MORE ON RC-LINDELÖF SETS AND ALMOST RC-LINDELÖF SETS

MOHAMMAD S. SARSAK

Received 25 November 2004; Revised 3 April 2006; Accepted 30 May 2006

We study new properties and characterizations of rc-Lindelöf sets and almost rc-Lindelöf sets; a special interest is given to the mapping properties of such sets. We also obtain some product theorems concerning rc-Lindelöf spaces.

1. Introduction and preliminaries

A subset A of a space X is called regular open if $A = \operatorname{Int}\overline{A}$, and regular closed if $X \backslash A$ is regular open, or equivalently, if $A = \overline{\operatorname{Int} A}$. A is called semiopen [16] (resp., preopen [17], semi-preopen [3], b-open [4]) if $A \subset \overline{\operatorname{Int} A}$ (resp., $A \subset \operatorname{Int}\overline{A}$, $A \subset \overline{\operatorname{Int}\overline{A}}$, $A \subset \overline{\operatorname{Int} A} \cup \operatorname{Int}\overline{A}$). The concept of a preopen set was introduced in [6] where the term locally dense was used and the concept of a semi-preopen set was introduced in [1] under the name β-open. It was pointed out in [3] that A is semi-preopen if and only if $P \subset A \subset \overline{P}$ for some preopen set P. Clearly, every open set is both semiopen and preopen, semiopen sets as well as preopen sets are b-open, and b-open sets are semi-preopen. A is called semiclosed (resp., preclosed, semi-preclosed, b-closed) if $X \backslash A$ is semiopen (resp., preopen, semi-preopen, b-open). A is called semiregular [8] if it is both semiopen and semiclosed, *or equivalently, if there exists a regular* open set U such that $U \subset A \subset \overline{U}$.

Clearly, every regular closed (regular open) set is semiregular. The semiclosure (resp., preclosure, semi-preclosure, b-closure) denoted by $\operatorname{scl} A$ (resp., $\operatorname{pcl} A$, $\operatorname{spcl} A$, $\operatorname{bcl} A$) is the intersection of all semiclosed (resp., preclosed, semi-preclosed, b-closed) subsets of X containing A, or equivalently, is the smallest semiclosed (resp., preclosed, semi-preclosed, b-closed) set containing A. Dually, the semi-interior (resp., preinterior, semi-preinterior, b-interior) denoted by $\operatorname{sint} A$ (resp., $\operatorname{pint} A$, $\operatorname{spint} A$, $\operatorname{bint} A$) is the union of all semiopen (resp., preopen, semi-preopen, b-open) subsets of X contained in A, or equivalently, is the largest semiopen (resp., preopen, semi-preopen, b-open) set contained in A.

A function f from a space X into a space Y is called almost open [20] if $f^{-1}(\overline{U}) \subset \overline{f^{-1}(U)}$ whenever U is open in Y, semicontinuous [16] if the inverse image of each

Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2006, Article ID 57918, Pages 1–9
DOI 10.1155/IJMMS/2006/57918

open set is semiopen, β-continuous [1] if the inverse image of each open set is β-open, weakly θ-irresolute [13] if the inverse image of each regular closed set is semiopen, rc-continuous [14] if the inverse image of each regular closed set is regular closed, and wrc-continuous [2] if the inverse image of each regular closed set is semi-preopen. We will use the term semiprecontinuous to indicate β-continuous. Clearly, every semicontinuous function is semi-precontinuous, every rc-continuous function is weakly θ-irresolute, and every weakly θ-irresolute function is wrc-continuous. It is also easy to see that a function that is both semicontinuous (resp., semi-precontinuous) and almost open is weakly θ-irresolute (resp., wrc-continuous).

A function f from a space X into a space Y is called somewhat continuous [12] if for each nonempty open set V in Y, $\operatorname{int} f^{-1}(V) \neq \phi$.

