

Proof. We have

$$
\begin{aligned}
\tau(\Phi_{\lambda,\varrho}) &= G_\lambda^{\hat{i}\hat{j}}(\nabla d\Phi_{\lambda,\varrho})_{\hat{i}\hat{j}}^{\hat{A}}\frac{\partial}{\partial Y_{\hat{A}}} = G_\lambda^{ij}(\nabla d\Phi_{\lambda,\varrho})_{ij}^a\frac{\partial}{\partial y_a} + G_\lambda^{ij}(\nabla d\Phi_{\lambda,\varrho})_{ij}^{\bar{a}}\frac{\partial}{\partial v_{\bar{a}}} \\
&\quad + G_\lambda^{\bar{i}j}(\nabla d\Phi_{\lambda,\varrho})_{\bar{i}j}^a\frac{\partial}{\partial y_a} + G_\lambda^{\bar{i}j}(\nabla d\Phi_{\lambda,\varrho})_{\bar{i}j}^{\bar{a}}\frac{\partial}{\partial v_{\bar{a}}} \\
&= g_1^{ij}(^1\nabla d\phi_1)_{ij}^a\frac{\partial}{\partial y_a} + \frac{1}{2}\lambda^{-2}g_2^{\bar{i}j}(\operatorname{grad}_{M_1}\lambda^2)^k\frac{\partial\phi_1^a}{\partial x_k}g_{2ij}\frac{\partial}{\partial y_a} \\
&\quad - \frac{1}{2}\lambda^{-2}g_2^{\bar{i}j}(\operatorname{grad}_{N_1}\varrho^2)^a\frac{\partial\phi_2^{\bar{b}}}{\partial u_{\bar{i}}} \cdot \frac{\partial\phi_2^{\bar{c}}}{\partial u_{\bar{j}}}h_{2\bar{b}\bar{c}}\frac{\partial}{\partial y_a} + \lambda^{-2}g_2^{\bar{i}j}(^2\nabla d\phi_2)_{\bar{i}j}^{\bar{a}}\frac{\partial}{\partial v_{\bar{a}}}.
\end{aligned}
\tag{2.12}
$$

By evaluating each term of the second hand member of the last equality, we obtain

$$
\begin{aligned}
g_1^{ij}(^1\nabla d\phi_1)_{ij}^a\frac{\partial}{\partial y_a} &= \tau(\phi_1), \\
\frac{1}{2}\lambda^{-2}g_2^{\bar{i}j}(\operatorname{grad}_{M_1}\lambda^2)^k\frac{\partial\phi_1^a}{\partial x_k}g_{2ij}\frac{\partial}{\partial y_a} &= \lambda^{-2}\frac{m_2}{2}[d\phi_1(\operatorname{grad}_{M_1}\lambda^2)], \\
\frac{1}{2}\lambda^{-2}g_2^{\bar{i}j}(\operatorname{grad}_{N_1}\varrho^2)^a\frac{\partial\phi_2^{\bar{b}}}{\partial u_{\bar{i}}} \cdot \frac{\partial\phi_2^{\bar{c}}}{\partial u_{\bar{j}}}h_{2\bar{b}\bar{c}}\frac{\partial}{\partial y_a} &= \lambda^{-2}\operatorname{e}(\phi_2)[(\operatorname{grad}_{N_1}\varrho^2)\circ\phi_1], \\
\lambda^{-2}g_2^{\bar{i}j}(^2\nabla d\phi_2)_{\bar{i}j}^{\bar{a}}\frac{\partial}{\partial v_{\bar{a}}} &= \lambda^{-2}\tau(\phi_2).
\end{aligned}
\tag{2.13}
$$

Thus we get the result. □

The following result is a consequence of the previous proposition.

COROLLARY 2.6. *Let $\phi_1 : (M_1,g_1) \to (N_1,h_1)$ and $\phi_2 : (M_2,g_2) \to (N_2,h_2)$ be two harmonic maps. The warped product $\Phi_{\lambda,\varrho}$ of ϕ_1 and ϕ_2, with warping functions $\lambda \in C^\infty(M_1)$ and $\varrho \in C^\infty(N_1)$, is harmonic if and only if*

$$
\frac{m_2}{2}d\phi_1(\operatorname{grad}_{M_1}\lambda^2) = \operatorname{e}(\phi_2)(\operatorname{grad}_{N_1}\varrho^2)\circ\phi_1.
\tag{2.14}
$$

2.3. Harmonic metrics on warped product spaces. Let us first recall the definition of harmonic metrics.

Defintion 2.7. Let (M,g) be a Riemannian manifold. A metric G on M is said to be harmonic with respect to g if the identity id : $(M,g) \to (M,G)$ is a harmonic map (see [6] for more information about harmonic metrics).

