UNIVERSAL SERIES BY TRIGONOMETRIC SYSTEM IN WEIGHTED L^1_μ SPACES

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We consider the question of existence of trigonometric series universal in weighted $L^1_\mu[0,2\pi]$ spaces with respect to rearrangements and in usual sense.

1. Introduction

Let X be a Banach space.

Definition 1.1. A series

$$\sum_{k=1}^{\infty} f_k, \quad f_k \in X, \tag{1.1}$$

is said to be universal in X with respect to rearrangements, if for any $f \in X$ the members of (1.1) can be rearranged so that the obtained series $\sum_{k=1}^{\infty} f_{\sigma(k)}$ converges to f by norm of X.

Definition 1.2. The series (1.1) is said to be universal (in X) in the usual sense, if for any $f \in X$ there exists a growing sequence of natural numbers n_k such that the sequence of partial sums with numbers n_k of the series (1.1) converges to f by norm of X.

Definition 1.3. The series (1.1) is said to be universal (in X) concerning partial series, if for any $f \in X$ it is possible to choose a partial series $\sum_{k=1}^{\infty} f_{n_k}$ from (1.1), which converges to the f by norm of X.

Note that many papers are devoted (see [1–10]) to the question on existence of various types of universal series in the sense of convergence *almost everywhere and on a measure*.

The first usual universal in the sense of convergence almost everywhere trigonometric series were constructed by Menshov [6] and Kozlov [5]. The series of the form

$$\frac{1}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \tag{1.2}$$

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was constructed just by them such that for any measurable-on-$[0,2\pi]$ function $f(x)$ there exists the growing sequence of natural numbers n_k such that the series (1.2) having the sequence of partial sums with numbers n_k converges to $f(x)$ almost everywhere on $[0,2\pi]$. (Note here that in this result, when $f(x) \in L^1_{[0,2\pi]}$, it is impossible to replace convergence almost everywhere by convergence in the metric $L^1_{[0,2\pi]}$).

This result was distributed by Talaljan on arbitrary orthonormal complete systems (see [8]). He also established (see [9]) that if $\{\phi_n(x)\}_{n=1}^{\infty}$—the normalized basis of space $L^p_{[0,1]}$, $p > 1$, then there exists a series of the form

$$\sum_{k=1}^{\infty} a_k \phi_k(x), \quad a_k \longrightarrow 0, \tag{1.3}$$

which has property: for any measurable function $f(x)$ the members of series (1.3) can be rearranged so that the again received series converge on a measure on [0,1] to $f(x)$.

Orlicz [7] observed the fact that there exist functional series that are universal with respect to rearrangements in the sense of a.e. convergence in the class of a.e. finite measurable functions.

It is also useful to note that even Riemann proved that every convergent numerical series which is not absolutely convergent is universal with respect to rearrangements in the class of all real numbers.

Let $\mu(x)$ be a measurable-on-$[0,2\pi]$ function with $0 < \mu(x) \le 1$, $x \in [0,2\pi]$, and let $L^1_{\mu}[0,2\pi]$ be a space of measurable functions $f(x)$, $x \in [0,2\pi]$, with

$$\int_0^{2\pi} |f(x)|\mu(x)dx < \infty. \tag{1.4}$$

Grigorian constructed a series of the form (see [3])

$$\sum_{k=-\infty}^{\infty} C_k e^{ikx} \quad \text{with} \quad \sum_{k=-\infty}^{\infty} |C_k|^q < \infty \ \forall q > 2, \tag{1.5}$$

which is universal in $L^1_{\mu}[0,2\pi]$ concerning partial series for some weighted function $\mu(x)$, $0 < \mu(x) \le 1$, $x \in [0,2\pi]$.

In [2] it is proved that for any given sequence of natural numbers $\{\lambda_m\}_{m=1}^{\infty}$ with $\lambda_m \nearrow \infty$ there exists a series by trigonometric system of the form

$$\sum_{k=1}^{\infty} C_k e^{ikx}, \qquad C_{-k} = \overline{C}_k, \tag{1.6}$$

with

$$\left| \sum_{k=1}^{m} C_k e^{ikx} \right| \le \lambda_m, \quad x \in [0,2\pi],\ m = 1,2,\dots, \tag{1.7}$$

so that for each $\varepsilon > 0$ a weighted function $\mu(x)$,

$$0 < \mu(x) \le 1, \quad |\{x \in [0,2\pi] : \mu(x) \ne 1\}| < \varepsilon, \tag{1.8}$$

can be constructed so that the series (1.6) is universal in the weighted space $L^1_\mu[0,2\pi]$ with respect simultaneously to rearrangements as well as to subseries.