A space X is called a weak P-space [18] if for each countable family $\{U_n : n \in \mathbb{N}\}$ of open subsets of X, $\overline{\cup U_n} = \cup \overline{U}_n$. Clearly, X is a weak P-space if and only if the countable union of regular closed subsets of X is regular closed (closed).

A space X is called rc-Lindelöf [15] (resp., nearly Lindelöf [5]) if every regular closed (resp., regular open) cover of X has a countable subcover, and called almost rc-Lindelöf [10] if every regular closed cover of X has a countable subfamily whose union is dense in X.

A subset A of a space X is called an S-set in X [7] if every cover of A by regular closed subsets of X has a finite subcover, and called an rc-Lindelöf set in X (resp., an almost rc-Lindelöf set in X) [9] if every cover of A by regular closed subsets of X admits a countable subfamily that covers A (resp., the closure of the union of whose members contains A). Obviously, every S-set is an rc-Lindelöf set and every rc-Lindelöf set is an almost rc-Lindelöf set; it is also clear that a subset A of a weak P-space X is rc-Lindelöf in X if and only if it is almost rc-Lindelöf in X.

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers. For the concepts not defined here, we refer the reader to Engelking [11].

In concluding this section, we recall the following facts for their importance in the material of our paper.

Theorem 1.1 [9]. *If A is an rc-Lindelöf (resp., almost rc-Lindelöf) set in a space X and B is a regular open subset of X, then $A \cap B$ is rc-Lindelöf (resp., almost rc-Lindelöf) in X. In particular, a regular open subset A of an rc-Lindelöf (resp., almost rc-Lindelöf) space X is rc-Lindelöf (resp., almost rc-Lindelöf) in X.*

Theorem 1.2 [9]. *Let A be a preopen subset of a space X and $B \subset A$. Then B is rc-Lindelöf (resp., almost rc-Lindelöf) in X if and only if B is rc-Lindelöf (resp., almost rc-Lindelöf) in A. In particular, a preopen subset A of a space X is rc-Lindelöf (resp., almost rc-Lindelöf) in X if and only if A is an rc-Lindelöf (resp., almost rc-Lindelöf) subspace.*

Proposition 1.3 [19]. *If A is an almost rc-Lindelöf set in a space X and $A \subset B \subset \overline{A}$, then B is almost rc-Lindelöf in X.*

Proposition 1.4 [9]. *The countable union of rc-Lindelöf (resp., almost rc-Lindelöf) sets in a space X is rc-Lindelöf (resp., almost rc-Lindelöf) in X.*

Proposition 1.5 [9]. *A subset A of a space X is rc-Lindelöf (resp., almost rc-Lindelöf) in X if and only if every cover of A by semiopen subsets of X admits a countable subfamily the union of the closures of whose members (resp., the closure of the union of whose members) contains A.*

Proposition 1.6 [19]. *Let A be a preopen, almost rc-Lindelöf set in a space X and B a regular closed subset of X, then $A \cap B$ is almost rc-Lindelöf in X. In particular, a regular closed subset A of an almost rc-Lindelöf space X is almost rc-Lindelöf in X.*

Lemma 1.7. *If A is a preopen subset of a space X and U is open in X, then $\overline{A \cap U} \cap A = \overline{U} \cap A$.*

2. Further properties

This section is devoted to study new properties concerning rc-Lindelöf sets and almost rc-Lindelöf sets. We obtain several characterizations of rc-Lindelöf sets and almost rc-Lindelöf sets.

The following proposition is an improvement of Proposition 1.6 and the fact of Theorem 1.1 that a regular open subset of an almost rc-Lindelöf space X is almost rc-Lindelöf in X.