Let $(M \times N, G_\lambda)$ be the warped product of two Riemannian manifolds (M,g) and (N,h) with warping function $\lambda \in C^\infty(M)$, $\lambda > 0$.

Let G_ϱ be the metric defined on $M \times N$ by $G_\varrho = \pi^*g + (\varrho \circ \pi)^2\sigma^*h$, with $\varrho \in C^\infty(M)$, $\varrho > 0$, where π and σ are as usual the projections of $M \times N$ on M and N, respectively. The following holds.

THEOREM 2.8. *Assume M is connected.*
The metric G_ϱ is harmonic with respect to G_λ if and only if

$$\lambda^2 - \varrho^2 \text{ is a constant function on } M. \tag{2.15}$$

Proof. Let $\mathscr{I}_{\lambda,\varrho} : (M \times N, G_\lambda) \to (M \times N, G_\varrho)$ be the warped product of the identity maps $\mathrm{id}_M : (M,g) \to (M,g)$ and $\mathrm{id}_N : (N,h) \to (N,h)$, with warping functions λ and ϱ.

The metric G_ϱ is harmonic with respect to G_λ if and only if $\mathscr{I}_{\lambda,\varrho}$ is a harmonic map.

Since id_M and id_N are harmonic, it follows from Corollary 2.6 that $\mathscr{I}_{\lambda,\varrho}$ is a harmonic map if and only if

$$\frac{n}{2} d(\mathrm{id}_M) \left[\mathrm{grad}_M \lambda^2 \right] = \mathrm{e}(\mathrm{id}_N)(\mathrm{grad}_M \varrho^2) \circ \mathrm{id}_M. \tag{2.16}$$

But

$$\mathrm{e}(\mathrm{id}_N) = \frac{n}{2}, \qquad d(\mathrm{id}_M)\left[\mathrm{grad}_M \lambda^2\right] = \mathrm{grad}_M \lambda^2, \qquad (\mathrm{grad}_M \varrho^2) \circ \mathrm{id}_M = \mathrm{grad}_M \varrho^2. \tag{2.17}$$

Thus $\mathscr{I}_{\lambda,\varrho}$ is harmonic if and only if

$$\frac{n}{2} \mathrm{grad}_M \lambda^2 - \frac{n}{2} \mathrm{grad}_M \varrho^2 = 0. \tag{2.18}$$

That is,

$$\mathrm{grad}_M (\lambda^2 - \varrho^2) = 0. \tag{2.19}$$

Since M is assumed to be connected, we get the result.   □

Example 2.9. (1) Consider $\mathbb{S}^3 \setminus \{(\pm 1,0,0,0)\}$ with the metric G defined in local coordinates (t,θ,ϕ) by

$$ds_G^2 = dt^2 + \sin^2 t \, d\theta^2 + \sin^2 t \sin^2 \theta \, d\phi^2. \tag{2.20}$$

For any real $c < 0$, the metric G' defined by

$$ds_{G'}^2 = dt^2 + (\sin^2 t - c) d\theta^2 + (\sin^2 t - c) \sin^2 \theta \, d\phi^2 \tag{2.21}$$

is harmonic with respect to G.

Indeed, $\mathbb{S}^3 \setminus \{(+1,0,0,0)\}$ can be viewed as the warped product of the open interval $]0,\pi[$ and of $\mathbb{S}^2$ with warping function λ defined on $]0,\pi[$ by $\lambda(t) = \sin t$.

(2) On the space $\mathbb{H}^3 \setminus \{(0)\}$ with the metric G defined in local coordinates (t,θ,ϕ) by

$$ds_G^2 = dt^2 + \sinh^2 t\, d\theta^2 + \sinh^2 t \sin^2 \theta\, d\phi^2, \tag{2.22}$$

the metric G' defined by

$$ds_{G'}^2 = dt^2 + (\sinh^2 t - c)\, d\theta^2 + (\sinh^2 t - c)\sin^2 \theta\, d\phi^2 \quad \text{with } c < 0 \tag{2.23}$$

is harmonic with respect to G.