In this paper, we prove the following results.

Theorem 1.4. *There exists a series of the form*

$$\sum_{k=-\infty}^{\infty} C_k e^{ikx} \quad \text{with} \quad \sum_{k=-\infty}^{\infty} |C_k|^q < \infty \ \forall q > 2 \tag{1.9}$$

such that for any number $\varepsilon > 0$ *a weighted function* $\mu(x)$, $0 < \mu(x) \leq 1$, *with*

$$|\{x \in [0,2\pi] : \mu(x) \neq 1\}| < \varepsilon \tag{1.10}$$

can be constructed so that the series (1.9) is universal in $L^1_\mu[0,2\pi]$ *with respect to rearrangements.*

Theorem 1.5. *There exists a series of the form (1.9) such that for any number* $\epsilon > 0$ *a weighted function* $\mu(x)$ *with (1.10) can be constructed so that the series (1.9) is universal in* $L^1_\mu[0,2\pi]$ *in the usual sense.*

2. Basic lemma

Lemma 2.1. *For any given numbers* $0 < \varepsilon < 1/2$, $N_0 > 2$, *and a step function*

$$f(x) = \sum_{s=1}^{q} \gamma_s \cdot \chi_{\Delta_s}(x), \tag{2.1}$$

where Δ_s *is an interval of the form* $\Delta_m^{(i)} = [(i-1)/2^m,\ i/2^m]$, $1 \leq i \leq 2^m$, *and*

$$|\gamma_s| \cdot \sqrt{|\Delta_s|} < \epsilon^3 \cdot \left(8 \cdot \int_0^{2\pi} f^2(x)dx\right)^{-1}, \quad s = 1,2,\ldots,q, \tag{2.2}$$

there exists a measurable set $E \subset [0,2\pi]$ *and a polynomial* $P(x)$ *of the form*

$$P(x) = \sum_{N_0 \leq |k| < N} C_k e^{ikx}, \tag{2.3}$$

which satisfy the conditions

$$|E| > 2\pi - \varepsilon, \tag{2.4a}$$

$$\int_E |P(x) - f(x)| dx < \varepsilon, \tag{2.4b}$$

$$\sum_{N_0 \leq |k| < N} |C_k|^{2+\varepsilon} < \varepsilon, \qquad C_{-k} = \overline{C}_k, \tag{2.4c}$$

$$\max_{N_0 \leq m < N} \left[\int_e \left|\sum_{N_0 \leq |k| \leq m} C_k e^{ikx}\right| dx\right] < \varepsilon + \int_e |f(x)| dx \tag{2.4d}$$

for every measurable subset e of E.

Proof. Let $0 < \epsilon < 1/2$ be an arbitrary number.

Set

$$g(x) = 1 \quad \text{if } x \in [0, 2\pi] \setminus \left[\frac{\varepsilon \cdot \pi}{2}, \frac{3\varepsilon \cdot \pi}{2}\right], \tag{2.5}$$

$$g(x) = 1 - \frac{2}{\varepsilon} \quad \text{if } x \in \left[\frac{\varepsilon \cdot \pi}{2}, \frac{3\varepsilon \cdot \pi}{2}\right]. \tag{2.6}$$

We choose natural numbers ν_1 and N_1 so large that the following inequalities be satisfied:

$$\frac{1}{2\pi}\left|\int_0^{2\pi} g_1(t)e^{-ikt}dt\right| < \frac{\varepsilon}{16 \cdot \sqrt{N_0}}, \quad |k| < N_0, \tag{2.7}$$

where

$$g_1(x) = \gamma_1 \cdot g(\nu_1 \cdot x) \cdot \chi_{\Delta_1}(x). \tag{2.8}$$

(By $\chi_E(x)$ we denote the characteristic function of the set E.) We put

$$E_1 = \{x \in \Delta_s : g_s(x) = \gamma_s\}. \tag{2.9}$$

By (2.5), (2.8), and (2.9) we have

$$|E_1| > 2\pi \cdot (1 - \epsilon) \cdot |\Delta_1|, \qquad g_1(x) = 0, \quad x \notin \Delta_1, \tag{2.10}$$

$$\int_0^{2\pi} g_1^2(x)dx < \frac{2}{\epsilon} \cdot |\gamma_1|^2 \cdot |\Delta_1|. \tag{2.11}$$