Proposition 2.1. *Let A be a preopen, almost rc-Lindelöf set in a space X and B a semiregular subset of X, then $A \cap B$ is almost rc-Lindelöf in X. In particular, a semiregular subset A of an almost rc-Lindelöf space X is almost rc-Lindelöf in X.*

Proof. Since B is a semiregular subset of X, there exists a regular open subset U of X such that $U \subset B \subset \overline{U}$, thus by Lemma 1.7, it follows that $A \cap U \subset A \cap B \subset \overline{U} \cap A \subset \overline{A \cap U}$. Since A is almost rc-Lindelöf set in X, it follows from Theorem 1.1 that $A \cap U$ is almost rc-Lindelöf set in X. The result yields from Proposition 1.3. □

Proposition 2.2 [19]. *If A is a regular closed subset of a space X such that A is almost rc-Lindelöf in X, then A is an almost rc-Lindelöf.*

The following proposition includes an improvement of Proposition 2.2.

Proposition 2.3. *Let A be a semiopen subset of a space X and $B \subset A$. If B is rc-Lindelöf (resp., almost rc-Lindelöf) in X , then B is rc-Lindelöf (resp., almost rc-Lindelöf) in A. In particular, if A is a semiopen subset of a space X such that A is rc-Lindelöf (resp., almost rc-Lindelöf) in X, then A is an rc-Lindelöf (resp., almost rc-Lindelöf) subspace.*

Proof. Follows from Proposition 1.5 and the fact that if A is a semiopen subset of a space X and B is semiopen in A, then B is semiopen in X. □

Corollary 2.4 [2]. *Let X be an rc-Lindelöf weak P-space. If $U \subset A \subset \overline{U}$, where U is a regular open subset of X, then A is an rc-Lindelöf subspace.*

Proof. By Theorem 1.1, U is an rc-Lindelöf set in X and thus almost rc-Lindelöf in X. By Proposition 1.3, A is almost rc-Lindelöf in X, but X is a weak P-space, so A is rc-Lindelöf in X. Finally, since A is semiopen (it is moreover semiregular), it follows from Proposition 2.3 that A is an rc-Lindelöf subspace. □

The following theorem includes new characterizations of rc-Lindelöf sets and almost rc-Lindelöf sets.

Theorem 2.5. *Let A be a subset of a space X. Then the following are equivalent.*

(i) *A is rc-Lindelöf (resp., almost rc-Lindelöf) in X.*

(ii) *Every cover of A by semi-preopen subsets of X admits a countable subfamily the union of the closures of whose members (resp., the closure of the union of whose members) contains A.*

(iii) *Every cover of A by b-open subsets of X admits a countable subfamily the union of the closures of whose members (resp., the closure of the union of whose members) contains A.*

(iv) *Every cover of A by semiopen subsets of X admits a countable subfamily the union of the closures of whose members (resp., the closure of the union of whose members) contains A.*

(v) *Every cover of A by semiregular subsets of X admits a countable subfamily the union of the closures of whose members (resp., the closure of the union of whose members) contains A.*

Proof. (i)⇒(ii): follows since the closure of a semi-preopen set is regular closed.

(ii)⇒(iii)⇒(iv)⇒(v)⇒(i): follows from the following implications: regular closed⇒ semiregular⇒semiopen⇒ b-open⇒semi-preopen.

The following theorem also characterizes rc-Lindelöf sets and almost rc-Lindelöf sets, it is a direct consequence of Theorem 2.5 and the definition of rc-Lindelöf (almost rc-Lindelöf) sets. □

Theorem 2.6. *Let A be a subset of a space X. Then the following are equivalent.*

(i) *A is rc-Lindelöf (resp., almost rc-Lindelöf) in X.*

(ii) *If $U_{\sim} = \{U_\alpha : \alpha \in \Lambda\}$ is a family of regular open subsets of X satisfying that for any countable subcollection $U^*_{\sim}$ of $U_{\sim}$, $A \cap (\cap U^*_{\sim}) \neq \phi$ (resp., $A \cap \operatorname{int}(\cap U^*_{\sim}) \neq \phi$), then $A \cap (\cap U_{\sim}) \neq \phi$.*