Indeed, $\mathbb{H}^3 \setminus \{(0)\}$ can be viewed as the warped product of the open interval $]0,\infty[$ and of $\mathbb{S}^2$ with warping function λ defined on $]0,\infty[$ by $\lambda(t) = \sinh t$.

2.4. Nonharmonic biharmonic metrics on product spaces.

In this paragraph, we study the biharmonicity of the warped product of two identity maps with warping functions of which one is constant.

We obtain then necessary and sufficient conditions for warped metrics to be nonharmonic biharmonic on product spaces.

Before giving the definition of biharmonic metrics, let us point out that, due to the fact that we are interested in *nonharmonic biharmonic* metrics, we conserve the same places of the domain and the codomain of the identity maps as in the definition of harmonic metrics (see [6]), contrary to the authors in [1] where the places of the domain and codomain are reversed.

Defintion 2.10. Let (M,g) be a Riemannian manifold. A metric G on M is said to be biharmonic with respect to g if the identity $\mathrm{id} : (M,g) \to (M,G)$ is a biharmonic map. Let (M,g) and (N,h) be two Riemannian manifolds of dimensions m and n, respectively, and $(M \times N, G)$ the Riemannian product of (M,g) and (N,h), respectively; that is, $(M \times N, G) = (M,g) \times (N,h)$.

Let $\varrho \in C^\infty(M)$ be a strictly positive function on M and let $G_\varrho = \pi^* g + (\varrho \circ \pi)^2 \sigma^* h$ be the warped product metric of g and h with warping function ϱ.

We have the following result.

THEOREM 2.11. *The warped product metric G_ϱ is nonharmonic biharmonic with respect to G if and only if*

$$\mathrm{grad}\, \varrho \neq 0, \qquad \Delta_g \omega + \frac{n}{4} d(\|\omega\|^2) - 2(\mathrm{Ric}_g(\omega^\sharp))^\flat = 0, \tag{2.24}$$

with $\omega = d\varrho^2$ and $(\mathrm{Ric}_g(\omega^\sharp))^\flat(X) = \mathrm{Ric}_g(\omega^\sharp, X)$, for all $X \in TM$, where $\mathrm{grad}\, \Delta_g = dd^* + d^* d$, and Ric_g are, respectively, the gradient, the Laplacian, and the Ricci tensor on (M,g).

Proof. By Theorem 2.8 applied to $\lambda \equiv 1$ and the function ϱ, the metric G_ϱ is nonharmonic with respect to G if and only if $\mathrm{grad}\, \varrho^2 = 2\varrho\, \mathrm{grad}\, \varrho \neq 0$.

Since ϱ is a strictly positive function, G_ϱ is then nonharmonic with respect to G if and only if (i) $\mathrm{grad}\, \varrho \neq 0$.

Let τ be the tension field of the identity map $\mathrm{id} : (M \times N, G) \to (M \times N, G_\varrho)$, and let J be the Jacobi field along id. We have

$$\tau = -\frac{n}{2} \operatorname{grad} \varrho^2, \qquad J(\tau) = \Delta^{\mathrm{id}} \tau + \operatorname{trace}_G R_\varrho(\cdot, \tau), \tag{2.25}$$

with $\Delta^{\mathrm{id}} \tau = -\operatorname{trace}_G(\nabla \nabla \tau - \nabla_\nabla \tau)$ is the Laplacian on the sections of $T(M \times N)$ and where R_ϱ is the Riemannian curvature operator on $(M \times N, G_\varrho)$.

Let $(E_1, \dots, E_{m+n})$ be a geodesic orthonormal basis on $T(M \times N)$,

$$\Delta^{\mathrm{id}} \tau = -\sum_{\hat{I}=1}^{m+n} \left(\nabla_{E_{\hat{I}}} \nabla_{E_{\hat{I}}} \tau - \nabla_{\nabla_{E_{\hat{I}}} E_{\hat{I}}} \tau \right) = -\left(\sum_{\hat{I}=1}^{m} \nabla_{E_{\hat{I}}} \nabla_{E_{\hat{I}}} \tau + \sum_{\hat{I}=m+1}^{m+n} \nabla_{E_{\hat{I}}} \nabla_{E_{\hat{I}}} \tau \right). \tag{2.26}$$