Since the trigonometric system $\{e^{ikx}\}_{k=-\infty}^{\infty}$ is complete in $L^2[0, 2\pi]$, we can choose a natural number $N_1 > N_0$ so large that

$$\int_0^{2\pi}\left|\sum_{0 \le |k| < N_1} C_k^{(1)} e^{ikx} - g_1(x)\right| dx \le \frac{\varepsilon}{8}, \tag{2.12}$$

where

$$C_k^{(1)} = \frac{1}{2\pi}\int_0^{2\pi} g_1(t)e^{-ikt}dt. \tag{2.13}$$

Hence by (2.7), (2.8), and (2.12) we obtain

$$\int_0^{2\pi}\left|\sum_{N_0 \le |k| < N_1} C_k^{(1)} e^{ikx} - g_1(x)\right| dx \le \frac{\varepsilon}{8} + \left[\sum_{0 \le |k| < N_0} \left|C_k^{(1)}\right|^2\right]^{1/2} < \frac{\varepsilon}{4}. \tag{2.14}$$

Now assume that the numbers $\nu_1 < \nu_2 < \cdots \nu_{s-1}$, $N_1 < N_2 < \cdots < N_{s-1}$, functions $g_1(x)$, $g_2(x), \ldots, g_{s-1}(x)$, and the sets $E_1, E_2, \ldots, E_{s-1}$ are defined. We take sufficiently large natural numbers $\nu_s > \nu_{s-1}$ and $N_s > N_{s-1}$ to satisfy

$$\frac{1}{2\pi}\left|\int_0^{2\pi} g_s(t)e^{-ikt}dt\right| < \frac{\varepsilon}{16\cdot\sqrt{N_{s-1}}}, \quad 1\le s\le q,\ |k|<N_{s-1}, \tag{2.15}$$

$$\int_0^{2\pi}\left|\sum_{0\le|k|<N_s} C_k^{(s)}e^{ikx} - g_s(x)\right|dx \le \frac{\varepsilon}{4^{s+1}}, \tag{2.16}$$

where

$$g_s(x) = \gamma_s\cdot g(\nu_s\cdot x)\cdot\chi_{\Delta_s}(x), \qquad C_k^{(s)} = \frac{1}{2\pi}\int_0^{2\pi} g_s(t)e^{-ikt}dt. \tag{2.17}$$

Set

$$E_s = \{x\in\Delta_s : g_s(x) = \gamma_s\}. \tag{2.18}$$

Using the above arguments (see (2.19)–(2.21)), we conclude that the function $g_s(x)$ and the set E_s satisfy the conditions

$$|E_s| > 2\pi\cdot(1-\epsilon)\cdot|\Delta_s|; g_s(x) = 0, \quad x\notin\Delta_s, \tag{2.19}$$

$$\int_0^{2\pi} g_s^2(x)dx < \frac{2}{\epsilon}\cdot|\gamma_s|^2\cdot|\Delta_s|, \tag{2.20}$$

$$\int_0^{2\pi}\left|\sum_{N_{s-1}\le|k|<N_s} C_k^{(s)}e^{ikx} - g_1(x)\right|dx < \frac{\varepsilon}{2^{s+1}}. \tag{2.21}$$

Thus, by induction, we can define natural numbers $\nu_1<\nu_2<\cdots\nu_q$, $N_1<N_2<\cdots<N_q$, functions $g_1(x),g_2(x),\dots,g_q(x)$, and sets $E_1,E_2,\dots,E_q$ such that conditions (2.17)–(2.19) are satisfied for all s, $1\le s\le q$. We define a set E and a polynomial $P(x)$ as follows:

$$E = \bigcup_{s=1}^{q} E_s, \tag{2.22}$$

$$P(x) = \sum_{N_0\le|k|<N} C_k e^{ikx} = \sum_{s=1}^{q}\left[\sum_{N_{s-1}\le|k|<N_s} C_k^{(s)}e^{ikx}\right], \qquad C_{-k} = \overline{C}_k, \tag{2.23}$$

where

$$C_k = C_k^{(s)} \quad \text{for } N_{s-1}\le|k|<N_s,\ s=1,2,\dots,q,\ N = N_q - 1. \tag{2.24}$$

