CONNECTION THEORY ON DIFFERENTIABLE FIBRE BUNDLES: A CONCISE INTRODUCTION

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Received 5 May 2006; Revised 27 July 2006; Accepted 8 August 2006

The paper contains a partial review on the general connection theory on differentiable fibre bundles. Particular attention is paid on (linear) connections on vector bundles. The (local) representations of connections in frames adapted to holonomic and arbitrary frames are considered.

1. Introduction

This is a partial review of the connection theory on differentiable fibre bundles. From different view points, this theory can be found in many works, like [2–6, 9, 13–15, 18, 19, 24, 26, 27, 30, 31, 35–40, 42]. The presentation of the material in Sections 2–5, containing the grounds of the connection theory, follows some of the main ideas of [30, Chapters 1 and 2], but their realization here is quite different and follows the modern trends in differential geometry. Since in the physical literature one can find misunderstanding or not quite rigorous applications of known mathematical definitions and results, the text is written in a way suitable for direct application in some regions of theoretical physics.

The work is organized as follows. In Section 2 some introductory material is collected, like the notion of Lie derivatives and distributions on manifolds needed for our exposition. Here some of our notations are fixed too.

Section 3 is devoted to the general connection theory on bundles whose base and bundles spaces are differentiable manifolds. In Section 3.1 some coordinates and frames/bases on the bundle space which are compatible with the fibre structure of a bundle are reviewed. Section 3.2 deals with the general connection theory. A connection on a bundle is defined as a distribution on its bundle space which is complimentary to the vertical distribution on it. The notion of parallel transport generated by connection and of specialized frame is introduced. The fibre coefficients and fibre components of the curvature of a connection are defined via part of the components of the anholonomicity object of a specialized frame. Frames adapted to local bundle coordinates are introduced and the local (2-index) coefficients in them of a connection are defined; their transformation law is

Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2006, Article ID 56397, Pages 1–51
DOI 10.1155/IJMMS/2006/56397

derived and it is proved that a geometrical object with such transformation law uniquely defines a connection. The parallel transport equation in their terms is derived and it is demonstrated how from it the equation of geodesics on a manifold can be obtained.

In Section 4, the general connection theory from Section 3 is specified on vector bundles. The most important structures in/on them are the ones that are consistent/compatible with the vector space structure of their fibres. The vertical lifts of sections of a vector bundle and the horizontal lifts of vector fields on its base are investigated in more details in Section 4.1. The general results are specified on the (co)tangent bundle over a manifold in Section 4.2; Section 4.3 is devoted to linear connections on vector bundles, that is, connections such that the parallel transport assigned to them is a linear mapping. It is proved that the 2-index coefficients of a linear connection are linear in the fibre coordinates, which leads to the introduction of the (3-index) coefficients of the connection, the latter coefficients being defined on the base space. The transformations of different objects under a change of vector bundle coordinates are explored. The covariant derivatives are introduced and investigated in Section 4.4. They are defined via the Lie derivatives and a mapping realizing an isomorphism between the vertical vector fields on the bundle space and the sections of the bundle. The equivalence of that definition with the widespread one, defining them as mappings on the module of sections of the bundle with suitable properties, is proved. Some properties of the covariant derivatives are explored. In Section 4.5, the affine connections on vector bundles are considered briefly.

Section 5 deals briefly with morphisms between bundles with connections defined on them.

In Section 6, some of the results of the previous sections are generalized when frames more general than the ones generated by local coordinates on the bundle space are employed. The most general of such frames, compatible with the fibre structure, and the frames adapted to them are investigated. The main differential-geometric objects, introduced in the previous sections, are considered in such general frames. Particular attention is paid to the case of a vector bundle. In vector bundles, a bijective correspondence between the mentioned general frames and pairs of bases, in the vector fields over the base and in the sections of the bundle, is proved. The (3-index) coefficients of a connection in such pairs of frames and their transformation laws are considered. The covariant derivatives are also mentioned on this context.

Section 7 closes the paper with some concluding remarks.

2. Preliminaries

This section contains an introductory material, notation, and so forth, that will be needed for our exposition. The reader is referred for details to standard books on differential geometry, like [19, 20, 29, 40].

A differentiable finite-dimensional manifold over a field $\mathbb{K}$ will be denoted typically by M. Here $\mathbb{K}$ stands for the field $\mathbb{R}$ of real or the field $\mathbb{C}$ of complex numbers, $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The manifolds we consider are supposed to be smooth of class C^2. (Some of our definitions or/and results are valid also for C^1 or even C^0 manifolds, but we do not want to overload the material with continuous counting of the required degree of differentiability of the manifolds involved. Some parts of the text admit generalizations on more general spaces, like the topological ones, but this is out of the subject of the present work.) The sets of vector fields, realized as first-order differential operators, and of differential k-forms, $k \in \mathbb{N}$, over M will be denoted by $\mathscr{X}(M)$ and $\Lambda^k(M)$, respectively. The space tangent (resp., cotangent) to M at $p \in M$ is $T_p(M)$ (resp., $T_p^*(M)$) and $(T(M),\pi_T,M)$ (resp., $(T^*(M),\pi_{T^*},M)$) will stand for the tangent (resp., cotangent) bundle over M. The value of $X \in \mathscr{X}(M)$ at $p \in M$ is $X_p \in T_p(M)$ and the action of X on a C^1 function $\varphi : M \to \mathbb{K}$ is a function $X(\varphi) : M \to \mathbb{K}$ with $X(\varphi)|_p := X_p(\varphi) \in \mathbb{K}$.

If M and $\overline{M}$ are manifolds and $f : \overline{M} \to M$ is a C^1 mapping, then $f_* := \mathrm{d}f := T(f) : T(\overline{M}) \to T(M)$ denotes the induced tangent mapping (or differential) of f such that, for $p \in \overline{M}$, $f_*|_p := \mathrm{d}f|_p := T_p(f) : T_p(\overline{M}) \to T_{f(p)}(M)$ and, for a C^1 function g on M, $(f_*(X))(g) := X(g \circ f) : p \mapsto f_*|_p(g) = X_p(g \circ f)$, with $\circ$ being the composition of mappings sign. Respectively, the induced cotangent mapping is $f^* := T^*(f) : T^*(M) \to T^*(\overline{M})$. If $h : N \to \overline{M}$, N being a manifold, we have the chain rule $\mathrm{d}(f \circ h) = \mathrm{d}f \circ \mathrm{d}h$, which is an abbreviation for $\mathrm{d}(f \circ h)_q = (\mathrm{d}f)_{f(q)} \circ (\mathrm{d}h)_q$ for $q \in N$.

By $J \subseteq \mathbb{R}$ will be denoted an arbitrary real interval that can be opened or closed at one or both its ends. The notation $\gamma : J \to M$ represents an arbitrary path in M. For a C^1 path $\gamma : J \to M$, the vector tangent to γ at $s \in J$ will be denoted by $\dot\gamma(s) := \mathrm{d}/\mathrm{d}t|_{t=s}(\gamma(t)) = \gamma_*(\mathrm{d}/\mathrm{d}r|_s) \in T_{\gamma(s)}(M)$, where r in $\mathrm{d}/\mathrm{d}r|_s$ is the standard coordinate function on $\mathbb{R}$, that is, $r : \mathbb{R} \to \mathbb{R}$ with $r(s) := s$ for all $s \in \mathbb{R}$ and hence $r = \mathrm{id}_{\mathbb{R}}$ is the identity mapping of $\mathbb{R}$. If $s_0 \in J$ is an end point of J and J is closed at s_0, the derivative in the definition of $\dot\gamma(s_0)$ is regarded as a one-sided derivative at s_0.

The Lie derivative relative to $X \in \mathscr{X}(M)$ will be denoted by $\mathscr{L}_X$. It is defined on arbitrary geometrical objects on M [41], but below we will be interested in its action on tensor fields [19, Chapter I, Section 2] (see also [21]). If f, Y, and θ are, C^1, respectively, function, vector field, and 1-form on M, then

$$\mathscr{L}_X(f) = X(f), \tag{2.1a}$$

$$\mathscr{L}_X(Y) = [X,Y]_\_, \tag{2.1b}$$

$$(\mathscr{L}_X(\theta))(Y) = X(\theta(Y)) - \theta([X,Y]_\_) = (\mathrm{d}\theta)(X,Y) + Y(\theta(X)), \tag{2.1c}$$

where $[A,B]_\_ = A \circ B - B \circ A$ is the commutator of operators A and B (with common domain) and d denotes the exterior derivative operator.

Since $\mathscr{L}_X$ is a derivation of the tensor algebra over the vector fields on M, for a tensor field $S : \Lambda^1(M) \times \cdots \times \Lambda^1(M) \times \mathscr{X}(M) \times \cdots \times \mathscr{X}(M) \to \mathfrak{F}(M)$, with $\mathfrak{F}(M)$ being the algebra of functions on M, we have

$$\begin{aligned}(\mathscr{L}_X S)(\theta,\ldots;Y,\ldots) = X(S(\theta,\ldots;Y,\ldots)) - S(\mathscr{L}_X\theta,\ldots;Y,\ldots)\\ - \cdots - S(\theta,\ldots;\mathscr{L}_X Y,\ldots) - \cdots,\end{aligned} \tag{2.2}$$

which defines $\mathscr{L}_X S$ explicitly, due to (2.1).

Let the Greek indices $\lambda,\mu,\nu,\ldots$ run over the range $1,\ldots,\dim M$ and let $\{E_\mu\}$ be a C^1 frame in $T(M)$, that is, let $E_\mu \in \mathscr{X}(M)$ be of class C^1 and, for each $p \in M$, let the set $\{E_\mu|_p\}$ be a basis of the vector space $T_p(M)$. (There are manifolds, like the even-dimensional spheres $\mathbb{S}^{2k}$, $k \in \mathbb{N}$, which do not admit global, continuous (and moreover C^k for $k \geq 1$), and nowhere vanishing vector fields [34]. If this is the case, the considerations must be localized over an open subset of M on which such fields exist. We will not overload our exposition with such details.) Let $\{E^\mu\}$ be the coframe dual to $\{E_\mu\}$, that is, $E^\mu \in \Lambda^1(M)$, let $\{E^\mu|_p\}$ be a basis in $T_p^*(M)$, and $E^\mu(E_\nu) = \delta_\nu^\mu$ with δ_μ^ν being the Kronecker deltas ($\delta_\mu^\nu = 1$ for $\mu = \nu$ and $\delta_\mu^\nu = 0$ for $\mu \neq \nu$). Assuming the Einstein's summation convention (summation on indices repeated on different levels over the whole range of their values), we define the *components* $(\Gamma_X)^\mu{}_\nu$ of $\mathscr{L}_X$ in (relative to) $\{E_\mu\}$ via the expansion

$$\mathscr{L}_X E_\mu =: (\Gamma_X)^\nu{}_\mu E_\nu, \tag{2.3}$$

which is equivalent to

$$\mathscr{L}_X E^\mu = -(\Gamma_X)^\mu{}_\nu E^\nu, \tag{2.3$'$}$$

by virtue of $E^\mu(E_\nu) = \delta_\nu^\mu$ and the commutativity of the Lie derivatives and contraction operators. (The sign before $(\Gamma_X)^\mu{}_\nu$ in (2.3) or (2.3′) is conventional and we have chosen it in a way similar to the accepted convention for the components of a covariant derivative (or, equivalently, the coefficients of a linear connection—see Section 4).) Sometimes, it is convenient for (2.3) and (2.3′) to be written in a matrix form

$$\mathscr{L}_X E = E \cdot \Gamma_X, \qquad \mathscr{L}_X E^* = -\Gamma_X \cdot E^*, \tag{2.4}$$

where $\Gamma_X := [(\Gamma_X)^\mu{}_\nu]_{\mu,\nu=1}^{\dim M}$, $E := (E_1,\dots,E_{\dim M})$, and $E^* := (E^1,\dots,E^{\dim M})^\top$, with $\top$ being the matrix transposition sign, and the matrix multiplication is explicitly denoted by centered dot $\cdot$ as otherwise $E \cdot \Gamma_X$ may be confused with $E\Gamma_X = E(\Gamma_X) = (E_1(\Gamma_X),\dots) = ([E_1((\Gamma_X)^\nu{}_\mu)],\dots)$. From (2.3) and (2.1b), we get

$$(\Gamma_X)^\nu{}_\mu = -E_\mu(X^\nu) - C_{\mu\lambda}^\nu X^\lambda, \tag{2.5}$$

in $\{E_\mu\}$, where $X = X^\mu E_\mu$ and the functions $C_{\mu\lambda}^\nu$, known as the *components of the anholonomicity object of* $\{E_\mu\}$, are defined by

$$[E_\mu, E_\nu]_- =: C_{\mu\nu}^\lambda E_\lambda, \tag{2.6}$$

or, equivalently, by its dual (see (2.1c))

$$dE^\lambda = -\frac{1}{2} C_{\mu\nu}^\lambda E^\mu \wedge E^\nu, \tag{2.6$'$}$$

with $\wedge$ being the exterior (wedge) product sign. (If M is a Lie group and $\{E_\mu\}$ is a basis of its Lie algebra (:= {left invariant vector fields in $\mathscr{X}(M)$}), then $C_{\mu\nu}^\lambda$ are constants, called structure constants of M, and (2.6) and (2.6′) are known as the structure equations of M.) For a tensor field S of type (r,s), $r,s \in \mathbb{N} \cup \{0\}$, with components $S_{\nu_1,\dots,\nu_s}^{\mu_1,\dots,\mu_r}$ relative to the tensor frame induced by $\{E_\mu\}$ and $\{E^\mu\}$, we get, from (2.2), the components of $\mathscr{L}_X S$ as

$$(\mathscr{L}_X S)^{\mu_1,\dots,\mu_r}_{\nu_1,\dots,\nu_s} = X(S^{\mu_1,\dots,\mu_r}_{\nu_1,\dots,\nu_s}) + \sum_{a=1}^{r} (\Gamma_X)^{\mu_a}{}_{\lambda} S^{\mu_1,\dots,\mu_{a-1},\lambda,\mu_{a+1},\dots,\mu_r}_{\nu_1,\dots,\nu_s} - \sum_{b=1}^{s} (\Gamma_X)^{\lambda}{}_{\nu_b} S^{\mu_1,\dots,\mu_r}_{\nu_1,\dots,\nu_{b-1},\lambda,\nu_{b+1},\dots,\nu_s}. \tag{2.7}$$

A frame $\{E_\mu\}$ or its dual coframe $\{E^\mu\}$ is called *holonomic* (*anholonomic*) if $C^\lambda_{\mu\nu} = 0$ ($C^\lambda_{\mu\nu} \neq 0$) for all (some) values of the indices μ, ν, and λ. For a holonomic frame always exist local coordinates $\{x^\mu\}$ on M such that *locally* $E_\mu = \partial/\partial x^\mu$ and $E^\mu = \mathrm{d}x^\mu$. Conversely, if $\{x^\mu\}$ are local coordinates on M, then the local frame $\{\partial/\partial x^\mu\}$ and local coframe $\{\mathrm{d}x^\mu\}$ are defined and holonomic on the domain of $\{x^\mu\}$.