By Lemma 2.1 we have

$$\sum_{\hat{I}=1}^{m} \nabla_{E_{\hat{I}}} \nabla_{E_{\hat{I}}} \tau = \sum_{\hat{I}=1}^{m} \nabla^g_{E_{\hat{I}}} \nabla^g_{E_{\hat{I}}} \tau, \quad \text{where } \nabla^g \text{ is the Levi-Civita connection on } (M, g). \tag{2.27}$$

Otherwise for $\hat{I} \in \{m+1, \dots, m+n\}$,

$$\nabla_{E_{\hat{I}}} \tau = \frac{\tau \cdot \varrho}{\varrho} E_{\hat{I}} \quad (\text{see } [10]), \tag{2.28}$$

and then

$$\begin{aligned}
\nabla_{E_{\hat{I}}}(\nabla_{E_{\hat{I}}} \tau) &= \nabla_{E_{\hat{I}}} \left(\frac{\tau \cdot \varrho}{\varrho} E_{\hat{I}} \right) \\
&= \left(\frac{\tau \cdot \varrho}{\varrho} \right) \nabla_{E_{\hat{I}}} E_{\hat{I}} + \left(E_{\hat{I}} \cdot \frac{\tau \cdot \varrho}{\varrho} \right) E_{\hat{I}} \\
&= \frac{\tau \cdot \varrho}{\varrho} \left[-\varrho \langle E_{\hat{I}}, E_{\hat{I}} \rangle \operatorname{grad} \varrho + \nabla^h_{E_{\hat{I}}} E_{\hat{I}} \right], \\
&\quad \text{since } E_{\hat{I}} \cdot \frac{\tau \cdot \varrho}{\varrho} = 0 \text{ and by Lemma 2.1} \\
&= n \| \operatorname{grad} \varrho \|^2 \varrho h_{\hat{I}\hat{I}} \operatorname{grad} \varrho, \quad \text{since } \nabla^h_{E_{\hat{I}}} E_{\hat{I}} = \sigma_* \left(\nabla_{E_{\hat{I}}} E_{\hat{I}} \right) = 0,
\end{aligned} \tag{2.29}$$

where ∇^h is the Levi-Civita connection on (N, h).

Thus

$$\sum_{\hat{I}=m+1}^{m+n} \nabla_{E_{\hat{I}}} \nabla_{E_{\hat{I}}} \tau = \frac{n^2}{2} \| \operatorname{grad} \varrho \|^2 \operatorname{grad} \varrho^2. \tag{2.30}$$

We obtain then

$$\Delta^{\mathrm{id}} \tau = -\sum_{\hat{I}=1}^{m} \nabla^g_{E_{\hat{I}}} \nabla^g_{E_{\hat{I}}} \tau - \frac{n^2}{2} \| \operatorname{grad} \varrho \|^2 \operatorname{grad} \varrho^2. \tag{2.31}$$

On the other hand,

$$\sum_{\hat{I}=1}^{m+n} R_\varrho(E_{\hat{I}}, \tau) E_{\hat{I}} = \sum_{\hat{I}=1}^{m} R_\varrho(E_{\hat{I}}, \tau) E_{\hat{I}} + \sum_{\hat{I}=m+1}^{m+n} R_\varrho(E_{\hat{I}}, \tau) E_{\hat{I}}$$

$$= \sum_{\hat{I}=1}^{m} R_g(E_{\hat{I}}, \tau) E_{\hat{I}} + \sum_{\hat{I}=m+1}^{m+n} \varrho < E_{\hat{I}}, E_{\hat{I}} > \nabla_\tau(\mathrm{grad}\,\varrho),$$

by [10, Proposition 40, page 210]   (2.32)

$$= \sum_{\hat{I}=1}^{m} R_g(E_{\hat{I}}, \tau) E_{\hat{I}} + n\varrho \nabla_\tau(\mathrm{grad}\,\varrho)$$

$$= -\sum_{\hat{I}=1}^{m} R_g(\tau, E_{\hat{I}}) E_{\hat{I}} - \frac{n^2 \varrho^2}{2} \mathrm{grad}\left(\|\mathrm{grad}\,\varrho\|^2\right),$$

since

$$\nabla_\tau(\mathrm{grad}\,\varrho) = -\frac{n}{2} \nabla_{\mathrm{grad}\,\varrho^2}(\mathrm{grad}\,\varrho) = -n\varrho \nabla_{\mathrm{grad}\,\varrho}(\mathrm{grad}\,\varrho),$$

$$\nabla_{\mathrm{grad}\,\varrho}(\mathrm{grad}\,\varrho) = \frac{1}{2} \mathrm{grad}\left(\|\mathrm{grad}\,\varrho\|^2\right).$$