(iii) *If $U_{\sim} = \{U_\alpha : \alpha \in \Lambda\}$ is a family of semi-preclosed subsets of X satisfying that for any countable subcollection $U^*_{\sim}$ of $U_{\sim}$, $A \cap (\cap\{\operatorname{int} U : U \in U^*_{\sim}\}) \neq \phi$ (resp., $A \cap \operatorname{int}(\cap U^*_{\sim}) \neq \phi$), then $A \cap (\cap U_{\sim}) \neq \phi$.*

(iv) *If $U_{\sim} = \{U_\alpha : \alpha \in \Lambda\}$ is a family of b-closed subsets of X satisfying that for any countable subcollection $U^*_{\sim}$ of $U_{\sim}$, $A \cap (\cap\{\operatorname{int} U : U \in U^*_{\sim}\}) \neq \phi$ (resp., $A \cap \operatorname{int}(\cap U^*_{\sim}) \neq \phi$), then $A \cap (\cap U_{\sim}) \neq \phi$.*

(v) *If $U_{\sim} = \{U_\alpha : \alpha \in \Lambda\}$ is a family of semiclosed subsets of X satisfying that for any countable subcollection $U^*_{\sim}$ of $U_{\sim}$, $A \cap (\cap\{\operatorname{int} U : U \in U^*_{\sim}\}) \neq \phi$ (resp., $A \cap \operatorname{int}(\cap U^*_{\sim}) \neq \phi$), then $A \cap (\cap U_{\sim}) \neq \phi$.*

(vi) *If $U_{\sim} = \{U_\alpha : \alpha \in \Lambda\}$ is a family of semiregular subsets of X satisfying that for any countable subcollection $U^*_{\sim}$ of $U_{\sim}$, $A \cap (\cap\{\operatorname{int} U : U \in U^*_{\sim}\}) \neq \phi$ (resp., $A \cap \operatorname{int}(\cap U^*_{\sim}) \neq \phi$), then $A \cap (\cap U_{\sim}) \neq \phi$.*

3. Invariance properties

In this section, we mainly study several types of functions that preserve the property of being an rc-Lindelöf (almost rc-Lindelöf) set.

Definition 3.1 [19]. A function f from a space X into a space Y is said to be slightly continuous if $f(\overline{U}) \subset \overline{f(U)}$ whenever U is open in X.

In [19], it was shown that if a function $f : X \to Y$ is slightly continuous and weakly θ-irresolute, then $f(A)$ is almost rc-Lindelöf in Y whenever A is almost rc-Lindelöf set in X. The following theorem is analogous to this result; it has a similar proof that we will mention for the convenience of the reader.

THEOREM 3.2. *Let $f : X \to Y$ be a slightly continuous and weakly θ-irresolute function. If A is rc-Lindelöf set in X, then $f(A)$ is rc-Lindelöf in Y.*

Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of $f(A)$ by regular closed subsets of X. Then $\{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a cover of A by semiopen subsets of X (as f is weakly θ-irresolute). Since A is rc-Lindelöf in X, it follows from Proposition 1.5 that there exist $\alpha_1, \alpha_2, \ldots \in \Lambda$ such that $A \subset \bigcup_{i=1}^{\infty} \overline{f^{-1}(U_{\alpha_i})}$. For each $i \in \mathbb{N}$, there is an open subset V_i of X such that $V_i \subset f^{-1}(U_{\alpha_i}) \subset \overline{V}_i$ and thus $\bigcup_{i=1}^{\infty} \overline{f^{-1}(U_{\alpha_i})} = \bigcup_{i=1}^{\infty} \overline{V_i}$. Since f is slightly continuous, it follows that $f(A) \subset \bigcup_{i=1}^{\infty} \overline{f(V_i)} \subset \bigcup_{i=1}^{\infty} \overline{U_{\alpha_i}} = \bigcup_{i=1}^{\infty} U_{\alpha_i}$. Hence $f(A)$ is rc-Lindelöf in Y. □

COROLLARY 3.3. *Let $f : X \to Y$ be a slightly continuous, semicontinuous, and almost open function. If A is rc-Lindelöf (resp., almost rc-Lindelöf) in X, then $f(A)$ is rc-Lindelöf (resp., almost rc-Lindelöf) in Y.*

COROLLARY 3.4. *Let $f : X \to Y$ be a surjective, slightly continuous, semicontinuous, and almost open function. If X is rc-Lindelöf, then Y is rc-Lindelöf.*

It will be seen later that the condition slightly continuous of Corollary 3.4 is not essential for preserving the almost rc-Lindelöf property.