By Bessel's inequality and (2.5), (2.17) for all $s\in[1,q]$ we get

$$\begin{aligned}\left[\sum_{N_{s-1}\le|k|<N_s}\left|C_k^{(s)}\right|^2\right]^{1/2} &\le \left[\int_o^{2\pi} g_s^2(x)dx\right]^{1/2}\\ &\le \frac{2}{\sqrt{\varepsilon}}\cdot|\gamma_s|\cdot\sqrt{|\Delta_s|}, \quad s=1,2,\dots,q.\end{aligned} \tag{2.25}$$

From (2.5), (2.15), and (2.16), it follows that

$$|E| > 2\pi - \varepsilon. \tag{2.26}$$

Taking relations (2.1), (2.5), (2.13), (2.15), (2.21)–(2.24), we obtain

$$\int_E |P(x) - f(x)|\,dx \le \sum_{s=1}^{q}\left[\int_E \left|\sum_{N_{s-1}\le|k|<N_s} C_k^{(s)} e^{ikx} - g_s(x)\right| dx\right] < \varepsilon. \tag{2.27}$$

By (2.1), (2.2), (2.23)-(2.24) for any $k \in [N_0, N]$, we have

$$\begin{aligned}
\sum_{N_0\le|k|<N} |C_k|^{2+\epsilon} &\le \max_{N_0\le k\le N} |C_k|^{\epsilon} \cdot \sum_{k=N_0}^{N} |C_k|^2 \\
&\le \max_{1\le s\le q}\left[\sqrt{\frac{8}{\epsilon}}\cdot|\gamma_s|\cdot\sqrt{|\Delta_s|}\right]\cdot\sum_{s=1}^{q}\left[\sum_{N_{s-1}\le|k|<N_s}\left|C_k^{(s)}\right|^2\right] \\
&\le \max_{1\le s\le q}\left[\sqrt{\frac{8}{\epsilon}}\cdot|\gamma_s|\cdot\sqrt{|\Delta_s|}\right]\cdot\frac{8}{\epsilon}\cdot\sum_{s=1}^{q}|\gamma_s|^2\cdot|\Delta_s| \\
&\le \max_{1\le s\le q}\left[\sqrt{\frac{8}{\epsilon}}\cdot|\gamma_s|\cdot\sqrt{|\Delta_s|}\right]\cdot\frac{8}{\epsilon}\cdot\left[\int_0^1 f^2(x)dx\right] < \epsilon;
\end{aligned} \tag{2.28}$$

that is, the statements (2.4a)–(2.4c) of Lemma 2.1 are satisfied. Now we will check the fulfillment of statement (2.4d) of Lemma 2.1. Let $N_0 \le m < N$, then for some $s_0, 1 \le s_0 \le q$, $(N_{s_0} \le m < N_{s_0+1})$ we will have (see (2.23) and (2.24))

$$\sum_{N_0\le|k|\le m} C_k e^{ikx} = \sum_{s=1}^{s_0}\left[\sum_{N_{s-1}\le|k|<N_s} C_k^{(s)} e^{ikx}\right] + \sum_{N_{s_0-1}\le|k|\le m} C_k^{(s_0+1)} e^{ikx}. \tag{2.29}$$

Hence and from (2.1), (2.2), (2.5), (2.21), (2.22), and (2.25) for any measurable set $e \subset E$, we obtain

$$\begin{aligned}
&\int_e \left|\sum_{N_{s-1}\le|k|\le m} C_k e^{ikx}\right| dx \\
&\quad\le \sum_{s=1}^{s_0}\left[\int_e\left|\sum_{N_{s-1}\le|k|<N_s} C_k^{(s)} e^{ikx} - g_s(x)\right| dx\right] \\
&\quad\quad+ \sum_{s=1}^{s_0}\int_e |g_s(x)|\,dx + \int_e\left|\sum_{N_{s_0-1}\le|k|\le m} C_k^{(s_0+1)} e^{ikx}\right| dx \\
&\quad< \sum_{s=1}^{s_0}\frac{\varepsilon}{2^{s+1}} + \int_e |f(x)|\,dx + \frac{2}{\sqrt{\varepsilon}}\cdot|\gamma_{s_0+1}|\cdot\sqrt{|\Delta_{s_0+1}|} < \int_e |f(x)|\,dx + \varepsilon.
\end{aligned} \tag{2.30}$$