(2.33)

It follows

$$J(\tau) = -\frac{n}{2}\left\{ -\sum_{\hat{I}=1}^{m} \nabla_{E_{\hat{I}}}^{g} \nabla_{E_{\hat{I}}}^{g} \mathrm{grad}\,\varrho^2 - \sum_{\hat{I}=1}^{m} R_g\left(\mathrm{grad}\,\varrho^2, E_{\hat{I}}\right) E_{\hat{I}} \right.$$

$$\left. + n\|\mathrm{grad}\,\varrho\|^2 \, \mathrm{grad}\,\varrho^2 + n\varrho^2 \, \mathrm{grad}\left(\|\mathrm{grad}\,\varrho\|^2\right) \right\}$$

$$= -\frac{n}{2}\left\{ -\sum_{\hat{I}=1}^{m} \nabla_{E_{\hat{I}}}^{g} \nabla_{E_{\hat{I}}}^{g} \mathrm{grad}\,\varrho^2 - \sum_{\hat{I}=1}^{m} R_g\left(\mathrm{grad}\,\varrho^2, E_{\hat{I}}\right) E_{\hat{I}} \right.$$

$$\left. + \frac{n}{4} \mathrm{grad}\left(\|\mathrm{grad}\,\varrho^2\|^2\right) \right\},$$

(2.34)

since

$$\|\mathrm{grad}\,\varrho\|^2 \mathrm{grad}\,\varrho^2 + \varrho^2 \, \mathrm{grad}\left(\|\mathrm{grad}\,\varrho\|^2\right) = \frac{1}{4} \mathrm{grad}\left(\|\mathrm{grad}\,\varrho^2\|^2\right). \tag{2.35}$$

Thus G_ϱ is biharmonic with respect to G if and only if

$$-\sum_{\hat{I}=1}^{m} \nabla_{E_{\hat{I}}}^{g} \nabla_{E_{\hat{I}}}^{g} \mathrm{grad}\,\varrho^2 - \sum_{\hat{I}=1}^{m} R_g\left(\mathrm{grad}\,\varrho^2, E_{\hat{I}}\right) E_{\hat{I}} + \frac{n}{4} \mathrm{grad}\left(\|\mathrm{grad}\,\varrho^2\|^2\right) = 0. \tag{$*$}$$

By the Weitzenböck formula,

$$\Delta_g \omega = -\operatorname{trace} \nabla_g^2 \omega + \left(\operatorname{Ric}_g \left(\omega^\sharp \right) \right)^\flat, \tag{2.36}$$

where $\sharp$ and $\flat$ are the usual musical isomorphisms.

Hence the relation $(\star)$ is equivalent to
(ii)

$$\Delta_g \omega + \frac{n}{4} d \left(\| \omega \|^2 \right) - 2 \left(\operatorname{Ric}_g \left(\omega^\sharp \right) \right)^\flat = 0. \tag{2.37}$$

From (i) and (ii) we get the result. □

When (M,g) is Einstein, the one-form $(\operatorname{Ric}_g(\omega^\sharp))^\flat$ is proportional to ω. Thus as an immediate consequence of the previous theorem, we have the following.

COROLLARY 2.12. *Let $\varrho \in C^\infty(M)$, $\varrho > 0$, with $\operatorname{grad}\varrho \neq 0$. Assume that (M,g) is an Einstein manifold and that $\operatorname{grad}(\|\operatorname{grad}\varrho\|)$ is parallel to $\operatorname{grad}\varrho$. If G_ϱ is harmonic with respect to G, then ϱ is an isoparametric function on M.*

Recall that a function f on M is said to be isoparametric if there exist real functions α and β such that $\|df\|^2 = \alpha \circ f$ and $\Delta f = \beta \circ f$ or equivalently if $\operatorname{grad}(\|\operatorname{grad} f\|)$ and $\operatorname{grad}(\Delta f)$ are parallel to $\operatorname{grad} f$, with $\operatorname{grad} f \neq 0$ (see [1, 2] for more information about isoparametric functions).

Acknowledgments

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