A straightforward calculation by means of (2.6) reveals that a change

$$\{E_\mu\} \longrightarrow \{\overline{E}_\nu = B^\mu_\nu E_\mu\} \tag{2.8}$$

of the frame $\{E_\mu\}$, where $B = [B^\nu_\mu]$ is a nondegenerate matrix-valued function, entails the transformation

$$C^\lambda_{\mu\nu} \longmapsto \overline{C}^\lambda_{\mu\nu} = (B^{-1})^\lambda_\varrho (B^\sigma_\mu E_\sigma(B^\varrho_\nu) - B^\sigma_\nu E_\sigma(E^\varrho_\mu) + B^\sigma_\mu B^\tau_\nu C^\varrho_{\sigma\tau}). \tag{2.9}$$

Besides, from (2.5) and (2.9), we see that the quantities $(\Gamma_X)^\nu{}_\mu$ undergo the change

$$(\Gamma_X)^\nu{}_\mu \longmapsto (\overline{\Gamma}_X)^\nu{}_\mu = (B^{-1})^\mu_\varrho ((\Gamma_X)^\varrho{}_\sigma B^\sigma_\nu + X(B^\sigma_\nu)) \tag{2.10}$$

when (2.8) takes place. Setting $\Gamma_X := [(\Gamma_X)^\nu{}_\mu]$ and $\overline{\Gamma}_X := [(\overline{\Gamma}_X)^\nu{}_\mu]$, we can rewrite (2.10) in a more compact matrix form as

$$\Gamma_X \longmapsto \overline{\Gamma}_X = B^{-1} \cdot (\Gamma_X \cdot B + X(B)). \tag{2.11}$$

If $n \in \mathbb{N}$ and $n \leq \dim M$, an n-dimensional *distribution* Δ on M is defined as a mapping $\Delta : p \mapsto \Delta_p$ assigning to each $p \in M$ an n-dimensional subspace Δ_p of the tangent space $T_p(M)$ of M at p, $\Delta_p \subseteq T_p(M)$. A *solution* (resp., *first integral*) of a distribution Δ on M is an immersion $\varphi : N \to M$ (resp., submersion $\psi : M \to N$), N being a manifold, such that $\operatorname{Im}\varphi_* \subseteq \Delta$ (resp., $\operatorname{Ker}\psi_* \supset \Delta$), that is, for each $q \in N$ (resp., $p \in M$), $\varphi_*(T_q(N)) \subseteq \Delta_{\varphi(q)}$ (resp., $\psi_*(\Delta_p) = 0_{\psi(q)} \in T_{\psi(q)}(N)$). A distribution is *integrable* if there is a submersion $\psi : M \to N$ such that $\operatorname{Ker}\psi_* = \Delta$; if the commutators of the vector fields in Δ are vector fields also in Δ, then Δ is locally integrable and if Δ is integrable, then the commutators of the vector fields in Δ belong to Δ. We say that a vector field $X \in \mathscr{X}(M)$ is in Δ and write $X \in \Delta$ if $X_p \in \Delta_p$ for all $p \in M$. A *basis on* $U \subseteq M$ *for* Δ is a set $\{X_1,\dots,X_n\}$ of n linearly independent (relative to functions $U \to \mathbb{K}$) vector fields in $\Delta|_U$, that is, $\{X_1|_p,\dots,X_n|_p\}$ is a basis for Δ_p for all $p \in U$.

A distribution is convenient to be described in terms of (global) frames or/and coframes over M. In fact, if $p \in M$ and $\varrho = 1,\dots,n$, in each $\Delta_p \subseteq T_p(M)$, we can choose a basis $\{X_\varrho|_p\}$ and hence a frame $\{X_\varrho\}$, $X_\varrho : p \mapsto X_\varrho|_p$, in $\{\Delta_p : p \in M\} \subseteq T(M)$; we say that $\{X_\rho\}$ is a basis for/in Δ. Conversely, any collection of n linearly independent (relative to functions $M \to \mathbb{K}$) vector fields X_ϱ on M defines a distribution $p \mapsto \{\sum_{\varrho=1}^n f^\varrho X_\varrho|_p : f^\varrho \in \mathbb{K}\}$. Consequently, a frame in $T(M)$ can be formed by adding to a basis for Δ a set of $(\dim M - n)$ new linearly independent vector fields (forming a frame in $T(M) \setminus \{\Delta_p : p \in M\}$) and vice versa, by selecting n linearly independent vector fields on M, we can define a distribution Δ on M. Equivalently, one can use $\dim M - n$ linearly independent 1-forms ω^a, $a = n+1,\dots,\dim M$, which are annihilators for it, $\omega^a|_{\Delta_p} = 0$ for all $p \in M$. For instance, if $\{X_\mu : \mu = 1,\dots,\dim M\}$ is a frame in $T(M)$ and $\{X_\varrho : \varrho = 1,\dots,n\}$ is a basis for Δ, then one can define ω^a to be elements in the coframe $\{\omega^\mu\}$ dual to $\{X_\mu\}$. We call $\{\omega^a\}$ a *cobasis* for Δ.

Ending this section, we will make a remark concerning the (non)local character of the considerations in this paper. In general, when some coordinate system(s) is (are) used, the quantities connected with it (them) are local in a sense that they are defined on, possibly, a subset of the (intersection of the) domain(s) of this (these) coordinate system(s). Other case when the considerations are possibly local is when one deals with continuous or differentiable vector fields which generally exist on proper subsets of a manifold. But, for instance, the concepts of a connection on a manifold and the associated to it parallel transport are global ones: the connection is an object defined on the whole manifold and the parallel transport (resp., along a path) is a mapping defined on the set of all C^1 paths (resp., on the whole fibre over the initial point of the path). However, the representations of these objects in a coordinate system or a frame are generally local.

3. Connections on bundles

Before presenting the general connection theory in Section 3.2, we at first fix some notation and concepts concerning fibre bundles in Section 3.1.

3.1. Frames and coframes on the bundle space. Let (E,π,M) be a bundle with bundle space E, projection $\pi : E \to M$, and base space M. Suppose that the spaces M and E are manifolds of finite dimensions $n \in \mathbb{N}$ and $n+r$, for some $r \in \mathbb{N}$, respectively; so the dimension of the fibre $\pi^{-1}(x)$, with $x \in M$, that is, the fibre dimension of (E,π,M), is r. Besides, let these manifolds be C^2 differentiable if the opposite is not stated explicitly. (Most of our considerations are valid also if C^1 differentiability is assumed and even some of them hold on C^0 manifolds. By assuming C^2 differentiability, we skip the problem of counting the required differentiability class of the whole material that follows. Sometimes, the C^2 differentiability is required explicitly, which is a hint that a statement or definition is not valid otherwise. If we want to emphasize that some text is valid under a C^1 differentiability assumption, we indicate that fact explicitly.)

Let the Greek indices $\lambda,\mu,\nu,\dots$ run from 1 to $n = \dim M$, let the Latin indices $a,b,c,\dots$ take the values from $n+1$ to $n+r = \dim E$, and let the uppercase Latin indices $I,J,K,\dots$ take values in the whole set $\{1,\dots,n+r\}$. One may call these types of indices, respectively, base, fibre, and bundle indices.

Suppose $\{u^I\} = \{u^\mu,u^a\} = \{u^1,\dots,u^{n+r}\}$ are local *bundle coordinates* on an open set $U \subseteq E$, that is, on the set $\pi(U) \subseteq M$ there are local coordinates $\{x^\mu\}$ such that $u^\mu = x^\mu \circ \pi$;

(On a bundle or fibred manifold, these coordinates are known also as adapted coordinates [32, Definition 1.1.5].) the coordinates $\{u^\mu\}$ (resp., $\{u^a\}$) are called *basic* (resp., *fibre*) *coordinates* [29]. (If (U,v) is a bundle chart, with $v : U \to \mathbb{K}^n \times \mathbb{K}^r$ and $e^a : \mathbb{K}^r \to \mathbb{K}$ are such that $e^a(c_1,\dots,c_r) = c_a \in \mathbb{K}$, then one can put $u^a = e^a \circ \mathrm{pr}_2 \circ v$, where $\mathrm{pr}_2 : \mathbb{K}^n \times \mathbb{K}^r \to \mathbb{K}^r$ is the projection on the second multiplier $\mathbb{K}^r$.)

Further only coordinate changes

$$\{u^\mu, u^a\} \longmapsto \{\tilde{u}^\mu, \tilde{u}^a\} \tag{3.1a}$$

on E which respect the fibre structure, namely, the division into basic and fibre coordinates, will be considered. This means that

$$\begin{aligned} \tilde{u}^\mu(p) &= f^\mu(u^1(p),\dots,u^n(p)),\\ \tilde{u}^a(p) &= f^a(u^1(p),\dots,u^n(p),u^{n+1}(p),\dots,u^{n+r}(p)) \end{aligned} \tag{3.1b}$$

for $p \in U \cap \tilde{U}$, with $\tilde{U} \subseteq E$ being the domain of the coordinates $\{\tilde{u}^I\}$, and some functions f^I. The bundle coordinates $\{u^\mu, u^a\}$ induce the (local) frame $\{\partial_\mu := \partial/\partial u^\mu, \partial_a := \partial/\partial u^a\}$ and coframe $\{\mathrm{d}u^\mu, \mathrm{d}u^a\}$ over U in, respectively, the tangent $T(E)$ and cotangent $T^*(E)$ bundle spaces of the tangent and cotangent bundles over the bundle space E. Since a change (3.1) of the coordinates on E implies $\partial_I \mapsto \tilde{\partial}_I := \partial/\partial\tilde{u}^I = \partial u^J/\partial\tilde{u}^I \partial_J$ and $\mathrm{d}u^I \mapsto \mathrm{d}\tilde{u}^I = (\partial\tilde{u}^I/\partial u^J)\mathrm{d}u^J$, the transformation (3.1) leads to

$$(\partial_\mu, \partial_a) \longmapsto (\tilde{\partial}_\mu, \tilde{\partial}_a) = (\partial_\nu, \partial_b) \cdot A, \tag{3.2a}$$

$$(\mathrm{d}u^\mu, \mathrm{d}u^a)^\top \longmapsto (\mathrm{d}\tilde{u}^\mu, \mathrm{d}\tilde{u}^a)^\top = A^{-1} \cdot (\mathrm{d}u^\nu, \mathrm{d}u^b)^\top. \tag{3.2b}$$

Here expressions like $(\partial_\mu, \partial_a)$ are shortcuts for ordered $(n+r)$-tuples like $(\partial_1,\dots,\partial_{n+r}) = ([\partial_\mu]_{\mu=1}^n, [\partial_a]_{a=n+1}^{n+r})$, $\top$ is the matrix transpositions sign, the centered dot $\cdot$ stands for the matrix multiplication, and the transformation matrix A is

$$A := \left[\frac{\partial u^I}{\partial\tilde{u}^J}\right]_{I,J=1}^{n+r} = \begin{pmatrix} \left[\dfrac{\partial u^\nu}{\partial\tilde{u}^\mu}\right] & 0_{n\times r} \\ \left[\dfrac{\partial u^b}{\partial\tilde{u}^\mu}\right] & \left[\dfrac{\partial u^b}{\partial\tilde{u}^a}\right] \end{pmatrix} =: \begin{bmatrix} \dfrac{\partial u^\nu}{\partial\tilde{u}^\mu} & 0 \\ \dfrac{\partial u^b}{\partial\tilde{u}^\mu} & \dfrac{\partial u^b}{\partial\tilde{u}^a} \end{bmatrix}, \tag{3.3}$$

where $0_{n\times r}$ is the $n \times r$ zero matrix. Besides, in expressions of the form $\partial_I a^I$, like the one in the right-hand side of (3.2a), the summation excludes differentiation, that is, $\partial_I a^I := a^I \partial_I = \sum_I a^I \partial_I$; if a differentiation really takes place, we write $\partial_I(a^I) := \sum_I \partial_I(a^I)$. This rule allows a lot of formulae to be written in a compact matrix form, like (3.2a). The explicit form of the matrix inverse to (3.3) is $A^{-1} = [\partial\tilde{u}^I/\partial u^J] = \cdots$, and it is obtained from (3.3) via the change $u \leftrightarrow \tilde{u}$.

The formulae (3.2) can be generalized for arbitrary frame $\{e_I\} = \{e_\mu, e_a\}$ in $T(E)$ and its dual coframe $\{e^I\} = \{e^\mu, e^a\}$ in $T^*(E)$ which respect the fibre structure in a sense that their *admissible changes* are given by

$$(e_I) = (e_\mu, e_a) \longmapsto (\tilde{e}_I) = (\tilde{e}_\mu, \tilde{e}_a) = (e_\nu, e_b) \cdot A, \tag{3.4a}$$

$$\begin{pmatrix} e^\mu \\ e^a \end{pmatrix} \longmapsto \begin{pmatrix} \tilde{e}^\mu \\ \tilde{e}^a \end{pmatrix} = A^{-1} \cdot \begin{pmatrix} e^\nu \\ e^b \end{pmatrix}. \tag{3.4b}$$

Here $A = [A^I_J]$ is a nondegenerate matrix-valued function with a block structure similar to (3.3), namely,

$$A = \begin{pmatrix} [A^\nu_\mu]^n_{\mu,\nu=1} & 0_{n\times r} \\ [A^b_\mu]_{\substack{\mu=1,\dots,n \\ b=n+1,\dots,n+r}} & [A^b_a]^{n+r}_{a,b=n+1} \end{pmatrix} =: \begin{bmatrix} A^\nu_\mu & 0 \\ A^b_\mu & A^b_a \end{bmatrix} \tag{3.5a}$$

with inverse matrix

$$A^{-1} = \begin{pmatrix} [A^\nu_\mu]^{-1} & 0 \\ -[A^a_b]^{-1} \cdot [A^a_\mu] \cdot [A^\nu_\mu]^{-1} & [A^a_b]^{-1} \end{pmatrix}. \tag{3.5b}$$

Here $A^a_\mu : U \to \mathbb{K}$ and $[A^\nu_\mu]$ and $[A^a_b]$ are nondegenerate matrix-valued functions on U such that $[A^\nu_\mu]$ is constant on the fibres of E, that is, for $p \in U$, $A^\nu_\mu(p)$ depends only on $\pi(p) \in M$, which is equivalent to any one of the equations

$$A^\nu_\mu = B^\nu_\mu \circ \pi, \qquad \frac{\partial A^\nu_\mu}{\partial u^a} = 0, \tag{3.6}$$

with $[B^\nu_\mu]$ being a nondegenerate matrix-valued function on $\pi(U) \subseteq M$. Obviously, (3.2) corresponds to (3.4) with $e_I = \partial/\partial u^I$, $\tilde{e}_I = \partial/\partial \tilde{u}^I$, and $A^J_I = \partial u^J/\partial \tilde{u}^I$.