COROLLARY 3.5 [2]. *Let $f : X \to Y$ be a surjective, continuous, and almost open function. If X is rc-Lindelöf, then Y is rc-Lindelöf.*

Obviously, every continuous function is both semicontinuous and slightly continuous. However, the converse is not true as the following example tells.

Example 3.6. Let $X = \{a,b,c\}$, $\tau = \{X, \phi, \{a\}\}$, $\tau^* = \{X, \phi, \{a,b\}\}$. Then the identity function from (X,τ) onto (X,τ^*) is a semicontinuous, slightly continuous, and almost open surjection. However, it is not continuous.

PROPOSITION 3.7. *Let $f : X \to Y$ be a semicontinuous function. If X is extremally disconnected (i.e., every regular closed subset of X is open), then f is slightly continuous.*

Proof. Let U be open in X. Then $\mathrm{scl}(U) = U \cup \mathrm{int}\,\overline{U} = \overline{U}$ (as X is extremally disconnected). Since f is semicontinuous, it follows that $f(\mathrm{scl}(U)) = f(\overline{U}) \subset \overline{f(U)}$. Hence f is slightly continuous. □

The following corollary is an immediate consequence of Corollary 3.4 and Proposition 3.7.

COROLLARY 3.8 [2]. *Let $f : X \to Y$ be a semicontinuous, almost open surjection, where X is extremally disconnected. If X is rc-Lindelöf, then Y is rc-Lindelöf.*

The following example shows that if X is extremally disconnected and $f : X \to Y$ is slightly continuous, almost open surjection, then f need not be semicontinuous.

Example 3.9. Let $X = \{a,b,c\}$, $\tau = \{X,\phi,\{a,b\}\}$, $\tau^* = \{X,\phi,\{a\}\}$. Then (X,τ) is extremally disconnected, also the identity function from (X,τ) onto (X,τ^*) is slightly continuous and almost open ; it is, however, not semicontinuous.

PROPOSITION 3.10 [10]. (i) *Let $f : X \to Y$ be a somewhat continuous and weakly θ-irresolute function. If X is almost rc-Lindelöf, then Y is almost rc-Lindelöf.*

(ii) *Let $f : X \to Y$ be a surjective, semicontinuous, and weakly θ-irresolute function. If X is almost rc-Lindelöf, then Y is almost rc-Lindelöf.*

COROLLARY 3.11. *Let $f : X \to Y$ be a surjective, semicontinuous, and almost open function. If X is almost rc-Lindelöf, then Y is almost rc-Lindelöf.*

The following corollary is an immediate consequence of Corollary 3.11 and the fact that for a weak P-space, the concepts of being rc-Lindelöf and almost rc-Lindelöf coincide.

COROLLARY 3.12 [2]. *Let $f : X \to Y$ be a surjective, semicontinuous, and almost open function, where Y is a weak P-space. If X is rc-Lindelöf, then Y is rc-Lindelöf.*

Definition 3.13. A function $f : X \to Y$ is said to be somewhat precontinuous if for each nonempty open set V in Y, $p\operatorname{int} f^{-1}(V) \neq \phi$.

Remark 3.14. It was pointed out in [10] that every surjective semicontinuous function is somewhat continuous, a similar result that may be pointed out here asserts that every surjective semi-precontinuous function is somewhat precontinuous. However, the converses of these two facts are not true as the following two examples tell.

Example 3.15. Let $X = \{a,b,c\}$, $\tau = \{X,\phi,\{a,b\},\{c\}\}$, $\tau^* = \{X,\phi,\{a,c\}\}$. Then the identity function from (X,τ) onto (X,τ^*) is somewhat continuous; it is, however, not semicontinuous.