□

3. Proof of theorems

Proof of Theorem 1.5. Let

$$f_1(x), f_2(x), \dots, f_n(x), \quad x \in [0, 2\pi], \tag{3.1}$$

be a sequence of all step functions, values, and constancy interval endpoints of which are rational numbers. Applying lemma consecutively, we can find a sequence $\{E_s\}_{s=1}^{\infty}$ of sets and a sequence of polynomials

$$P_s(x) = \sum_{N_{s-1} \le |k| < N_s} C_k^{(s)} e^{ikx}, \\ 1 = N_0 < N_1 < \cdots < N_s < \cdots, \quad s = 1, 2, \dots, \tag{3.2}$$

which satisfy the conditions

$$|E_s| > 1 - 2^{-2(s+1)}, \qquad E_s \subset [0, 2\pi], \tag{3.3}$$

$$\int_{E_s} |P_s(x) - f_s(x)|\, dx < 2^{-2(s+1)}, \tag{3.4}$$

$$\sum_{N_{s-1} \le |k| < N_s} |C_k^{(s)}|^{2+2^{-2s}} < 2^{-2s}, \qquad C_{-k}^{(s)} = \overline{C}_k^{(s)}, \tag{3.5}$$

$$\max_{N_{s-1} \le p < N_s} \left[\int_e \left| \sum_{N_{s-1} \le |k| \le p} C_k e^{ikx} \right| dx \right] < 2^{-2(s+1)} + \int_e |f_s(x)|\, dx \tag{3.6}$$

for every measurable subset e of E_s.

Denote

$$\sum_{k=-\infty}^{\infty} C_k e^{ikx} = \sum_{s=1}^{\infty} \left[\sum_{N_{s-1} \le |k| < N_s} C_k^{(s)} e^{ikx} \right], \tag{3.7}$$

where $C_k = C_k^{(s)}$ for $N_{s-1} \le |k| < N_s$, $s = 1, 2, \dots$.

Let ε be an arbitrary positive number. Setting

$$\Omega_n = \bigcap_{s=n}^{\infty} E_s, \quad n = 1, 2, \dots, \tag{3.8}$$

$$E = \Omega_{n_0} = \bigcap_{s=n_0}^{\infty} E_s, \quad n_0 = [\log_{1/2} \varepsilon] + 1, \tag{3.9}$$

$$B = \bigcup_{n=n_0}^{\infty} \Omega_n = \Omega_{n_0} \bigcup \left(\bigcup_{n=n_0+1}^{\infty} \Omega_n \setminus \Omega_{n-1} \right). \tag{3.10}$$

It is clear (see (3.3)) that $|B| = 2\pi$ and $|E| > 2\pi - \varepsilon$.

We define a function $\mu(x)$ in the following way:

$$\begin{aligned} \mu(x) &= 1 \quad \text{for } x \in E \cup ([0,2\pi] \setminus B), \\ \mu(x) &= \mu_n \quad \text{for } x \in \Omega_n \setminus \Omega_{n-1},\ n \geq n_0 + 1, \end{aligned} \tag{3.11}$$

where

$$\mu_n = \left[2^{4n} \cdot \prod_{s=1}^{n} h_s \right]^{-1}, \tag{3.12}$$

$$h_s = \|f_s(x)\|_C + \max_{N_{s-1} \leq p < N_s} \left\| \sum_{N_{s-1} \leq |k| \leq p} C_k^{(s)} e^{ikx} \right\|_C + 1, \tag{3.13}$$

where

$$\|g(x)\|_C = \max_{x \in [0,2\pi]} |g(x)|, \tag{3.14}$$

$g(x)$ is a continuous function on $[0,2\pi]$.

From (3.5), (3.7)–(3.12), we obtain the following.

(A) $0 < \mu(x) \leq 1$, $\mu(x)$ is a measurable function and

$$|\{x \in [0,2\pi] : \mu(x) \neq 1\}| < \varepsilon. \tag{3.15}$$

(B) $\sum_{k=1}^{\infty} |C_k|^q < \infty$ for all $q > 2$.