All frames on E connected via (3.4)-(3.5), which are (locally) obtainable from holonomic ones, induced by bundle coordinates via admissible changes, will be referred as *bundle frames*. Only such frames will be employed in the present work.

If we deal with a *vector* bundle (E, π, M) endowed with vector bundle coordinates $\{u^I\}$ [29], then the new fibre coordinates $\{\tilde{u}^a\}$ in (3.1) must be *linear and homogeneous* in the old ones $\{u^a\}$, that is,

$$\tilde{u}^a = (B^a_b \circ \pi) \cdot u^b, \qquad u^a = \left((B^{-1})^a_b \circ \pi\right) \cdot \tilde{u}^b, \tag{3.7}$$

with $B = [B^a_b]$ being a nondegenerate matrix-valued function on $\pi(U) \subseteq M$. In that case, the matrix (3.3) and its inverse take the form

$$A = \begin{bmatrix} \dfrac{\partial u^\mu}{\partial \tilde{u}^\nu} & 0 \\ \left(\dfrac{\partial (B^{-1})^a_b}{\partial \tilde{x}^\nu} \circ \pi\right) \cdot \tilde{u}^b & (B^{-1})^b_a \circ \pi \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} \dfrac{\partial \tilde{u}^\nu}{\partial u^\mu} & 0 \\ \left(\dfrac{\partial B^b_a}{\partial x^\mu} \circ \pi\right) \cdot u^a & B^b_a \circ \pi \end{bmatrix}. \tag{3.8}$$

More generally, in the vector bundle case, transformations (3.4) are admissible with matrices like

$$A = \begin{pmatrix} [A^{\mu}_{\nu}] & 0 \\ [A^{b}_{c\mu}\widetilde{u}^{c}] & [A^{a}_{b}] \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} [A^{\mu}_{\nu}]^{-1} & 0 \\ -[A^{a}_{b}]^{-1} \cdot [A^{b}_{c\mu}\widetilde{u}^{c}] \cdot [A^{\mu}_{\nu}]^{-1} & [A^{a}_{b}]^{-1} \end{pmatrix} \tag{3.9}$$

with $A^{a}_{b\mu} : U \to \mathbb{K}$ being functions on U which are constant on the fibres of E,

$$A^{a}_{b\mu} = B^{a}_{b\mu} \circ \pi, \qquad \frac{\partial A^{a}_{b\mu}}{\partial u^{c}} = 0 \tag{3.10}$$

for some functions $B^{a}_{b\mu} : \pi(U) \to \mathbb{K}$. Obviously, (3.9) corresponds to (3.5) with $A^{b}_{\mu} = A^{b}_{c\mu}\widetilde{u}^{c}$ and the setting $A^{J}_{I} = \partial u^{J}/\partial\widetilde{u}^{I}$ reduces (3.9) to (3.8) due to (3.7).

3.2. Connection theory. From a number of equivalent definitions of a connection on differentiable manifold [24, Sections 2.1 and 2.2], we will use the following one.

Defintion 3.1. A *connection on a bundle* (E,π,M) is an $n = \dim M$-dimensional distribution Δ^{h} on E such that, for each $p \in E$ is fulfilled,

$$\Delta^{v}_{p} \oplus \Delta^{h}_{p} = T_{p}(E), \tag{3.11}$$

where the *vertical distribution* Δ^{v} is defined by

$$\Delta^{v} : p \longmapsto \Delta^{v}_{p} := T_{\imath(p)}(\pi^{-1}(\pi(p))) \cong T_{p}(\pi^{-1}(\pi(p))), \tag{3.12}$$

with $\imath : \pi^{-1}(\pi(p)) \to E$ being the inclusion mapping, $\Delta^{h} : p \mapsto \Delta^{h}_{p} \subseteq T_{p}(E)$, and $\oplus$ is the direct sum sign. The distribution Δ^{h} is called *horizontal*, and symbolically we write $\Delta^{v} \oplus \Delta^{h} = T(E)$.

A *vector* at a point $p \in E$ (resp., a *vector field* on E) is said to be *horizontal* or *vertical* if it (resp., its value at p) belongs to Δ^{h}_{p} or Δ^{v}_{p}, respectively, for the given (resp., any) point p. A vector $Y_{p} \in T_{p}(E)$ (resp., vector field $Y \in \mathcal{X}(E)$) is called a *horizontal lift of a vector* $X_{\pi(p)} \in T_{\pi(p)}(M)$ (resp., *vector field* $X \in \mathcal{X}(M)$ on $M = \pi(E)$) if $\pi_{*}(Y_{p}) = X_{\pi(p)}$ for the given (resp., any) point $p \in E$. Since $\pi_{*}|_{\Delta^{h}_{p}} : \Delta^{h}_{p} \to T_{\pi(p)}(M)$ is a vector space isomorphism for all $p \in E$ [29, Section 1.24], any vector in $T_{\pi(p)}(M)$ (resp., vector field in $\mathcal{X}(M)$) has a unique horizontal lift in $T_{p}(E)$ (resp., $\mathcal{X}(E)$).

As a result of (3.11), any vector $Y_{p} \in T_{p}(E)$ (resp., vector field $Y \in \mathcal{X}(E)$) admits a unique representation $Y_{p} = Y^{v}_{p} \oplus Y^{h}_{p}$ (resp., $Y = Y^{v} \oplus Y^{h}$) with $Y^{v}_{p} \in \Delta^{v}_{p}$ and $Y^{h}_{p} \in \Delta^{h}_{p}$ (resp., $Y^{v} \in \Delta^{v}$ and $Y^{h} \in \Delta^{h}$). If the distribution $p \mapsto \Delta^{h}_{p}$ is differentiable of class C^{m}, $m \in \mathbb{N} \cup \{0,\infty,\omega\}$, it is said that the *connection* Δ^{h} *is (differentiable) of class* C^{m}. A connection Δ^{h} is of class C^{m} if and only if, for every C^{m} vector field Y on E, the vertical Y^{v} and horizontal Y^{h} vector fields are of class C^{m}.

A C^{1} *path* $\beta : J \to E$ is called *horizontal* (*vertical*) if its tangent vector $\dot{\beta}$ is horizontal (vertical) vector along β, that is, $\dot{\beta}(s) \in \Delta^{h}_{\beta(s)}$ ($\dot{\beta}(s) \in \Delta^{v}_{\beta(s)}$) for all $s \in J$. A *lift* $\overline{\gamma} : J \to E$ of a path $\gamma : J \to M$, that is, $\pi \circ \overline{\gamma} = \gamma$, is called *horizontal* if $\overline{\gamma}$ is a horizontal path, that is, when the vector field $\dot{\overline{\gamma}}$ tangent to $\overline{\gamma}$ is horizontal or, equivalently, if $\dot{\overline{\gamma}}$ is a horizontal lift of $\dot{\gamma}$. Since $\pi^{-1}(\gamma(J))$ is an $(r+1)$- dimensional submanifold of E for any C^1 path γ, the distribution $p \mapsto \Delta_p^h \cap T_p(\pi^{-1}(\gamma(J)))$ is one-dimensional and, consequently, is integrable for an arbitrary C^1 path γ. The integral paths of that distribution are horizontal lifts of γ and, *for each $p \in \pi^{-1}(\gamma(J))$, there is (locally) a unique horizontal lift $\overline{\gamma}_p$ of γ passing through p*. (In this sense, a connection Δ^h is an Ehresmann connection [9, page 314] and *vice versa* [32, pages 85–89]).

Defintion 3.2. Let $\gamma : [\sigma, \tau] \to M$, with $\sigma, \tau \in \mathbb{R}$ and $\sigma \leq \tau$, and let $\overline{\gamma}_p$ be the unique horizontal lift of γ in E passing through $p \in \pi^{-1}(\gamma([\sigma,\tau]))$. The *parallel transport (translation, displacement)* generated by (assigned to, defined by) a connection Δ^h is a mapping $P : \gamma \mapsto P^\gamma$, assigning to the path γ a mapping

$$P^\gamma : \pi^{-1}(\gamma(\sigma)) \to \pi^{-1}(\gamma(\tau)) \quad \gamma : [\sigma,\tau] \to M \tag{3.13}$$

such that, for each $p \in \pi^{-1}(\gamma(\sigma))$,

$$P^\gamma(p) := \overline{\gamma}_p(\tau). \tag{3.14}$$

In vector bundles the *linear* connections for which is required the parallel transport assigned to them to be linear in a sense that the mapping (3.13) is linear for every path γ are important (see Section 4.3).

Let us now look on the connections Δ^h on a bundle (E,π,M) from a view point of (local) frames and their dual coframes on E. Let $\{e_\mu\}$ be a basis for Δ^h, that is, $e_\mu \in \Delta^h$ and $\{e_\mu|_p\}$ is a basis for Δ_p^h for all $p \in E$, and let $\{e^a\}$ be the coframe for Δ^h, that is, a collection of 1-forms e^a, $a = n+1,\dots,n+r$, which are linearly independent (relative to functions $E \to \mathbb{K}$) and such that $e^a(X) = 0$ if $X \in \Delta^h$.

Defintion 3.3. A frame $\{e_I\}$ in $T(E)$ over E is called *specialized* for a connection Δ^h if the first $n = \dim M$ of its vector fields form a basis $\{e_\mu\}$ for the horizontal distribution Δ^h and its last $r = \dim \pi^{-1}(x)$, $x \in M$, vector fields form a basis $\{e_a\}$ for the vertical distribution Δ^v. Respectively, a coframe $\{e^I\}$ on E is called *specialized* if $\{e^a\}$ is a cobasis for Δ^h and $\{e^\mu\}$ is a cobasis for Δ^v.

The horizontal lifts of vector fields and 1-forms can easily be described in specialized (co)frames. Indeed, let $\{e_I\}$ and $\{e^I\}$ be, respectively, a specialized frame and its dual coframe. Define a frame $\{E_\mu\}$ and its dual coframe $\{E^\mu\}$ on M which are π-related to $\{e_I\}$ and $\{e^I\}$, that is, $E_\mu := \pi_*(e_\mu)$ and $e^\mu := \pi^*(E^\mu) = E^\mu \circ \pi_*$. (Recall, $\pi_*|_{\Delta_p^h} : \Delta_p^h \to T_{\pi(p)}(M)$ is a vector space isomorphism.) If $Y = Y^\mu E_\mu \in \mathcal{X}(M)$ and $\phi = \phi_\mu e^\mu \in \Lambda^1(M)$, then their horizontal lifts (from M to E), respectively, are

$$\overline{Y} = (Y^\mu \circ \pi) e_\mu, \qquad \overline{\phi} = (\phi_\mu \circ \pi) e^\mu. \tag{3.15}$$

The specialized (co)frames transform into each other according to the general rules (3.4) in which the transformation matrix and its inverse have the following block structure:

$$A = \begin{pmatrix} [A_\mu^\nu] & 0_{n\times r} \\ 0_{r\times n} & [A_a^b] \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} [A_\mu^\nu]^{-1} & 0_{n\times r} \\ 0_{r\times n} & [A_a^b]^{-1} \end{pmatrix}, \tag{3.16}$$

where $A_\mu^\nu, A_a^b : E \to \mathbb{K}$ and the functions A_μ^ν are constant on the fibres of the bundle (E, π, M), that is, we have

$$A_\mu^\nu = B_\mu^\nu \circ \pi \qquad \text{or} \qquad \frac{\partial A_\mu^\nu}{\partial u^a} = 0 \tag{3.17}$$

for some nondegenerate matrix-valued function $[B_\mu^\nu]$ on M. Besides, in a case of vector bundle, the functions A_b^a are also constant on the fibres of the bundle (E, π, M), that is,

$$A_a^b = B_a^b \circ \pi \qquad \text{or} \qquad \frac{\partial A_a^b}{\partial u^a} = 0 \tag{3.18}$$

for some nondegenerate matrix-valued function $B = [B_a^b]$ on M. Changes like (3.4), with A given by (3.16), respect the fibre as well as the connection structure of the bundle.

Let E be a C^2 manifold and Δ^h a C^1 connection on (E, π, M). The components C_{IJ}^K of the anholonomicity object of a *specialized* frame $\{e_I\}$ are (local) functions on E defined by (see (2.6))

$$[e_I, e_J]_- =: C_{IJ}^K e_K \tag{3.19}$$

and are naturally divided into the following six groups (cf. [30, page 21]):

$$\{C_{\mu\nu}^\lambda\}, \qquad \{C_{\mu\nu}^a\}, \qquad \{C_{\mu b}^\lambda = 0\}, \qquad \{C_{ab}^\lambda = 0\}, \qquad \{C_{\mu b}^c\}, \qquad \{C_{ab}^c\}. \tag{3.20}$$

The functions $C_{\mu\nu}^\lambda$ are constant on the fibres of (E, π, M), precisely $C_{\mu\nu}^\lambda = f_{\mu\nu}^\lambda \circ \pi$, where $f_{\mu\nu}^\lambda$ are the components of the anholonomicity object for the π-related frame $\{\pi_*(e_\mu)\}$ on M, as the commutators of π-related vector fields are π-related [40, Section 1.55]. Besides, since the vertical distribution Δ^v is integrable (the space Δ_p^v is the space tangent to the fibre through $p \in E$ at p), we have

$$[e_a, e_b]_- = C_{ab}^c e_c \tag{3.21}$$

(so that $C_{ab}^\lambda = 0$), due to which C_{ab}^c are called components of the vertical anholonomicity object. To prove that $C_{\mu b}^\lambda = 0$, one should expand $\{e_I\}$ along $\{\partial_I = \partial/\partial u^I\}$, with $\{u^I\}$ being some bundle coordinates, namely, $e_\mu = e_\mu^\nu \partial_\nu + e_\mu^b \partial_b$ and $e_a = e_a^b \partial_b$, with some functions e_μ^ν, e_μ^b, and e_a^b, and to notice that e_μ^ν are constant on the fibres, that is, $\partial_a(e_\mu^\nu) = 0$.