Example 3.16. Let $X = \{a,b,c,d\}$, $\tau = \{X,\phi,\{b\},\{d\},\{b,d\},\{a,d\},\{a,b,d\}\}$, $\tau^* = \{X,\phi,\{a,b\}\}$. Then the identity function from (X,τ) onto (X,τ^*) is even somewhat continuous and thus somewhat precontinuous; it is, however, not semi-precontinuous since $\{a,b\}$ is not semi-preopen in (X,τ).

The following result is a slight improvement of Proposition 3.10(i), the similar proof follows from Theorem 2.5 and the fact that if A is a semiopen subset of a space X, then $\operatorname{pcl}(A) = \overline{A}$.

PROPOSITION 3.17. (i) *Let $f : X \to Y$ be a somewhat continuous and wrc-continuous function. If X is almost rc-Lindelöf, then Y is almost rc-Lindelöf.*

(ii) *Let $f : X \to Y$ be a somewhat precontinuous and weakly θ-irresolute function. If X is almost rc-Lindelöf, then Y is almost rc-Lindelöf.*

Remark 3.18. Clearly, every somewhat continuous function is somewhat precontinuous and every weakly θ-irresolute function is wrc-continuous. However, the following two examples show that the property of being both somewhat continuous and wrc-continuous and the property of being both somewhat precontinuous and weakly θ-irresolute are independent.

Example 3.19. Let $X = \{a,b,c\}$, $\tau = \{X,\phi,\{a,b\}\}$, $\tau^* = \{X,\phi,\{a,c\}\}$. Then the identity function from (X,τ) onto (X,τ^*) is somewhat precontinuous and weakly θ-irresolute; it is, however, not somewhat continuous.

Example 3.20. Let $X = \{a,b,c,d\}$, $\tau = \{X,\phi,\{a\},\{b,c\},\{d\},\{a,b,c\},\{a,d\},\{b,c,d\}\}$, $\tau^* = \{X,\phi,\{a,b\},\{d\},\{a,b,d\}\}$. Then the identity function from (X,τ) onto (X,τ^*) is somewhat continuous and wrc-continuous; it is, however, not weakly θ-irresolute (observe that $\{d,c\}$ is regular closed in (X,τ^*) but not semiopen in (X,τ)).

The following result is a slight improvement of Proposition 3.10(ii), it is a direct consequence of Remark 3.14 and Proposition 3.17.

Corollary 3.21. (i) *Let $f : X \to Y$ be a surjective, semicontinuous, and wrc-continuous function. If X is almost rc-Lindelöf, then Y is almost rc-Lindelöf.*

(ii) *Let $f : X \to Y$ be a surjective, semi-precontinuous, and weakly θ-irresolute function. If X is almost rc-Lindelöf, then Y is almost rc-Lindelöf.*

Corollary 3.22 [2]. *Let $f : X \to Y$ be a somewhat continuous and wrc-continuous surjection, where Y is a weak P-space. If X is rc-Lindelöf, then Y is rc-Lindelöf.*

Corollary 3.22 is still true even if the function f is not surjective.

4. Product theorems

In this section, we study some types of functions that inversely preserve the property of being an rc-Lindelöf (almost rc-Lindelöf) set. We mainly obtain some product theorems concerning rc-Lindelöf spaces.

Definition 4.1 [19]. A function f from a space X into a space Y is said to be regular open if it maps regular open subsets onto regular open subsets.

Definition 4.2 [19]. (i) A subset A of a space X is said to be an rc-F_σ subset if A is the countable union of regular closed subsets.

(ii) A function f from a space X into a space Y is said to be weakly almost open if $f^{-1}(\overline{A}) \subset \overline{f^{-1}(A)}$ whenever A is an rc-F_σ subset of Y.

In [19], it was shown that every almost open function is weakly almost open, but not conversely.