Hence, obviously, we have

$$\lim_{k \to \infty} C_k = 0. \tag{3.16}$$

It follows from (3.9)–(3.12) that for all $s \geq n_0$ and $p \in [N_{s-1}, N_s)$,

$$\begin{aligned} \int_{[0,2\pi] \setminus \Omega_s} \left| \sum_{N_{s-1} \leq |k| \leq p} C_k^{(s)} e^{ikx} \right| \mu(x) dx &= \sum_{n=s+1}^{\infty} \left[\int_{\Omega_n \setminus \Omega_{n-1}} \left| \sum_{N_{s-1} \leq |k| \leq p} C_k^{(s)} e^{ikx} \right| \mu_n dx \right] \\ &\leq \sum_{n=s+1}^{\infty} 2^{-4n} \left[\int_0^{2\pi} \left| \sum_{N_{s-1} \leq |k| \leq p} C_k^{(s)} e^{ikx} \right| h_s^{-1} dx \right] < 2^{-4s}. \end{aligned} \tag{3.17}$$

By (3.4), (3.9)–(3.12) for all $s \geq n_0$, we have

$$\begin{aligned} &\int_0^{2\pi} |P_s(x) - f_s(x)| \mu(x) dx \\ &\quad = \int_{\Omega_s} |P_s(x) - f_s(x)| \mu(x) dx + \int_{[0,2\pi] \setminus \Omega_s} |P_s(x) - f_s(x)| \mu(x) dx \\ &\quad = 2^{-2(s+1)} + \sum_{n=s+1}^{\infty} \left[\int_{\Omega_n \setminus \Omega_{n-1}} |P_s(x) - f_s(x)| \mu_n dx \right] \end{aligned}$$

$$\leq 2^{-2(s+1)} + \sum_{n=s+1}^{\infty} 2^{-4s}\left[\int_0^{2\pi}\left(|f_s(x)| + \left|\sum_{N_{s-1}\leq|k|<N_s} C_k^{(s)}e^{ikx}\right|\right)h_s^{-1}dx\right]$$
$$< 2^{-2(s+1)} + 2^{-4s} < 2^{-2s}. \tag{3.18}$$

Taking relations (3.6), (3.9)–(3.12), and (3.17) into account we obtain that for all $p \in [N_{s-1}, N_s)$ and $s \geq n_0 + 1$,

$$\begin{aligned}
&\int_0^{2\pi}\left|\sum_{N_{s-1}\leq|k|\leq p} C_k^{(s)}e^{ikx}\right|\mu(x)dx \\
&\quad= \int_{\Omega_s}\left|\sum_{N_{s-1}\leq|k|\leq p} C_k^{(s)}e^{ikx}\right|\mu(x)dx + \int_{[0,2\pi]\setminus\Omega_s}\left|\sum_{N_{s-1}\leq|k|\leq p} C_k^{(s)}e^{ikx}\right|\mu(x)dx \\
&\quad< \sum_{n=n_0+1}^{s}\left[\int_{\Omega_n\setminus\Omega_{n-1}}\left|\sum_{N_{s-1}\leq|k|\leq p} C_k^{(s)}e^{ikx}\right|dx\right]\cdot\mu_n + 2^{-4s} \\
&\quad< \sum_{n=n_0+1}^{s}\left(2^{-2(s+1)} + \int_{\Omega_n\setminus\Omega_{n-1}}|f_s(x)|dx\right)\mu_n + 2^{-4s} \\
&\quad= 2^{-2(s+1)}\cdot\sum_{n=n_0+1}^{s}\mu_n + \int_{\Omega_s}|f_s(x)|\mu(x)dx + 2^{-4s} \\
&\quad< \int_0^{2\pi}|f_s(x)|\mu(x)dx + 2^{-4s}.
\end{aligned} \tag{3.19}$$

Let $f(x) \in L_\mu^1[0,2\pi]$, that is, $\int_0^{2\pi}|f(x)|\mu(x)dx < \infty$.

It is easy to see that we can choose a function $f_{\nu_1}(x)$ from the sequence (3.1) such that

$$\int_0^{2\pi}|f(x) - f_{\nu_1}(x)|\mu(x)dx < 2^{-2}, \quad \nu_1 > n_0 + 1. \tag{3.20}$$

Hence, we have

$$\int_0^{2\pi}|f_{\nu_1}(x)|\mu(x)dx < 2^{-2} + \int_0^{2\pi}|f(x)|\mu(x)dx. \tag{3.21}$$

From (2.1), (A), (3.18), and (3.20), we obtain with $m_1 = 1$,

$$\begin{aligned}
&\int_0^{2\pi}|f(x) - [P_{\nu_1}(x) + C_{m_1}e^{im_1x}]|\mu(x)dx \\
&\quad\leq \int_0^{2\pi}|f(x) - f_{\nu_1}(x)|\mu(x)dx + \int_0^{2\pi}|f_{\nu_1}(x) - P_{\nu_1}(x)|\mu(x)dx \\
&\qquad+ \int_0^{2\pi}|C_{m_1}e^{im_1x}|\mu(x)dx < 2\cdot 2^{-2} + 2\pi\cdot|C_{m_1}|.
\end{aligned} \tag{3.22}$$