The nontrivial mixed "vertical-horizontal" components between (3.20), namely, $C^a_{\mu\nu}$ and $C^a_{\mu b}$, are important characteristics of the connection Δ^h. The functions

$$ {}^{\circ}\Gamma^a_{b\mu} := +C^a_{b\mu} = -C^a_{\mu b}, \tag{3.22a} $$

$$ R^a_{\mu\nu} := +C^a_{\mu\nu} = -C^a_{\nu\mu}, \tag{3.22b} $$

which enter into the commutators

$$ \mathcal{L}_{e_\mu} e_b = [e_\mu, e_b]_{\_} = {}^{\circ}\Gamma^a_{b\mu} e_a, \tag{3.23a} $$

$$ \mathcal{L}_{e_\mu} e_\nu = [e_\mu, e_\nu]_{\_} = R^a_{\mu\nu} e_a + C^\lambda_{\mu\nu} e_\lambda, \tag{3.23b} $$

are called, respectively, the *fibre coefficients of* Δ^h (or *components of the connection object of* Δ^h) and *fibre components of the curvature of* Δ^h (or *components of the curvature (object) of* Δ^h) in $\{e_I\}$. Under a change (3.4), with matrix (3.16), of the specialized frame, functions (3.22) transform into, respectively,

$$ {}^{\circ}\widetilde{\Gamma}^a_{b\mu} = A^\nu_\mu \big([A^f_e]^{-1} \big)^a_d \big({}^{\circ}\Gamma^d_{c\nu} A^c_b + e_\nu (A^d_b) \big), \tag{3.24a} $$

$$ \widetilde{R}^a_{\mu\nu} = \big([A^f_e]^{-1} \big)^a_b A^\lambda_\mu A^\varrho_\nu R^b_{\lambda\varrho}, \tag{3.24b} $$

which formulae are direct consequences of (3.23). If we put $\overline{A} := [A^b_a]$, ${}^{\circ}\Gamma_\nu := [{}^{\circ}\Gamma^d_{c\nu}]$, and ${}^{\circ}\widetilde{\Gamma}_\nu := [{}^{\circ}\widetilde{\Gamma}^d_{c\nu}]$, then (3.24a) is tantamount to

$$ \begin{aligned} {}^{\circ}\widetilde{\Gamma}_\mu &= A^\nu_\mu \overline{A}^{-1} \cdot \big({}^{\circ}\Gamma_\nu \cdot \overline{A} + e_\nu(\overline{A}) \big) \\ &= A^\nu_\mu \big(\overline{A}^{-1} \cdot {}^{\circ}\Gamma_\nu - e_\nu(\overline{A}^{-1}) \big) \cdot \overline{A}. \end{aligned} \tag{3.25} $$

Up to a meaning of the matrices $[A^\nu_\mu]$ and $\overline{A}$ and the size of the matrices ${}^{\circ}\Gamma_\nu$ and $\overline{A}$, the last equation is identical with the one expressing the transformed matrices of the coefficients of a linear connection (covariant derivative operator) in a vector bundle [17, equation (3.5)] on which we will return later in this work (see Section 4, in particular (4.37)). Equation (3.24b) indicates that $R^a_{\mu\nu}$ are components of a tensor, namely,

$$ \Omega := \frac{1}{2} R^a_{\mu\nu} e_a \otimes e^\mu \wedge e^\nu, \tag{3.26} $$

called *curvature tensor* of the connection Δ^h. By (3.23a), the horizontal distribution Δ^h is (locally) integrable if and only if its curvature tensor vanishes, $\Omega = 0$.

Defintion 3.4. A connection with vanishing curvature tensor is called *flat*, *integrable*, or *curvature free.*

Proposition 3.5. *The flat connections are the only ones that may admit holonomic specialized frames.*

Proof. See Definition 3.4 and (3.23b). □

The above considerations of specialized (co)frames for a connection Δ^h on a bundle (E,π,M) were *global* as we supposed that these (co)frames are defined on the whole bundle space E, which is always possible if no smoothness conditions on Δ^h are imposed. Below we will show how *local* specialized (co)frames can be defined via local bundle coordinates on E.

Let $\{u^I\}$ be local bundle coordinates on an open set $U \subseteq E$. They define on $T(U) \subseteq T(E)$ the local basis $\{\partial_I := \partial/\partial u^I\}$, so that any vector can be expended along its vectors. In particular, we can do so with any basic vector field e_I^U of a *specialized* frame $\{e_I\}$ restricted to U, $e_I^U := e_I|_U$. Since $\{\partial_a|_p\}$, with $p \in U$, is a basis for Δ_p^v, we can write

$$(e_\mu^U, e_a^U) = (A_\mu^\nu \partial_\mu + A_\mu^a \partial_a, A_a^b \partial_b) = (\partial_\nu, \partial_b) \cdot \begin{pmatrix} [A_\mu^\nu] & 0 \\ [A_\mu^b] & [A_a^b] \end{pmatrix}, \tag{3.27}$$

where $[A_\mu^\nu]$ and $[A_a^b]$ are nondegenerate matrix-valued functions on U. (The nondegeneracy of $[A_\mu^\nu]$ follows from the fact that the vector fields $\pi_*|_{\Delta^h}(e_\mu^U) = A_\mu^\nu \pi_*(\partial/\partial u^\mu)$ form a basis for $\mathscr{X}(\pi(U)) \subseteq \mathscr{X}(M)$).

Defintion 3.6. A frame $\{X_I\}$ over U in $T(U)$ is called *adapted (to the coordinates $\{u^I\}$ in U)* for a connection Δ^h if it is the specialized frame obtained from (3.27) via admissible transformation (3.4) with the matrix $A = \begin{pmatrix} [A_\mu^\nu]^{-1} & 0 \\ 0 & [A_a^b]^{-1} \end{pmatrix}$.

Exercise 3.7. An arbitrary *specialized* frame $\{e_I^U\}$ in $T(E)$ over U enters in the definition of a frame $\{X_I\}$ adapted to bundle coordinates $\{u^I\}$ on U. Prove that $\{X_I\}$ is independent of the particular choice of the frame $\{e_I^U\}$. (Hint: apply Definition 3.6 and (3.4a) with A given by (3.16).) The below-derived equality (3.34) is an indirect proof of that fact too.

According to (3.4) and Definition 3.6, the adapted frame $\{X_I\}$ and its corresponding *adapted coframe* $\{\omega^I\}$ are given by

$$X_\mu = \partial_\mu + \Gamma_\mu^a \partial_a, \qquad X_a = \partial_a, \tag{3.28a}$$

$$\omega^\mu = \mathrm{d}u^\mu, \qquad \omega^a = \mathrm{d}u^a - \Gamma_\mu^a \mathrm{d}u^\mu. \tag{3.28b}$$

Here the functions $\Gamma_\mu^a : U \to \mathbb{K}$ are defined via

$$[\Gamma_\mu^a] = +[A_\nu^a] \cdot [A_\mu^\nu]^{-1} \tag{3.29}$$

and are called *(2-index) coefficients* of Δ^h. In a matrix form, (3.28) can be written as

$$(X_\mu, X_a) = (\partial_\nu, \partial_b) \cdot \begin{bmatrix} \delta_\mu^\nu & 0 \\ +\Gamma_\mu^b & \delta_a^b \end{bmatrix}, \qquad \begin{pmatrix} \omega^\mu \\ \omega^a \end{pmatrix} = \begin{bmatrix} \delta_\nu^\mu & 0 \\ -\Gamma_\nu^a & \delta_b^a \end{bmatrix} \cdot \begin{pmatrix} \mathrm{d}u^\nu \\ \mathrm{d}u^b \end{pmatrix}. \tag{3.30}$$

The operators $X_\mu = \partial_\mu + \Gamma_\mu^a \partial_a$ are known as *covariant derivatives on* $T(U)$ and the plus sign in (3.28a) before Γ_μ^a (hence in the right-hand side of (3.29)) is conventional.

If $\{u^I\}$ and $\{\tilde{u}^I\}$ are local coordinates on open sets $U \subseteq E$ and $\tilde{U} \subseteq E$, respectively, and $U \cap \tilde{U} \neq \varnothing$, then, on the overlapping set $U \cap \tilde{U}$, a problem arises: how are the adapted frames corresponding to these coordinates connected? Let us mark with a tilde all quantities that refer to the coordinates $\{\tilde{u}^I\}$. Since the adapted frames are, by definitions, specialized ones, we can write (see (3.4))

$$(\tilde{X}_\mu, \tilde{X}_a) = (X_\nu, X_b) \cdot A, \qquad \begin{pmatrix} \tilde{\omega}^\mu \\ \tilde{\omega}^a \end{pmatrix} = A^{-1} \cdot \begin{pmatrix} \omega^\nu \\ \omega^b \end{pmatrix}, \tag{3.31a}$$

where the transformation matrix A and its inverse have the form (3.16). Recalling (3.2) and (3.3), from these equalities, we get

$$A = \operatorname{diag}\left(\left[\frac{\partial u^\nu}{\partial \tilde{u}^\mu}\right], \left[\frac{\partial u^b}{\partial \tilde{u}^a}\right] \right) = \begin{pmatrix} \left[\frac{\partial u^\nu}{\partial \tilde{u}^\mu}\right] & 0 \\ 0 & \left[\frac{\partial u^b}{\partial \tilde{u}^a}\right] \end{pmatrix}. \tag{3.31b}$$

Combining (3.29) and (3.31), one can easily prove the following.

Proposition 3.8. *A change $\{u^I\} \mapsto \{\tilde{u}^I\}$ of the local bundle coordinates implies the following transformation of the 2-index coefficients of the connection:*

$$\Gamma^a_\mu \longmapsto \tilde{\Gamma}^a_\mu = \left(\frac{\partial \tilde{u}^a}{\partial u^b} \Gamma^b_\nu + \frac{\partial \tilde{u}^a}{\partial u^\nu} \right) \frac{\partial u^\nu}{\partial \tilde{u}^\mu}. \tag{3.32}$$

It is obvious that a connection Δ^h is of class C^m, $m \in \mathbb{N} \cup \{0\}$, if and only if its coefficients Γ^a_μ are C^m functions on U, provided ∂_I are C^m vector fields on U (which is the case when E is a C^{m+1} manifold). By virtue of (3.32), the C^{m+1} changes of the local bundle coordinates preserve the C^m differentiability of Γ^a_μ. Thus the C^{m+1} differentiability of the base M and bundle E spaces is a necessary condition for existence of C^m connections on (E, π, M); as we assumed $m = 1$ in this work, the connections considered here can be at most of differentiability class C^1.

The next proposition states that a connection on a bundle is locally equivalent to a geometric object whose components transform like (3.32).

Proposition 3.9. *To any connection Δ^h in a bundle (E, π, M) can be assigned, according to the procedure described above, a geometrical object on E whose components Γ^a_μ in bundle coordinates $\{u^I\}$ on E transform according to (3.32) under a change $\{u^I\} \mapsto \{\tilde{u}^I\}$ of the bundle coordinates on the intersection of the domains of $\{u^I\}$ and $\{\tilde{u}^I\}$. Conversely, given a geometrical object on E with local transformation law (3.32), there is a unique connection Δ^h in (E, π, M) which generates the components of that object as described above.*

Proof. The first part of the statement was proved above, when we constructed the adapted frame (3.28a) and derived (3.32). To prove the second part, choose a point $p \in E$ and some local coordinates $\{u^I\}$ on an open set U in E containing p in which the geometrical object mentioned has local components Γ^a_μ. Define a local frame $\{X_I\} = \{X_\mu, X_a\}$ on U by (3.28a). The required connection is then $\Delta^h : q \mapsto \Delta^h_q := \{r^\mu X_\mu|_q : r^\mu \in \mathbb{K}\}$ for any $q \in U$, which means that Δ^h_q is the linear cover of $\{X_\mu|_q\}$. The transformation law (3.32) insures the independence of Δ^h from the local coordinates employed in its definition. $\square$

From the construction of an adapted frame $\{X_I\}$, as well as from the proof of Proposition 3.9, follows that the set of vectors $\{X_\mu\}$ is a basis for the horizontal distribution Δ^h, and the set $\{X_a\}$ is a basis for the vertical distribution Δ^v. The matrix of the restricted tangent projection $\pi_*|_{\Delta^h}$ in bundle coordinate system $\{u^\mu = x^\mu \circ \pi, u^a\}$ on E, where the x^μ are local the coordinates on M, is the identity matrix as $(\pi_*|_{\Delta^h_p})^\nu_\mu = \partial(x^\mu \circ \pi)/\partial u^\mu|_p = \delta^\nu_\mu$ for any point p in the domain of $\{u^I\}$. Hereof we get

$$\pi_*|_{\Delta^h}(X_\mu) = \frac{\partial}{\partial x^\mu}\left(\Longleftrightarrow \pi_*|_{\Delta^h_p}(X_\mu|_p) = \frac{\partial}{\partial x^\mu}\bigg|_{\pi(p)}\right). \tag{3.33}$$

In particular, from here follows that $\pi_*|_{\Delta^h_p} : \Delta^h_p \to T_{\pi(p)}(M)$ is a vector space isomorphism. The inverse to (3.33), namely,

$$X_\mu = (\pi_*|_{\Delta^h})^{-1}\left(\frac{\partial}{\partial x^\mu}\right) = (\pi_*|_{\Delta^h})^{-1} \circ \pi_*\left(\frac{\partial}{\partial u^\mu}\right), \tag{3.34}$$

can be used in an equivalent definition of a frame $\{X_I\}$ adapted to local coordinates $\{u^I\}$, namely, this is the frame $((\pi_*|_{\Delta^h})^{-1} \circ \pi_*(\partial/\partial u^\mu), \partial/\partial u^a)$. If one accepts such a definition of an adapted frame for Δ^h, the (2-index) coefficients of Δ^h have to be defined via expansion (3.28a); the only changes this may entail are in the proofs of some results, like (3.31) and (3.32).

It is useful to be recorded also the simple fact that, by construction, we have

$$\pi_*(X_a) = 0. \tag{3.35}$$

Let E be a C^2 manifold and let Δ^h be a C^1 connection. The *adapted frames are generally anholonomic* as the commutators between the basic vector fields of the adapted frame (3.28a) are (cf. (3.31) and (3.32))

$$[X_\mu, X_\nu]_- = R^a_{\mu\nu}X_a, \qquad [X_\mu, X_b]_- = {}^\circ\Gamma^a_{b\mu}X_a, \qquad [X_a, X_b]_- = 0, \tag{3.36}$$

with

$$R^a_{\mu\nu} = \partial_\mu(\Gamma^a_\nu) - \partial_\nu(\Gamma^a_\mu) + \Gamma^b_\mu\partial_b(\Gamma^a_\nu) - \Gamma^b_\nu\partial_b(\Gamma^a_\mu) = X_\mu(\Gamma^a_\nu) - X_\nu(\Gamma^a_\mu) \tag{3.37a}$$

$${}^\circ\Gamma^a_{b\mu} = -\partial_b(\Gamma^a_\mu) = -X_b(\Gamma^a_\mu) \tag{3.37b}$$

being the fibre components of the curvature and fibre coefficients of the connection. An obvious result from (3.36) is stated as follows.