Theorem 4.3 [19]. *Let f be a weakly almost open and regular open function from a space X onto a space Y. Then the following hold.*

(i) *If for each $y \in Y$, $f^{-1}(y)$ is an S-set in X, then X is almost rc-Lindelöf whenever Y is almost rc-Lindelöf.*

(ii) *If for each $y \in Y$, $f^{-1}(y)$ is rc-Lindelöf in X, then X is almost rc-Lindelöf whenever Y is almost rc-Lindelöf provided that X is a weak P-space.*

We point out here that in the result of Theorem 4.3(ii), X being almost rc-Lindelöf may be replaced by rc-Lindelöf since X is a weak P-space.

Theorem 4.3 may be improved in the following form.

Theorem 4.4. *Let f be a weakly almost open and regular open function from a space X onto a space Y. Then the following hold.*

(i) *If for each $y \in Y$, $f^{-1}(y)$ is an S-set in X, then $f^{-1}(A)$ is almost rc-Lindelöf in X whenever A is almost rc-Lindelöf in Y.*

(ii) *If for each $y \in Y$, $f^{-1}(y)$ is rc-Lindelöf in X, then $f^{-1}(A)$ is rc-Lindelöf in X whenever A is almost rc-Lindelöf in Y provided that X is a weak P-space.*

The following theorem shows that the assumption weakly almost open of Theorem 4.4 is not essential for the inverse preservation of the rc-Lindelöf set property.

Theorem 4.5. *Let f be a regular open function from a space X onto a space Y. Then the following hold.*

(i) *If for each $y \in Y$, $f^{-1}(y)$ is an S-set in X, then $f^{-1}(A)$ is rc-Lindelöf in X whenever A is rc-Lindelöf in Y.*

(ii) *If for each $y \in Y$, $f^{-1}(y)$ is rc-Lindelöf in X, then $f^{-1}(A)$ is rc-Lindelöf in X whenever A is rc-Lindelöf in Y provided that X is a weak P-space.*

The proof of the following proposition is straightforward and thus omitted.

Proposition 4.6. *Let X be a nearly Lindelöf space and Y a weak P-space. Then the projection function $p : X \times Y \to Y$ sends regular closed sets onto closed sets.*

Corollary 4.7. *Let X, Y be two spaces such that Y is rc-Lindelöf and $X \times Y$ is extremally disconnected. Then the following hold.*

(i) *If X is compact, then $X \times Y$ is rc-Lindelöf [2].*

(ii) *If X is Lindelöf, then $X \times Y$ is rc-Lindelöf provided that $X \times Y$ is a weak P-space.*

Proof. We will show (ii), the other part is similar. Consider the projection function $p : X \times Y \to Y$. Since $X \times Y$ is a weak P-space, it follows that Y is a weak P-space, but X is Lindelöf and thus nearly Lindelöf, so by Proposition 4.6, $p : X \times Y \to Y$ sends regular closed sets onto closed sets, but $X \times Y$ is extremally disconnected, so every regular open subset of $X \times Y$ is regular closed and thus $p : X \times Y \to Y$ sends regular open sets onto closed sets, but p is an open function, so p is regular open. Also for each $y \in Y$, $p^{-1}(y) = X \times \{y\}$ is rc-Lindelöf in $X \times Y$ (as X is Lindelöf and $X \times Y$ is extremally disconnected). Finally, since Y is rc-Lindelöf, it follows immediately from Theorem 4.5(ii) that $X \times Y$ is rc-Lindelöf.

The following result is an improvement of Corollary 4.7, it follows from Theorem 1.2, Proposition 1.4, Corollary 4.7, and the fact that the properties of being extremally disconnected (a weak P-space) are hereditary with respect to open subsets. □

Corollary 4.8. *Let X, Y be two rc-Lindelöf spaces such that $X \times Y$ is extremally disconnected. Then the following hold.*

(i) *If X is locally compact, that is, for each $x \in X$, there exists an open set U_x containing x such that $\overline{U_x}$ is compact, then $X \times Y$ is rc-Lindelöf.*

(ii) *If X is locally Lindelöf, that is, for each $x \in X$, there exists an open set U_x containing x such that $\overline{U_x}$ is Lindelöf, then $X \times Y$ is rc-Lindelöf provided that $X \times Y$ is a weak P-space.*

Acknowledgment

The author is grateful to the referee for his/her careful reading of the manuscript and for the valuable suggestions.