Assume that numbers $\nu_1 < \nu_2 < \cdots < \nu_{q-1}$, $m_1 < m_2 < \cdots < m_{q-1}$ are chosen in such a way that the following condition is satisfied:

$$\int_0^{2\pi} \left| f(x) - \sum_{s=1}^{j} \left[P_{\nu_s}(x) + C_{m_s} e^{i m_s x} \right] \right| \mu(x) dx < 2 \cdot 2^{-2j} + 2\pi \cdot |C_{m_j}|, \quad 1 \le j \le q-1. \tag{3.23}$$

We choose a function $f_{\nu_q}(x)$ from the sequence (3.1) such that

$$\int_0^{2\pi} \left| \left(f(x) - \sum_{s=1}^{q-1} \left[P_{\nu_s}(x) + C_{m_s} e^{i m_s x} \right] \right) - f_{n_q}(x) \right| \mu(x) dx < 2^{-2q}, \tag{3.24}$$

where $\nu_q > \nu_{q-1}$; $\nu_q > m_{q-1}$

This, with (3.23), implies

$$\int_0^{2\pi} |f_{\nu_q}(x)| \mu(x) dx < 2^{-2q} + 2 \cdot 2^{-2(q-1)} + 2\pi \cdot |C_{m_{q-1}}| = 9 \cdot 2^{-2q} + 2\pi \cdot |C_{M_{q-1}}|. \tag{3.25}$$

By (3.18), (3.19), and (3.25) we obtain

$$\begin{gathered} \int_0^{2\pi} |f_{\nu_q}(x) - P_{\nu_q}(x)| \mu(x) dx < 2^{-2\nu_q}, \\ P_{\nu_q}(x) = \sum_{N_{\nu_q - 1} \le |k| < N_{\nu_q}} C_k^{(\nu_q)} e^{ikx}, \end{gathered} \tag{3.26}$$

$$\max_{N_{\nu_q-1} \le p < N\nu_q} \int_0^{2\pi} \left| \sum_{k=N_{\nu_q-1}}^{p} C_k^{(\nu_q)} e^{ikx} \right| \mu(x) dx < 10 \cdot 2^{-2q} + 2\pi \cdot |C_{m_{q-1}}|. \tag{3.27}$$

Denote

$$m_q = \min \left\{ n \in N : n \notin \left\{ \left\{ \{k\}_{k=N_{\nu_s-1}}^{N_{\nu_s}-1} \right\}_{s=1}^{q} \cup \{m_s\}_{s=1}^{q-1} \right\} \right\}. \tag{3.28}$$

From (2.1), (A), (3.24), and (3.26), we have

$$\begin{aligned} &\int_0^{2\pi} \left| f(x) - \sum_{s=1}^{q} \left[P_{\nu_s}(x) + C_{m_s} e^{i m_s x} \right] \right| \mu(x) dx \\ &\quad \le \int_0^{2\pi} \left| \left(f(x) - \sum_{s=1}^{q-1} \left[P_{\nu_s}(x) + C_{m_s} e^{i m_s x} \right] \right) - f_{\nu_q}(x) \right| \mu(x) dx \end{aligned}$$

$$+\int_0^{2\pi} |f_{\nu_q}(x)-P_{\nu_q}(x)|\mu(x)dx$$
$$+\int_0^{2\pi} |C_{m_q}e^{im_qx}|\mu(x)dx < 2\cdot 2^{-2q}+2\pi\cdot |C_{m_q}|. \tag{3.29}$$

Thus, by induction we, on q, can choose from series (3.7) a sequence of members

$$C_{m_q}e^{im_qx}, \quad q=1,2,\ldots, \tag{3.30}$$

and a sequence of polynomials

$$P_{\nu_q}(x)=\sum_{N_{\nu_q-1}\le|k|<N_{\nu_q}} C_k^{(\nu_q)}e^{ikx}, \quad N_{n_q-1}>N_{n_{q-1}},\ q=1,2,\ldots \tag{3.31}$$

such that conditions (3.27)–(3.29) are satisfied for all $q\ge 1$.