Proposition 3.10. *An adapted frame is holonomic if and only if*

$$R^a_{\mu\nu} = 0(\Longleftrightarrow \Omega = 0), \qquad {}^\circ\Gamma^a_{b\mu} = 0. \tag{3.38}$$

Therefore, only the flat (integrable) C^1 connections, for which $\Omega = 0$, may admit holonomic adapted frames (cf. Proposition 3.5). Besides, as a consequence of (3.37b) and (3.38), such connections admit holonomic adapted frames on $U \subseteq E$ if and only if there are local coordinates on U in which the coefficients Γ^a_μ are constant on the fibres passing through U, that is, if and only if $\Gamma^a_\mu = G^a_\mu \circ \pi$ for some functions $G^a_\mu : \pi(U) \to \mathbb{K}$, which is equivalent to $\partial_b(\Gamma^a_\mu) = 0$.

Example 3.11 (horizontal lifting of a path). Recall, the horizontal lift of a C^1 path $\gamma : J \to M$ passing through a point $p \in \pi^{-1}(\gamma(t_0))$ for some $t_0 \in J$ is the unique path $\overline{\gamma}_p : J \to E$ such that $\pi \circ \overline{\gamma}_p = \gamma$, $\overline{\gamma}_p(t_0) = p$, and $\dot{\overline{\gamma}}_p(t) \in \Delta^h_{\overline{\gamma}_p(t)}$ for all $t \in J$. As in a specialized frame $\{e_I\}$, the relation $X_p \in \Delta^h_p$ is equivalent to $e^a(X) = 0$ for any $X \in \mathscr{X}(M)$; in an adapted coframe, given by (3.28b), the horizontal lift $\overline{\gamma}_p$ of γ is the unique solution of the initial value problem

$$\omega^a(\dot{\overline{\gamma}}_p) = 0, \tag{3.39a}$$

$$\overline{\gamma}_p(t_0) = p, \tag{3.39b}$$

which is tantamount to any one of the initial-value problems ($t \in J$)

$$\dot{\overline{\gamma}}^a_p(t) - \Gamma^a_\mu(\overline{\gamma}_p(t))\dot{\overline{\gamma}}^\mu_p(t) = 0, \tag{3.39a$'$}$$

$$\overline{\gamma}^I_p(t_0) = p^I := u^I(p), \tag{3.39b$'$}$$

$$\frac{\mathrm{d}(u^a \circ \overline{\gamma}_p(t))}{\mathrm{d}t} - \Gamma^a_\mu(\overline{\gamma}_p(t))\frac{\mathrm{d}(x^\mu \circ \gamma(t))}{\mathrm{d}t} = 0, \tag{3.39a$''$}$$

$$u^I(\overline{\gamma}_p(t_0)) = u^I(p), \tag{3.39b$''$}$$

where x^μ are the local coordinates in the base space that induce the basic coordinates u^μ on the bundle space, $u^\mu = x^\mu \circ \pi$. (Note that the quantities $\mathrm{d}(x^\mu \circ \gamma(t))/\mathrm{d}t$, entering into (3.39a$''$), are the components of the vector $\dot{\gamma}$ tangent to γ at parameter value t.) One may call (3.39a), or any one of its versions (3.39a$'$) or (3.39a$''$), the *parallel transport equation* in an adapted frame as it uniquely determines the parallel transport along the restriction of γ to any closed subinterval in J (see Definition 3.2).

Example 3.12 (the equation of geodesic paths). A connection Δ^h on the tangent bundle $(T(M), \pi_T, M)$ of a manifold M is called a *connection on* M. In this case, (3.39) defines also the geodesics (relative to Δ^h) in M. A C^2 path $\gamma : J \to M$ in a C^2 manifold M is called a *geodesic path* if its tangent vector field $\dot{\gamma}$ undergoes parallel transport along the same path γ, that is, $P^{\gamma|[\sigma,\tau]}(\dot{\gamma}(\sigma)) = \dot{\gamma}(\tau)$ for all $\sigma, \tau \in J$, which means that the lifted path $\dot{\gamma} : J \to T(M)$ is a *horizontal* lift of γ (relative to Δ^h). So, if x^μ are local coordinates on $\pi(U) \in M$ and the bundle coordinates on $U \subseteq E$ are such that [40, Section 1.25] $u^\mu = x^\mu \circ \pi$ and $u^{n+\mu} = \mathrm{d}x^\mu$ ($\mu, \nu, \ldots = 1, \ldots, n = \dim M$), then (3.39a$''$) transforms into the *geodesic equation* (on M)

$$\frac{\mathrm{d}^2(x^\mu \circ \gamma(t))}{\mathrm{d}t^2} - \Gamma^{n+\mu}_\nu(\dot{\gamma}(t))\frac{\mathrm{d}(x^\nu \circ \gamma(t))}{\mathrm{d}t} = 0, \quad t \in J, \tag{3.42}$$

which (locally) defines all geodesics in M. (With obvious renumbering of the indices, one usually writes Γ^μ_ν for $\Gamma^{n+\mu}_\nu$.) Of course, a particular geodesic is specified by fixing some initial values for $\gamma(t_0)$ and $\dot{\gamma}(t_0)$ for some $t_o \in J$. Notice that (3.42) is an equation for a path γ in M, while (3.39a$''$) is an equation for the lifted path $\overline{\gamma}$ in $T(M)$ provided the path γ in M is known; for a geodesic path $\gamma : J \to M$, evidently, we have $\dot{\gamma} = \overline{\gamma}$. That is, the vector $\dot{\gamma}(t)$ tangent to γ at t is identical with the value $\overline{\gamma}(t)$ of the horizontal lifting $\overline{\gamma} : J \to T(M)$ of γ (in the tangent bundle space) at t for all $t \in J$.

4. Connections on vector bundles

In this section, by (E,π,M) we will denote an arbitrary *vector* bundle [29]. A peculiarity of such bundles is that their fibres are isomorphic vector spaces, which leads to a natural description of the vertical distribution Δ^v on their fibre spaces, as well as to existence of a kind of differentiation of their sections (known as covariant differentiation).

In the vector bundles, as we will do in this section, the so-called vector bundle coordinates which are linear on their fibres and are constructed as follows are used (cf. [32, page 30]). Here the linearity means that the fibre coordinate functions u^a with domain $U \subseteq E$ are linear on the intersection $U \cap \pi^{-1}(p)$ for all $p \in U$, that is, $u^a|_{U\cap\pi^{-1}(p)} : U \cap \pi^{-1}(p) \to \mathbb{K}$ are linear functions on the vector space $U \cap \pi^{-1}(p)$.

Let $\{e_a\}$ be a frame in E over a subset $U_M \subseteq M$, that is, $\{e_a(x)\}$ is a basis in $\pi^{-1}(x)$ for all $x \in U_M$. Then, for each $p \in \pi^{-1}(U_M)$, we have a unique expansion $p = p^a e_a(\pi(p))$ for some numbers $p^a \in \mathbb{K}$. The *vector fibre coordinates* $\{u^a\}$ on $\pi^{-1}(U_M)$ induced (generated) by the frame $\{e_a\}$ are defined via $u^a(p) := p^a$ and hence can be identified with the elements of the coframe $\{e^a\}$ dual to $\{e_a\}$, that is, $u^a = e^a$. Conversely, if $\{u^I\}$ are coordinates on $\pi^{-1}(U_M)$ for some $U_M \subseteq M$ which are linear on the fibres over U_M, then there is a unique frame $\{e_a\}$ in $\pi^{-1}(U_M)$ which generates $\{u^a\}$ as just described; indeed, considering $u^{n+1},\dots,u^{n+r}$ as 1-forms on $\pi^{-1}(U_M)$, one should define the frame $\{e_a\}$ required as a one whose dual is $\{u^a\}$, that is, via the conditions $u^a(e_b) = \delta^a_b$.

A collection (u^I) of basic coordinates (u^μ) and vector fibre coordinates $\{u^a\}$ on $\pi^{-1}(U_M)$ is called *vector bundle coordinates* on $\pi^{-1}(U_M)$. Only such coordinates on E will be employed in this section.

4.1. Vertical lifts. The idea of describing the vertical distribution Δ^v on a vector bundle is that if L is a vector space, then to any $Y \in L$ there corresponds a "vertical" vector field $Y^v \in \mathscr{X}(L) = \mathrm{Sec}(T(L),\pi_T,L)$ whose value at $X \in L$ is the vector tangent to the path $t \mapsto X + tY \in L$, with $t \in \mathbb{R}$, at $t = 0$, that is, $Y^v|_X := \mathrm{d}/\mathrm{d}t|_{t=0}(X + tY)$. Here and below, with $\mathrm{Sec}(E,\pi,M)$ (resp., $\mathrm{Sec}^m(E,\pi,M)$ with $m \in \mathbb{N} \cup \{0\}$) we denote the module of sections (resp., C^m sections) of a bundle (E,π,M) (resp., of a C^{m+1} bundle (E,π,M)).

Let (E,π,M) be a vector bundle and Δ^v the vertical distribution on it, namely, for each $p \in E$, $\Delta^v : p \mapsto \Delta^v_p := T_p(\pi^{-1}(\pi(p)))$. To every $Y \in \mathrm{Sec}(E,\pi,M)$, we assign a *vertical* vector field $Y^v \in \Delta^v$ on E such that, for $p \in E$,

$$Y^v_p := Y^v|_p := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}(p + tY|_{\pi(p)}). \tag{4.1}$$

(The mapping $(p, Y_{\pi(p)}) \mapsto Y^v_p$ defines an isomorphism from the pullback bundle π^*E into the vertical bundle $\mathscr{V}(E)$—see [29, Sections 1.27 and 1.28] and also [32, page 41, Exercises 2.2.1 and 2.2.2].)

Lemma 4.1. *Let $\{u^a\}$ be vector fibre coordinates generated by a frame $\{e_a\}$ on M. If $Y \in \operatorname{Sec}(E,\pi,M)$ and $Y = Y^a e_a$, then*

$$Y^v = (Y^a \circ \pi)\frac{\partial}{\partial u^a}. \tag{4.2}$$

Proof. Using Definition 4.4, we get for $p \in E$,

$$\begin{aligned} Y^v|_p &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(p+tY|_{\pi(p)}) = \frac{\mathrm{d}(u^a(p+tY|_{\pi(p)}))}{\mathrm{d}t}\Big|_{t=0}\frac{\partial}{\partial u^a}\Big|_p \\ &= \frac{\mathrm{d}(p^a+tY^a(\pi(p)))}{\mathrm{d}t}\Big|_{t=0}\frac{\partial}{\partial u^a}\Big|_p = Y^a(\pi(p))\frac{\partial}{\partial u^a}\Big|_p = \left((Y^a\circ\pi)\cdot\frac{\partial}{\partial u^a}\right)\Big|_p. \end{aligned} \tag{4.3}$$

□

If $Y \in \operatorname{Sec}(E,\pi,M)$, the vector field $Y^v := v(Y) \in \Delta^v$, defined via (4.1), is called the *vertical lift of the section Y*. It is (locally) given by (4.2) in vector bundle coordinates. An evident corollary of Lemma 4.1 is stated as follows.

Corollary 4.2. *The commutator (Lie bracket) of the vertical lifts of any two sections is zero, that is, if $Y_1, Y_2 \in \operatorname{Sec}(E,\pi,M)$, then*

$$[Y_1^v, Y_2^v] \equiv 0. \tag{4.4}$$

Proposition 4.3. *The mapping*

$$\begin{gathered} v : \operatorname{Sec}(E,\pi,M) \longrightarrow \{\textit{vector fields in } \Delta^v\}, \\ v : Y \longmapsto Y^v : p \longmapsto Y^v|_p := \frac{d}{dt}\Big|_{t=0}(p+tY_{\pi(p)}) \end{gathered} \tag{4.5}$$

is a bijection and it and its inverse are linear mappings.

Proof. The linearity and injectivity of v follow directly from (4.1). Now we will prove that, for each $Z \in \Delta^v$, there is a $Y \in \operatorname{Sec}(E,\pi,M)$ such that $Y^v = Z$, that is, v is also surjective. Let $Z = Z^a\partial/\partial u^a$, with $\{u^I\}$ being (local) vector bundle coordinates on E and the functions Z^a being constant on the fibres of E, that is, $Z^I = z^I \circ \pi$ for some functions z^I on M. Define $Y = z^a e_a$ with $\{e_a\}$ being the frame in E over M generating $\{u^I\}$. By Lemma 4.1, we have $Y^v = (z^a\circ\pi)(\partial/\partial u^a) = Z^a(\partial/\partial u^a) = Z$. The linearity of v^{-1} follows from here too. □

Consider a section ω of the bundle dual to (E,π,M) [29]. Its *vertical lift* ω_v is a 1-form on Δ^v such that, for $Z \in \Delta^v$ and $p \in E$, $\omega_v(Z)|_p = \omega(Y)|_{\pi(p)}$ for the unique section $Y \in \operatorname{Sec}(E,\pi,M)$ with $Y^v = Z$ (see Proposition 4.3), that is, we have $\omega_v(\cdot)|_p = (\omega\circ v^{-1}(\cdot))|_{\pi(p)}$ which means that

$$\omega_v(Z) = (\omega\circ v^{-1}(Z))\circ\pi \quad\text{or}\quad \omega_v(Y^v)\big|_p = \omega(Y)|_{\pi(p)}(=\omega_{\pi(p)}(Y_{\pi(p)})). \tag{4.6}$$

If $\{e^a = u^a\}$ is the coframe dual to $\{e_a\}$ and $\omega = \omega_a e^a$, then in the coframe $\{\mathrm{d}u^a\}$ dual to $\{\partial/\partial u^a\}$, we can write (cf. (4.2))

$$\omega_v = (\omega_a \circ \pi)\mathrm{d}u^a. \tag{4.7}$$

It should be mentioned, "vertical" lifts of vector fields or 1-forms over the base space M are generally not defined unless $E = T(M)$ or $E = T^*(M)$, respectively. (Since $\pi_*(\Delta_p^v) = 0_{\pi(p)} \in T_{\pi(p)}(M)$, $p \in E$, we can say that only the zero vector field over M has vertical lifts relative to π and any vector field in Δ^v is its vertical lift. This conclusion is independent of the existence of a connection on (E,π,M) and depends only on the fibre structure of E induced by π.)