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb, and R. A. Mahmoud, *β-open sets and β-continuous mapping*, Bulletin of the Faculty of Science. Assiut University. A. Physics and Mathematics **12** (1983), no. 1, 77–90.

[2] B. Al-Nashef and K. Al-Zoubi, *A note on rc-Lindelof and related spaces*, Questions and Answers in General Topology **21** (2003), no. 2, 159–170.

[3] D. Andrijević, *Semi-preopen sets*, Matematichki Vesnik **38** (1986), no. 1, 24–32.

[4] ———, *On b-open sets*, Matematichki Vesnik **48** (1996), no. 1-2, 59–64.

[5] G. Balasubramanian, *On some generalizations of compact spaces*, Glasnik Matematički. Serija III **17(37)** (1982), no. 2, 367–380.

[6] H. H. Corson and E. Michael, *Metrizability of certain countable unions*, Illinois Journal of Mathematics **8** (1964), 351–360.

[7] G. Di Maio, *S-closed spaces, S-sets and S-continuous functions*, Atti della Accademia delle Scienze di Torino **118** (1984), no. 3-4, 125–134.

[8] G. Di Maio and T. Noiri, *On s-closed spaces*, Indian Journal of Pure and Applied Mathematics **18** (1987), no. 3, 226–233.

[9] K. Dlaska, *rc-Lindelöf sets and almost rc-Lindelöf sets*, Kyungpook Mathematical Journal **34** (1994), no. 2, 275–281.

[10] K. Dlaska and M. Ganster, *Almost rc-Lindelöf spaces*, Bulletin of the Malaysian Mathematical Sciences Society. Second Series **17** (1994), 51–56.

[11] R. Engelking, *General Topology*, Sigma Series in Pure Mathematics, vol. 6, Heldermann, Berlin, 1989.

[12] Z. Frolík, *Remarks concerning the invariance of Baire spaces under mappings*, Czechoslovak Mathematical Journal **11 (86)** (1961), 381–385.

[13] M. Ganster, T. Noiri, and I. L. Reilly, *Weak and strong forms of θ-irresolute functions*, Journal of Institute of Mathematics & Computer Sciences. (Mathematics Series) **1** (1988), no. 1, 19–29.

[14] D. S. Janković, *A note on mappings of extremally disconnected spaces*, Acta Mathematica Hungarica **46** (1985), no. 1-2, 83–92.

[15] D. S. Janković and C. Konstadilaki, *On covering properties by regular closed sets*, Mathematica Pannonica **7** (1996), no. 1, 97–111.

[16] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, The American Mathematical Monthly **70** (1963), no. 1, 36–41.

[17] A. S. Mashhour, M. E. Abd El-Monsef, and S. N. El-Deep, *On precontinuous and weak precontinuous mappings*, Proceedings of the Mathematical and Physical Society of Egypt (1982), no. 53, 47–53 (1983).

[18] T. K. Mukherji and M. Sarkar, *On a class of almost discrete spaces*, Matematichki Vesnik **3(16)(31)** (1979), no. 4, 459–474.

[19] M. S. Sarsak, *On almost rc-Lindelöf sets*, Acta Mathematica Hungarica **100** (2003), no. 1-2, 1–7.

[20] A. Wilansky, *Topics in Functional Analysis*, Lecture Notes in Mathematics, no. 45, Springer, Berlin, 1967.

Mohammad S. Sarsak: Department of Mathematics, Faculty of Science, The Hashemite University, P.O. Box 150459, Zarqa 13115, Jordan
E-mail address: sarsak@hu.edu.jo