Taking account the choice of $P_{\nu_q}(x)$ and $C_{m_q}e^{im_qx}$ (see (3.28) and (3.31)), we conclude that the series

$$\sum_{q=1}^{\infty}\left[\sum_{N_{\nu_q-1}\le|k|<N_{\nu_q}} C_k^{(\nu_q)}e^{ikx}+C_{m_q}e^{iqx}\right] \tag{3.32}$$

is obtained from the series (3.7) by rearrangement of members. Denote this series by $\sum C_{\sigma(k)}e^{i\sigma(k)x}$.

It follows from (3.16), (3.27), and (3.29) that the series $\sum C_{\sigma(k)}e^{i\sigma(k)x}$ converges to the function $f(x)$ in the metric $L^1_\mu[0,2\pi]$, that is, the series (3.7) is universal with respect to rearrangements (see Definition 1.1). □

Proof of Theorem 1.5. Applying Lemma 2.1 consecutively, we can find a sequence $\{E_s\}_{s=1}^{\infty}$ of sets and a sequence of polynomials

$$P_s(x)=\sum_{N_{s-1}\le|k|<N_s} C_k^{(s)}e^{ikx}, \qquad C_{-k}^{(s)}=\overline{C}_k^{(s)},$$
$$1=N_0<N_1<\cdots<N_s<\cdots, \quad s=1,2,\ldots, \tag{3.33}$$

which satisfy the conditions

$$|E_s|>1-2^{-2(s+1)}, \qquad E_s\subset[0,2\pi], \tag{3.34}$$

$$\sum_{N_{s-1}\le|k|<N_s} |C_k^{(s)}|^{2+2^{-2s}}<2^{-2s}, \tag{3.35}$$

$$\int_{E_n}\left|f_n(x)-\sum_{s=1}^{n}P_s(x)\right|dx<2^{-n}, \quad n=1,2,\ldots, \tag{3.36}$$

where $\{f_n(x)\}_{n=1}^{\infty}$, $x\in[0,2\pi]$, is a sequence of all step functions, values, and constancy interval endpoints of which are rational numbers.

Denote

$$\sum_{k=-\infty}^{\infty} C_k e^{ikx} = \sum_{s=1}^{\infty}\left[\sum_{N_{s-1}\le|k|<N_s} C_k^{(s)} e^{ikx}\right], \tag{3.37}$$

where $C_k = C_k^{(s)}$ for $N_{s-1} \le |k| < N_s$, $s = 1,2,\dots$.

It is clear (see (3.35)) that

$$\sum_{k=1}^{\infty} |C_k|^q < \infty \quad \forall q > 2. \tag{3.38}$$

Repeating reasoning of Theorem 1.4, a weighted function $\mu(x)$, $0 < \mu(x) \le 1$, can be constructed so that the following condition is satisfied:

$$\int_0^{2\pi} \left| f_n(x) - \sum_{s=1}^{n} P_s(x)\right| \cdot \mu(x)dx < 2^{-2n}, \quad n = 1,2,\dots. \tag{3.39}$$

For any function $f(x) \in L^1_\mu[0,1]$, we can choose a subsystem $\{f_{n_\nu}(x)\}_{\nu=1}^{\infty}$ from the sequence (3.1) such that

$$\int_0^{2\pi} |f(x) - f_{n_\nu}(x)|\mu(x)dx < 2^{-2\nu}. \tag{3.40}$$

From (3.37)–(3.40), we conclude

$$\begin{aligned}\int_0^{2\pi}\left|f(x) - \sum_{|k|\le M_\nu} C_k e^{ikx}\right|\mu(x)dx &= \int_0^{2\pi}\left|f(x) - \sum_{s=1}^{n_\nu}\left[\sum_{N_{s-1}\le|k|<N_s} C_k^{(s)} e^{ikx}\right]\right|\mu(x)dx \\ &\le \int_0^{2\pi} |f(x) - f_{\nu_k}(x)| \cdot \mu(x)dx \\ &\quad + \int_0^{2\pi}\left|f_{\nu_k}(x) - \sum_{s=1}^{\nu_k} P_s(x)\right| \cdot \mu(x)dx < 2^{-2k} + 2^{-2\nu_k},\end{aligned} \tag{3.41}$$

where $M_\nu = N_{n_\nu} - 1$.

Thus, the series (3.37) is universal in $L^1_\mu[0,1]$ in the sense of usual (see Definition 1.2).

Theorem 1.5 is proved. □

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