Let Δ^h be a connection on (E,π,M) and let $\varphi : E \to \mathbb{K}$ be a C^1 mapping. Since any $X \in \mathscr{X}(E)$ can uniquely be written as a direct sum $X = v(X) \oplus h(X)$, with $v(X) \in \Delta^v$ and $h(X) \in \Delta^h$, we have $\varphi_*(X) = \varphi_*(v(X)) + \varphi_*(h(X)) \in \mathscr{X}(\mathbb{K})$. If $\{Z_I\}$ is a specialized frame in $T(E)$ and $\{Z^I\}$ is its dual coframe of 1-forms on $\mathscr{X}(E)$, we immediately get

$$\varphi_* = (\varphi_*(Z_a))Z^a + (\varphi_*(Z_\mu))Z^\mu = (Z_a(\varphi))Z^a + (Z_\mu(\varphi))Z^\mu, \tag{4.8}$$

as $X = X^I Z_I$ entails $v(X) = X^a Z_a$ and $h(X) = X^\mu Z_\mu$; in particular, (4.8) holds in any adapted (co)frame (3.28) and/or a section φ of the bundle (E,π,M). If $\{u^I\}$ are vector bundle coordinates, in the (co)frame (3.28) adapted to them, we have $Z_\mu = X_\mu$, $Z_a = \partial_a$, $Z^\mu = \omega^\mu = \mathrm{d}u^\mu$, $Z^a = \omega^a$, and we can write the expansion $\varphi = \varphi_a u^a$ with $\varphi_a : E \to \mathbb{K}$. Thus (4.8) takes the form

$$\varphi_* = \varphi_a \omega^a + \big(X_\mu(\varphi_a u^a)\big)\omega^\mu = \varphi_v + \big(X_\mu(\varphi_a u^a)\big)\omega^\mu, \tag{4.9}$$

where (4.7) was applied.

A *section* Y *of* (E,π,M) and *section* ω *of the bundle dual to* (E,π,M) can be lifted *vertically* via the mappings

$$v : Y \longmapsto Y^v \in \Delta^v, \tag{4.10a}$$

$$\omega \longmapsto \omega_v, \tag{4.10b}$$

respectively, given by (4.5) and (4.6) (see also (4.2) and (4.7)). These mappings do not require a connection and arise only from the fibre structure of the bundle space induced from the projection $\pi : E \to M$.

If a connection Δ^h on (E,π,M) is given, it generates *horizontal lifts of the vector fields on the base space* M and of the *1-forms on the same base space* M into respectively, vector fields in Δ^h and linear mappings on the vector fields in Δ^h. Precisely, if $F \in \mathscr{X}(M)$ and $\phi \in \Lambda^1(M)$, their *horizontal lifts* are defined by the mappings (alternatively, one may define $\phi'_h = \phi \circ \pi_* = \pi^*(\phi)$; in this way the domain of ϕ_h, which is defined by (4.11b), is expanded on the whole space $\mathscr{X}(E)$; obviously, $\phi'_h(Z) = \phi_h(Z)$ for $Z \in \Delta^h \subseteq \mathscr{X}(E)$ and $\phi'_h(Z) = 0$ for $Z \in \mathscr{X}(E) \setminus \{X \in \Delta^h\}$)

$$F \longmapsto F^h \in \Delta^h \quad \text{with } F^h : p \longmapsto F_p^h := \big(\pi_*|_{\Delta_p^h}\big)^{-1}(F_{\pi(p)}), \quad p \in E, \tag{4.11a}$$

$$\phi \longmapsto \phi_h \quad \text{with } \phi_h := \phi \circ \pi_*|_{\Delta^h} : p \longmapsto \phi_h|_p = \phi|_{\pi(p)} \circ \big(\pi_*|_{\Delta_p^h}\big). \tag{4.11b}$$

The horizontal lift ϕ_h of ϕ can also be defined alternatively via

$$\phi_h(F^h)\big|_p = \phi(F)\big|_{\pi(p)}, \tag{4.12}$$

which equation is tantamount to (4.11b).

Let $\{u^\mu = x^\mu \circ \pi, u^a\}$ be vector bundle coordinate system and let $\{X_I\}$ (resp., $\{\omega_I\}$) be its adapted frame (resp., coframe) constructed from it according to (3.28). If $Y = Y^a e_a$, $\omega = \omega_a e^a$, $F = F^\mu(\partial/\partial x^\mu) \in \mathscr{X}(M)$, and $\phi = \phi_\mu \mathrm{d}x^\mu \in \Lambda^1(M)$, (4.2) and (4.7) imply

$$Y^v = (Y^a \circ \pi)X_a, \qquad \omega_v = (\omega_a \circ \pi)\omega^a, \tag{4.13}$$

while from (4.11) and (3.33), one gets

$$F^h = (F^\mu \circ \pi)X_\mu, \qquad \phi_h = (\phi_\mu \circ \pi)\omega^\mu, \tag{4.14}$$

which agree with (3.15).

4.2. The tangent and cotangent bundle cases. As an example, in the present subsection a connection Δ^h on the tangent bundle $(T(M), \pi_T, M)$ over a manifold M is considered.

A vector field $Y \in \mathscr{X}(M) = \mathrm{Sec}(T(M), \pi_T, M)$ has unique vertical lift $Y^v \in \Delta^v$ (which is independent of Δ^h) and unique *horizontal* lift given by (see (4.5))

$$Y^v := v(Y) \in \Delta^v, \qquad Y^h := \left((\pi_T)_*|_{\Delta^h}\right)^{-1}(Y) \in \Delta^h, \tag{4.15}$$

the last equality meaning that $Y^h_p := ((\pi_T)_*|_{\Delta^h_p})^{-1}(Y_p)$, which is correct as $(\pi_T)_*|_{\Delta^h_p} : \Delta^h_p \to T_{\pi(p)}(M)$ is an isomorphism. Respectively, if ω is 1-form on M, it has vertical lift ω_v (which is independent of Δ^h) and *horizontal* lift ω_h, which is 1-form on Δ^h, defined by (see (4.6))

$$\omega_v(Z) = (\omega \circ v^{-1}(Z)) \circ \pi_T, \qquad \omega_h := \omega \circ (\pi_T)_* = \pi_T^*(\omega). \tag{4.16}$$

The horizontal lift of ω has the properties

$$\omega_h(Y^v) = 0 \quad \text{for } Y \in \mathscr{X}(M), \tag{4.17a}$$

$$\omega_h(Y^h) = (\omega(Y)) \circ \pi_T \quad \text{for } Y \in \mathscr{X}(M), \tag{4.17b}$$

the first of which is equivalent to

$$\omega_h(Z) = 0 \quad \text{for } Z \in \Delta^v, \tag{4.15a$'$}$$

due to Proposition 4.3.

Thus there arises a lift $\mathscr{X}(M) \to \mathscr{X}(T(M))$ such that the *lift* of $Y \in \mathscr{X}(M)$ is $\overline{Y} \in \mathscr{X}(T(M))$ with

$$\overline{Y} := Y^v \oplus Y^h. \tag{4.18}$$

Obviously, this decomposition respects Definition 4.4 and

$$(\pi_T)_*(\overline{Y}) = (\pi_T)_*(Y^h) = Y. \tag{4.19}$$

The dual *lift* $\omega \mapsto \overline{\omega} \in \Lambda^1(T(M))$ of a 1-form $\omega \in \Lambda^1(M)$ is given by

$$\overline{\omega} = \omega_v \oplus \omega_h. \tag{4.20}$$

As a result of (4.6) and (4.17), we have

$$\overline{\omega}(\overline{Y}) = \omega_v(Y^v) + \omega_h(Y^h) = 2(\omega(Y)) \circ \pi_T. \tag{4.21}$$

At last, let us look on the vertical and/or horizontal lifts from the view point of local bases/frames.

In the case of the tangent bundle $(T(M), \pi_T, M)$ (resp., cotangent bundle $(T^*(M), \pi_T^*, M)$) over a manifold M, any coordinate system $\{x^\mu\}$ on an open set $U_M \subseteq M$ induces natural vector bundle coordinates in the bundle space [40, Section 1.25] (see also [32, pages 8, 43]). For the purpose, we put $e_\mu = \partial/\partial x^\mu$, so that $e^\mu = \mathrm{d}x^\mu$, and we get $(\lambda, \mu, \ldots = 1, \ldots, \dim M$ and $a, b = \dim M + 1, \ldots, 2\dim M)$

$$\{u^I\} = \{x^\mu \circ \pi_T, \mathrm{d}x^\nu\}, \quad \text{that is, } u^\mu = x^\mu \circ \pi_T\ u^a = \mathrm{d}x^{a-\dim M} \tag{4.22a}$$

on $\pi_T^{-1}(U_M)$, in the tangent bundle case, and

$$\{u^I\} = \left\{x^\mu \circ \pi_{T^*}, (\cdot)\left(\frac{\partial}{\partial x^\nu}\right)\right\} \quad \text{that is, } u^\mu = x^\mu \circ \pi_{T^*}\ u^{\dim M+\nu} : \xi \longmapsto \xi\left(\frac{\partial}{\partial x^\nu}\right) \tag{4.22b}$$

on $\pi_{T^*}^{-1}(U_M) \ni \xi$, in the cotangent bundle case. In connection with the higher order (co)tangent bundles, it is convenient the vector fibre coordinates to be denoted also as $u_1^\mu := \dot{x}^\mu := \mathrm{d}x^\mu$ in $T(M)$ and by $u_\mu^1(\cdot) = (\cdot)(\partial/\partial x^\mu)$ in $T^*(M)$.

Consider the vector bundle coordinates $\{u^\mu = x^\mu \circ \pi_T, u_1^\gamma = \mathrm{d}x^\gamma\}$ on $\pi_T^{-1}(U_M)$. They induce the frame $\{\partial_\mu = \partial/\partial u^\mu, \partial_\nu^1 = \partial/\partial u_1^\gamma\}$ and the coframe $\{\mathrm{d}u^\mu, \mathrm{d}u_1^\gamma\}$ on $\pi_T^{-1}(U_M)$ and $\pi_{T^*}^{-1}(U_M)$, respectively. According to (3.30), they induce the following adapted frame and its dual coframe:

$$(X_\mu, X_\mu^1) = (\partial_\nu, \partial_\nu^1) \cdot \begin{bmatrix} \delta_\mu^\nu & 0 \\ +\Gamma_\mu^\nu & \delta_\mu^\nu \end{bmatrix} = (\partial_\mu + \Gamma_\mu^\nu \partial_\nu^1, \partial_\mu^1), \tag{4.23a}$$

$$\begin{pmatrix} \omega^\mu \\ \omega_1^\mu \end{pmatrix} = \begin{bmatrix} \delta_\nu^\mu & 0 \\ -\Gamma_\nu^\mu & \delta_\nu^\mu \end{bmatrix} \cdot \begin{pmatrix} \mathrm{d}u^\nu \\ \mathrm{d}u_1^\mu \end{pmatrix} = \begin{pmatrix} \mathrm{d}u^\mu \\ \mathrm{d}u_1^\mu - \Gamma_\nu^\mu \mathrm{d}u^\nu \end{pmatrix}, \tag{4.23b}$$

where, as accepted in the (co)tangent bundle case, a fibre index, like a, is replaced with a base index, like μ, according to $a \mapsto \mu = a - \dim M$, which leads to identification like $\Gamma_\mu^\nu := \Gamma_\mu^{\dim M+\nu}$.

Consider a vector field $Y = Y^\mu(\partial/\partial x^\mu) \in \mathscr{X}(M)$ and 1-form $\eta = \eta_\mu \mathrm{d}x^\mu \in \Lambda^1(M)$. According to (4.2) and (4.7), their vertical lifts are

$$Y^v = (Y^\mu \circ \pi_T) X_\mu^1 \in \Delta^v, \qquad \eta^v = (\eta_\mu \circ \pi_{T^*}) \omega_1^\mu, \tag{4.24a}$$

and similarly, due to (4.14), the horizontal lifts of Y and η are

$$Y^h = (Y^\mu \circ \pi_T)X_\mu \in \Delta^h, \qquad \eta^h = (\eta_\mu \circ \pi_{T^*})\omega^\mu. \tag{4.24b}$$

4.3. Linear connections on vector bundles. The most valued structures in/on vector bundles are the ones which are compatible/consistent with the linear structure of the fibres of these bundles. Since a distribution $\Delta : p \mapsto \Delta_p \subseteq T_p(E)$, $p \in E$, on the bundle space E of a (vector) bundle (E,π,M) cannot be considered as a linear mapping without additional hypotheses, the concept of a linear connection arises from the one of the parallel transport assigned to a connection (see Definition 3.2). (For an alternative approach, see [24, page 42].)

Defintion 4.4. A connection on a vector bundle is called *linear* if its assigned parallel transport is a linear mapping along every path in the base space, that is, if the mapping (3.13) is linear for all paths $\gamma : [\sigma,\tau] \to M$ in the base.

The restriction on a connection to be linear is quite severe and is described locally by the following theorem.

Theorem 4.5 (cf. [30, Section 5.2]). *Let (E,π,M) be a vector bundle, let $\{u^I\}$ be vector bundle coordinate system on an open set $U \subseteq E$, and let Δ^h be a connection on it described in the frame $\{X_I\}$, adapted to $\{u^I\}$, by its 2-index coefficients Γ^a_μ (see (3.27)–(3.29)). The connection Δ^h is linear if and only if, for each $p \in U$,*

$$\Gamma^a_\mu(p) = -\Gamma^a_{b\mu}(\pi(p))u^b(p) = -\big(\big(\Gamma^a_{b\mu} \circ \pi\big) \cdot u^b\big)(p), \tag{4.25}$$

where $\Gamma^a_{b\mu} : \pi(U) \to \mathbb{K}$ are some functions on the set $\pi(U) \subseteq M$ and the minus sign before $\Gamma^a_{b\mu}$ in (4.25) is conventional.

Proof. Take a C^1 path $\gamma : [\sigma,\tau] \to \pi(U)$ and consider the parallel transport equation (3.39a″), namely,

$$\frac{\mathrm{d}\overline{\gamma}^a_p(t)}{\mathrm{d}t} = \Gamma^a_\mu\big(\overline{\gamma}_p(t)\big)\dot{\gamma}^\mu(t), \tag{4.26}$$

where $\overline{\gamma}_p : [\sigma,\tau] \to U$ is the horizontal lift of γ through $p \in \pi^{-1}(\gamma(\sigma))$, $\overline{\gamma}^a := u^a \circ \overline{\gamma}$, and $\dot{\gamma}^\mu(t) = \mathrm{d}(x^\mu \circ \gamma(t))/\mathrm{d}t = \mathrm{d}(u^\mu \circ \overline{\gamma}(t))/\mathrm{d}t$ as $u^\mu = x^\mu \circ \pi$ for some coordinate system $\{x^\mu\}$ on $\pi(U)$.

Sufficiency. If (4.25) holds, (4.26) can be transformed into

$$\frac{\mathrm{d}\overline{\gamma}^a_p(t)}{\mathrm{d}t} = -\Gamma^a_{b\mu}\big(\gamma(t)\big)\overline{\gamma}^b_p(t)\dot{\gamma}^\mu(t), \tag{4.27}$$

which is a system of r linear first-order ordinary differential equations for the r functions $\overline{\gamma}^{n+1}_p,\dots,\overline{\gamma}^{n+r}_p$. According to the general theorems of existence and uniqueness of the solutions of such systems [12], it has a unique solution

$$\overline{\gamma}^a_p(t) = Y^a_b(t)p^b \tag{4.28}$$

satisfying the initial condition $\overline{\gamma}_p^a(\sigma) = u^a(p) =: p^a$, where $Y = [Y_b^a]$ is the fundamental solution of (4.27), that is,

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = -\left[\Gamma_{b\mu}^a(\gamma(t))\dot{\gamma}^\mu(t)\right]_{a,b=n+1}^{n+r} \cdot Y(t), \quad Y(\sigma) = \mathbb{1}_{r\times r} = \left[\delta_b^a\right]. \tag{4.29}$$

The linearity of the mapping P^γ, defined by (3.14), with respect to p follows from (4.28) for $t = \tau$.

Necessity. Suppose (3.13) is linear in p for all paths $\gamma : [\sigma,\tau] \to \pi(U)$. Then $\overline{\gamma}_p(t) := P^{\gamma|[\sigma,t]}(p)$ is the horizontal lift of $\gamma \mid [\sigma,t]$ through p and (cf. (4.28)) $\overline{\gamma}_p^a(t) = A_b^a(\gamma(t))p^b$ for some C^1 functions $A_b^a : \pi(U) \to \mathbb{K}$. The substitution of this equation in (4.26) results into

$$\left.\frac{\partial A_b^a(x)}{\partial x^\mu}\right|_{x=\gamma(t)=\pi(\overline{\gamma}_p(t))} \cdot \dot{\gamma}^\mu(t)p^b = \Gamma_\mu^a(\overline{\gamma}_p(t))\dot{\gamma}^\mu(t). \tag{4.30}$$

Since $\gamma : [\sigma,\tau] \to M$, we get (4.25) from here, for $t = \sigma$, with $\Gamma_{b\mu}^a(x) = -\partial A_b^a(x)/\partial x^\mu$ for $x \in \pi(U)$. □

The functions $\Gamma_{b\mu}^a : \pi(U) \to \mathbb{K}$ will be referred as the *(local) 3-index coefficients* of the linear connection Δ^h in the adapted frame $\{X_I\}$. If there is no risk to confuse them with the 2-index coefficients $\Gamma_\mu^a : U \to \mathbb{K}$, they will be called simply coefficients of Δ^h. Note, the 2-index coefficients of a linear connections are defined on (a subset of) the bundle space E, while the 3-index ones are define on (a subset of) the base space M. Equation (4.27) is simply the *parallel transport equation* for the linear connection considered.

Example 4.6. Since u^a is replaced by $u_1^\mu = \mathrm{d}x^\mu$ in the tangent bundle case (see Section 4.2), the linear connections in $(T(M),\pi_T,M)$ have 2-index coefficients of the form

$$\Gamma_\mu^\nu = -(\Gamma_{\lambda\mu}^\nu \circ \pi_T) \cdot u_1^\lambda = -(\Gamma_{\lambda\mu}^\nu \circ \pi_T) \cdot \mathrm{d}x^\lambda, \tag{4.31}$$

and, consequently, they can be regarded as 1-forms on M.

Consider a linear connection Δ^h on a vector bundle (E,π,M). Let Γ_μ^a and $\Gamma_{b\mu}^a$ be its 2- and 3-index coefficients, respectively, in a frame $\{X_I\}$ adapted to vector bundle coordinates $\{u^I\}$.

Corollary 4.7. *The 3-index coefficients $\Gamma_{b\mu}^a$ of a linear connection Δ^h uniquely define the fibre coefficients of Δ^h in $\{X_I\}$ by*

$${}^\circ\Gamma_{b\mu}^a = \Gamma_{b\mu}^a \circ \pi = \pi^*(\Gamma_{b\mu}^a), \tag{4.32}$$

that is, the fibre coefficients of a linear connection are equal to the 3-index ones lifted by the projection π.

Proof. Since (3.28a) and (4.25) imply

$$[X_\mu, X_b]_- = (\Gamma_{b\mu}^a \circ \pi)X_a, \tag{4.33}$$

(4.32) follows from (3.22a) and (3.23a) or (3.37b) and (4.25). □

As the vector bundle coordinates u^I are, by definition, linear on the fibres of the bundle, the general change of such coordinates is

$$\{u^\mu, u^a\} \longmapsto \{\widetilde{u}^\mu = \widetilde{x}^\mu \circ \pi, \widetilde{u}^a = (B^a_b \circ \pi) \cdot u^b\}, \tag{4.34}$$

with $B = [B^a_b]$ being a nondegenerate matrix-valued function on $\pi(U)$. The change (4.34) entails the following transformation of the corresponding adapted frames:

$$\{X_\mu, X_a\} \longmapsto \{\widetilde{X}_\mu = (B^\nu_\mu \circ \pi) \cdot X_\nu, \widetilde{X}_a = (B^b_a \circ \pi) \cdot X_b\}, \tag{4.35}$$

where $[B^\nu_\mu] = [\partial x^\nu / \partial \widetilde{x}^\mu]$ is a nondegenerate matrix-valued function on the intersection of the domains of $\{x^\mu\}$ and $\{\widetilde{x}^\mu\}$. (In (4.35) we have used that $\partial u^\nu / \partial \widetilde{u}^\mu|_p = \partial(x^\nu \circ \pi)/\partial(\widetilde{x}^\mu \circ \pi)$ $|_p = \partial x^\nu / \partial \widetilde{x}^\mu|_{\pi(p)}$.)

Proposition 4.8. *The change (4.34) implies the following transformations of the 3-index coefficients of the linear connection:*

$$\Gamma^a_{b\mu} \longmapsto \widetilde{\Gamma}^a_{b\mu} = B^\nu_\mu \left(B^a_d \Gamma^d_{c\nu} - \frac{\partial B^a_c}{\partial x^\nu} \right) (B^{-1})^c_b. \tag{4.36}$$

Proof. Apply (4.35), (3.32), and (4.25). Alternatively, the same transformation law follows also from (3.24a) and (4.32). □

If we introduce the matrix-valued functions $\Gamma_\mu := [\Gamma^a_{b\mu}]$ and $\widetilde{\Gamma}_\mu := [\widetilde{\Gamma}^a_{b\mu}]$ on M, we can rewrite (4.36) as

$$\Gamma_\mu \longmapsto \widetilde{\Gamma}_\mu = B^\nu_\mu \left(B \cdot \Gamma_\nu - \frac{\partial B}{\partial x^\nu} \right) \cdot B^{-1} = B^\nu_\mu B \cdot \left(\Gamma_\nu \cdot B^{-1} + \frac{\partial B^{-1}}{\partial x^\nu} \right). \tag{4.37}$$

This relation corresponds to (3.25) with $[A^a_b] = B^{-1} \circ \pi$ (see also (4.32)) as the frame $\{e_a : M \to E\}$, relative to which the vector fibre coordinate system $\{u^a\}$ is defined ($E \ni p \mapsto u^a(p)$ with $p = u^a(p) e_a(\pi(p))$), transforms via the matrix inverse to $B \circ \pi$.

Let E be a C^2 manifold and Δ^h a C^1 connection on (E, π, M). Substituting (4.25) into (3.37a), we get the fibre components of the curvature of a linear connection as

$$R^a_{\mu\nu} = -(R^a_{b\mu\nu} \circ \pi) \cdot u^b, \tag{4.38}$$

where

$$R^a_{b\mu\nu} := \frac{\partial}{\partial x^\mu}(\Gamma^a_{b\nu}) - \frac{\partial}{\partial x^\nu}(\Gamma^a_{b\mu}) - \Gamma^c_{b\mu}\Gamma^a_{c\nu} + \Gamma^c_{b\nu}\Gamma^a_{c\mu}, \tag{4.39}$$

or in a matrix form

$$R_{\mu\nu} := [R^a_{b\mu\nu}] = \frac{\partial \Gamma_\nu}{\partial x^\mu} - \frac{\partial \Gamma_\mu}{\partial x^\nu} - \Gamma_\nu \cdot \Gamma_\mu + \Gamma_\mu \cdot \Gamma_\nu, \tag{4.35$'$}$$

are the *components of the curvature operator* (see (4.58)). As a result of (3.22b) and (4.38), the transformation (4.34) entails the change

$$R^a_{b\mu\nu} \longmapsto \widetilde{R}^a_{b\mu\nu} = B^\lambda_\mu B^\varrho_\nu (B^{-1})^a_c B^d_b R^c_{d\lambda\varrho}, \tag{4.40}$$

or in a matrix form

$$R_{\mu\nu} \longmapsto \widetilde{R}_{\mu\nu} = B^{\lambda}_{\mu} B^{\varrho}_{\nu} B^{-1} \cdot R_{\lambda\varrho} \cdot B, \tag{4.36$'$}$$

which corresponds to (3.24b) with $A = B^{-1} \circ \pi$ (see also (4.38)).

4.4. Covariant derivatives in vector bundles. A possibility for introduction of differentiation in vector bundles, endowed with connection, comes from the vector space structure of their fibres. This operation can be defined in many independent ways, leading to identical results. In one of them is involved the parallel transport induced by the connection: the idea is the values of sections to be parallel transported (along paths in the base) into a single fibre (over the paths), where one can work with the "transported" sections as with functions with values in a vector space. Other method uses the existence of natural vertical lifts of sections of the bundle and horizontal lifts of the vector fields on the base space; since the both lifts are vector fields on the bundle space, their commutator (or Lie derivative relative to each other) is well defined and can be used as a prototype of some sort of differentiation. We will realize below the second method mentioned, which seems to be first introduced in a rudimentary form in [30, page 31]. (In [30, page 31] it is proved that, for $F = \partial/\partial x^{\mu}$ and in our notation, the ath component of the right-hand sides of (4.50) and of (4.51) coincide in a frame $\{E_a\}$ in E.) (An equivalent alternative approach is realized in [29, Sections 2.49–2.52].) The first way, as well as the axiomatic approach, for introduction of covariant derivatives will be obtained as theorems in what follows.

Let (E,π,M) be a vector bundle on which a *linear* connection Δ^h is defined. Suppose $\{E_a\}$ is a frame in E to which vector fibre coordinates u^a are associated and $\{u^I\}$ is the corresponding vector bundle coordinate system. The frame adapted to $\{u^I\}$ will be denoted by $\{X_I\}$, and $\{\omega^I\}$ will be its dual coframe, both defined by (3.28) through the (2-index) coefficients Γ^a_{μ} of Δ^h.

Let $\hat{Z} = \hat{Z}^a X_a \in \Delta^v$ and $\overline{Z} = \overline{Z}^{\mu} X_{\mu} \in \Delta^h$ be, respectively, vertical and horizontal vector fields on E. Define a mapping $\hat{\nabla} : \Delta^v \oplus \Delta^h = T(E) \to \mathscr{X}(E)$ such that (the idea of the construction (4.41) is to drag the vertical vector field $\hat{Z}$ along the horizontal one $\overline{Z}$, which will give a vector field in $\mathscr{X}(E)$, and then to project the result onto the *vertical* distribution Δ^v by means of the invariant projection operator $\Pi = X_a \otimes \omega^a : \mathscr{X}(E) \to \mathscr{X}(E)$; evidently $\Pi^2 = \Pi \circ \Pi = \Pi$ and Π is the unit (identity) tensor in the tensor product of vector fields and 1-forms on E)

$$\hat{\nabla} : (\hat{Z},\overline{Z}) \longmapsto \hat{\nabla}_{\overline{Z}}(\hat{Z}) := \Pi(\mathscr{L}_{\overline{Z}}\hat{Z}) \in \mathscr{X}(E), \tag{4.41}$$

where the (1,1) tensor field

$$\Pi := \sum_a X_a \otimes \omega^a \tag{4.42}$$

is considered as a operator on the set of vector fields on E. Since (see (2.1b) and (2.7))

$$\mathscr{L}_{\overline{Z}}\hat{Z} = \overline{Z}(\hat{Z}^a)X_a + \overline{Z}^{\mu}\hat{Z}^a[X_{\mu},X_a]_{\_} \tag{4.43}$$

and $\omega^a(X_\mu) = \delta^a_\mu = 0$, from (3.36), (3.37b), and (4.41), we obtain

$$\widehat{\nabla}_{\overline{Z}}\widehat{Z} = \overline{Z}^\mu \{X_\mu(\widehat{Z}^a) - \widehat{Z}^b \partial_b(\Gamma^a_\mu)\} X_a, \tag{4.44}$$

from where one can prove, via direct calculation, the independence of $\widehat{\nabla}_{\overline{Z}}\widehat{Z}$ of the particular (co)frame used. For any particular point $p \in E$, the value of the vector field (4.44) at p is a vertical vector, $(\widehat{\nabla}_{\overline{Z}}\widehat{Z})|_p \in \Delta^v_p$, but generally $\widehat{\nabla}_{\overline{Z}}\widehat{Z}$ is not a vertical vector field. The reason is that a vertical vector field on E is a mapping $V : p \mapsto V_p \in \Delta^v_p := T_p(\pi^{-1}(\pi(p))) := T_{\imath(p)}(\pi^{-1}(\pi(p))) = (\pi_*|_p)^{-1}(0_{\pi(p)})$, with $\imath : \pi^{-1}(p) \to E$ being the inclusion mapping and $0_{\pi(p)} \in T_{\pi(p)}(M)$ being the zero vector, due to which V_p, and hence its components, must depend only on $\pi(p) \in M$. Therefore, we have

$$\widehat{\nabla}_{\overline{Z}}\widehat{Z} \in \Delta^v \iff \partial_b(\Gamma^a_\mu) = -\Gamma^a_{b\mu} \circ \pi \iff \Gamma^a_\mu = -(\Gamma^a_{b\mu} \circ \pi) \cdot u^b + G^a_\mu \circ \pi, \tag{4.45}$$

for some functions $\Gamma^a_{b\mu}, G^a_\mu : M \to \mathbb{K}$. Thus $\widehat{\nabla}_{\overline{Z}}\widehat{Z}$ is a vertical vector field if and only if the 2-index coefficients Γ^a_μ in $\{X_I\}$ of the connection Δ^h are of the form

$$\Gamma^a_\mu = -(\Gamma^a_{b\mu} \circ \pi) \cdot u^b + G^a_\mu \circ \pi. \tag{4.46}$$

This equality selects the set of *affine connections* among all connections (see Section 4.5); (usually the affine connections are defined on affine bundles [19, 24]) in particular, the linear connections for which $G^a_\mu = 0$ and $\Gamma^a_{b\mu}$ are their 3-index coefficients (see (4.25) are of this type). For connections with 2-index coefficients (4.46), (4.44) reduces to

$$\widehat{\nabla}_{\overline{Z}}\widehat{Z} = \overline{Z}^\mu \{X_\mu(\widehat{Z}^a) + \widehat{Z}^b(\Gamma^a_{b\mu} \circ \pi)\} X_a \in \Delta^v. \tag{4.47}$$

Now the idea of introduction of a covariant derivative of a section $Y \in \mathrm{Sec}(E,\pi,M)$ along a vector field $F \in \mathscr{X}(M)$ is to "lower" the operator $\widehat{\nabla}$ from $T(E)$ to $T(M)$.

Defintion 4.9. A *covariant derivative* or *covariant derivative operator*, associated to a linear (or affine) connection Δ^h on a vector bundle (E,π,M), is a mapping

$$\begin{aligned} \nabla &: \mathscr{X}(M) \times \mathrm{Sec}^1(E,\pi,M) \longrightarrow \mathrm{Sec}^0(E,\pi,M), \\ \nabla &: (F,Y) \longmapsto \nabla_F Y \end{aligned} \tag{4.48}$$

such that, for $F \in \mathscr{X}(M)$ and $Y \in \mathrm{Sec}^1(E,\pi,M)$, $\nabla_F Y$ is the unique section of (E,π,M) whose vertical lift is $\widehat{\nabla}_{F^h} Y^v$, with $\widehat{\nabla}$ defined by (4.41) (or (4.47)), namely,

$$(\nabla_F Y)^v := \widehat{\nabla}_{F^h} Y^v \tag{4.49}$$

or

$$\nabla_F Y = v^{-1} \circ \widehat{\nabla}_{(\pi_*|_{\Delta^h})^{-1}(F)}(v(Y)) = (v^{-1} \circ \widehat{\nabla}_{(\pi_*|_{\Delta^h})^{-1}(F)} \circ v)(Y), \tag{4.50}$$

where $F^h \in \Delta^h$ and $Y^v \in \Delta^v$ are, respectively, the horizontal and vertical lifts of F and Y.

Remark 4.10. Definition 4.9 and the rest of this subsection are valid also for affine connections for which (4.46) holds, not only for the linear ones. For some details, see Section 4.5.

PROPOSITION 4.11. *Let $\{E_a\}$ be a frame in E and $\{x^\mu\}$ local coordinate system on M. If $Y = Y^a E_a \in \mathrm{Sec}^1(E,\pi,M)$ and $F = F^\mu(\partial/\partial x^\mu) \in \mathfrak{X}(M)$, then the following explicit local expression holds:*

$$\nabla_F Y = F^\mu \left(\frac{\partial Y^a}{\partial x^\mu} + \Gamma^a_{b\mu} Y^b \right) E_a. \tag{4.51}$$

Proof. Apply (4.49), (4.13), (4.14), (4.47), and (4.2). □

PROPOSITION 4.12. *Let Δ^h be a linear connection on (E,π,M) and let P be generated its parallel transport. Let $x \in M$, $\gamma : [\sigma,\tau] \to M$, $\gamma(t_0) = x$ for some $t_0 \in [\sigma,\tau]$, and $\dot\gamma(t_0) = F_x$, that is, γ being the integral path of $F \in \mathfrak{X}(M)$ through x. Then*

$$(\nabla_F Y)\big|_x = \lim_{s\to t_0} \frac{P^\gamma_{s\to t_0}(Y_{\gamma(s)}) - Y_{\gamma(t_0)}}{s - t_0} = \lim_{\varepsilon\to 0} \frac{P^\gamma_{t_0+\varepsilon\to t_0}(Y_{\gamma(t_0+\varepsilon)}) - Y_{\gamma(t_0)}}{\varepsilon}, \tag{4.52}$$

where $Y \in \mathrm{Sec}^1(E,\pi,M)$ and

$$P^\gamma_{s\longrightarrow t} := \begin{cases} P^{\gamma|[s,t]} & \text{for } s \le t, \\ \left(P^{\gamma|[t,s]}\right)^{-1} & \text{for } s \ge t. \end{cases} \tag{4.53}$$

Proof. Use Definition 3.2 and apply the parallel transport equation (4.27) with initial value $\overline{\gamma}_{Y_{\gamma(s)}}(s) = Y_{\gamma(s)}$ at the point $t = s \in [\sigma,\tau]$. □

By Proposition 4.12, (4.52) can be used as an equivalent definition of a covariant derivative associated with a linear connection.

PROPOSITION 4.13. *Let $F,G \in \mathfrak{X}(M)$, $Y,Z \in \mathrm{Sec}^1(E,\pi,M)$, and let $f : M \to \mathbb{K}$ be a C^1 function. Then*

$$\nabla_{F+G} Y = \nabla_F Y + \nabla_G Y, \tag{4.54a}$$

$$\nabla_{fF} Y = f \nabla_F Y, \tag{4.54b}$$

$$\nabla_F (Y + Z) = \nabla_F Y + \nabla_F Z, \tag{4.54c}$$

$$\nabla_F (fY) = F(f) \cdot Y + f \cdot \nabla_F Y. \tag{4.54d}$$

Proof. Apply (4.51). □

PROPOSITION 4.14. *If a mapping (4.48) satisfies (4.54), there exists a unique linear connection Δ^h, the assigned to which covariant derivative is exactly ∇.*

Proof. Define local functions $\Gamma^a_{b\mu}$ on M, called *components* of ∇, by the decomposition

$$\nabla_{\partial/\partial x^\mu} E_b =: \Gamma^a_{b\mu} E_a. \tag{4.55}$$

A simple verification proves that they transform according to (4.36) and hence the quantities (4.25) transform by (3.32). Proposition 3.9 ensures the existence of a unique linear connection whose 2-index (3-index) coefficients are Γ^a_μ ($\Gamma^a_{b\mu}$). Thus the covariant derivative of $Y \in \mathrm{Sec}(E,\pi,M)$ relative to $F \in \mathscr{X}(M)$ is given by the right-hand side of (4.51). On another hand, (4.54) entail (4.51), with $\Gamma^a_{b\mu}$ defined by (4.55), so that ∇ is exactly the covariant derivative operator assigned to the connection with 3-index coefficients $\Gamma^a_{b\mu}$. □

Consequently, (4.54) and (4.55) provide a third equivalent definition of a covariant derivative (covariant derivative operator). Moreover, since Proposition 4.14 establishes a bijective correspondence between linear connections and operators (4.48) satisfying (4.54), quite often such operators are called linear connections. (See also [29, Sections 2.15 and 2.52].) As it is clear from the proof of Proposition 4.14, the bijection between linear connections and covariant derivative operators is locally given by the coincidence of their (3-index) coefficients and components, respectively.

Exercise 4.15. A C^1 section $\omega = \omega_a E^a$ of the bundle dual to (E,π,M) can be differentiated covariantly similarly as the sections of (E,π,M). Show that the corresponding operator, say ∇^*, can equivalently be defined by (the "Leibnitz rule")

$$(\nabla^*_F \omega)(Y) = F(\omega(Y)) - \omega(\nabla_F Y) \tag{4.56}$$

and locally the equation

$$\nabla^*_F \omega = F^\mu \left(\frac{\partial \omega_a}{\partial x^\mu} - \Gamma^b_{a\mu}\omega_b \right) E^a, \tag{4.57}$$

is valid.

Equipped with the covariant derivative ∇ assigned to a C^1 linear connection Δ^h, we define the *curvature operator* of Δ^h (or ∇) by

$$\begin{aligned} R &: \mathscr{X}(M) \times \mathscr{X}(M) \longrightarrow \mathrm{End}\,(\mathrm{Sec}(E,\pi,M)), \\ R &: (F,G) \longmapsto R(F,G) := \nabla_F \circ \nabla_G - \nabla_G \circ \nabla_F - \nabla_{[F,G]_-}, \end{aligned} \tag{4.58}$$

with $\mathrm{End}(\cdots)$ denoting the set of endomorphisms of $(\cdots)$.

Exercise 4.16. Prove that locally

$$(R(F,G))(Y) = (R^a_{b\mu\nu} Y^b F^\mu G^\nu) E_a, \tag{4.59}$$

where the functions $R^a_{b\mu\nu} : M \to \mathbb{K}$, called the *components* of the curvature operator R in the pair of frames $(\{\partial/\partial x^\mu\}, \{E_a\})$, are defined by

$$R\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)(E_b) =: R^a_{b\mu\nu} E_a \tag{4.60}$$

and are explicitly expressed through the coefficients of ∇(= 3-index coefficients of Δ^h) via (4.39).

A linear connection or covariant derivative operator is called *flat* or *curvature free* if

$$R = 0 (\iff R^a_{b\mu\nu} = 0). \tag{4.61}$$

Obviously, the flatness of Δ^h or ∇ is a necessary and sufficient condition for the (local) integrability of the horizontal distribution $\Delta^h : p \mapsto \Delta^h_p \subseteq T_p(E)$, $p \in E$ (see (3.23b) and (4.38)).

THEOREM 4.17. *Let Y be a C^1 section of a vector bundle (E,π,M) endowed with a linear connection Δ^h. The following three conditions are equivalent:*

(i) *Y is covariantly constant, namely, if $F \in \mathcal{X}(M)$, then*

$$\nabla_F Y = 0; \tag{4.62}$$

(ii) *Y is a solution of Δ^h, that is,*

$$\operatorname{Im} Y_* \subset \Delta^h \left(\iff Y_*|_x(T_x(M)) \subseteq \Delta^h_{Y_x} \quad \text{for } x \in M \right); \tag{4.63}$$

(iii) *Y is parallelly transported along any path $\gamma : [\sigma,\tau] \to M$,*

$$P^\gamma\left(Y_{\gamma(\sigma)}\right) = Y_{\gamma(\tau)}. \tag{4.64}$$

Proof. Since $Y = u^a(Y)e_a$, $\pi \circ Y = \mathrm{id}_M$, and $\omega^a = \mathrm{d}u^a - \Gamma^a_\mu \mathrm{d}u^\mu$, we have for $x \in M$,

$$\begin{aligned} \omega^a \circ Y\left(\frac{\partial}{\partial x^\mu}\right) &= \omega^a\left(\frac{\partial(u^\nu \circ Y)}{\partial x^\mu}\bigg|_x \frac{\partial}{\partial u^\nu}\bigg|_{Y_x} + \frac{\partial(u^a \circ Y)}{\partial x^\mu}\bigg|_x \frac{\partial}{\partial u^a}\bigg|_{Y_x}\right) \\ &= -\frac{\partial(x^\nu \circ \pi \circ Y)}{\partial x^\mu}\bigg|_x \Gamma^a_\nu(Y_x) + \frac{\partial Y^a}{\partial x^\mu}\bigg|_x = \left(\frac{\partial Y^a}{\partial x^\mu} - \Gamma^a_\mu \circ Y\right)(x). \end{aligned} \tag{4.65}$$

The equivalence of (i) and (ii) follows from here, (4.25), (4.51), and that Δ^h annihilates the 1-forms ω^a, $\omega^a(Z) = 0 \iff Z \in \Delta^h$.

If we rewrite the parallel transport equation (4.27) as (see (4.51))

$$\left(\nabla_{\dot\gamma(t)}\overline{\gamma}\right)\big|_{\gamma(t)} = 0, \tag{4.66}$$

the equivalence of (i) and (iii) follows from Definition 3.2 of a parallel transport and the arbitrariness of γ in (4.66). □

Exercise 4.18. Formulate and prove a theorem dual to Theorem 4.17; for example, a section $\varphi = \varphi_a u^a$ of the bundle dual to (E,π,M) is a first integral of Δ^h, that is, $\operatorname{Ker}\varphi_* \supseteq \Delta^h$ ($\iff \varphi_*|_p(\Delta^h_p) = 0_{\varphi(p)} \in T_{\varphi(p)}(\mathbb{K})$ for $p \in E$), if and only if

$$\nabla^* \varphi = 0. \tag{4.67}$$

PROPOSITION 4.19 (cf. [30, page 32]). *Let a linear connection Δ^h on a vector bundle be given and let $\Gamma^a_{b\mu}$ be its (3-index) coefficients. The following conditions are (locally) equivalent:*

(a) *Δ^h is integrable;*

(b) *Δ^h is flat;*

(c) *there exists a solution of the system of partial differential equations*

$$\frac{\partial U^a}{\partial x^\mu} + \Gamma^a_{b\mu} U^b = 0 \tag{4.68}$$

relative to U^a *and the solution of (4.68) satisfying* $U^a|_{x=x_0} = U_0^a$ *is* $U^a = B_b^a U_0^b$, *where* $B = [B_b^a]$ *is the fundamental solution of (4.68), namely,*

$$\frac{\partial B_b^a}{\partial x^\mu} + \Gamma_{c\mu}^a B_b^c = 0, \qquad B_b^a|_{x=x_0} = \delta_b^a; \tag{4.69}$$

(d) *there is an integrating matrix* B^{-1} *for the 1-forms* ω^a, *that is,* $(B^{-1}\circ\pi)_b^a\omega^b = df^a$, *where the functions* $f^a : E \to \mathbb{K}$ *are first integrals of* Δ^h, *that is,* $\operatorname{Ker} df^a \supset \Delta^h$;

(e) *the coefficients of* Δ^h *have the form*

$$\Gamma_\mu := [\Gamma_{b\mu}^a] = B\cdot\frac{\partial B^{-1}}{\partial x^\mu} = -\frac{\partial B}{\partial x^\mu}\cdot B^{-1} \tag{4.70}$$

for some matrix-valued function B *on* M.

Proof. (a)⇔(b). See (4.61) and the comment after it.

(c)⇔(e). The matrix form of the equation in (4.69), that is,

$$\frac{\partial B}{\partial x^\mu} + \Gamma_\mu\cdot B = 0, \tag{4.63$'$}$$

is tantamount to (4.70).

(b)⇔(c). The flatness of Δ^h, that is, $R_{\mu\nu} = 0$ (see (4.35′)), is the integrability condition for (4.63′) as an equation relative to B—see [17, Lemma 2.1].

(c)⇔(d). Since (3.36) and the first equality in (2.1c) entail

$$\mathcal{L}_{X_\mu}(\varphi_a\omega^a) = -\varphi_a R_{\mu\nu}^a\omega^\nu + \{X_\mu(\varphi_a) - {}^\circ\Gamma_{a\mu}^b\varphi_b\}\,\omega^a, \tag{4.71}$$

we have (see also (4.32)) for a flat linear connection

$$\begin{aligned}\mathcal{L}_{X_\mu}((B^{-1})_b^a\omega^b) &= \left\{\left(\frac{\partial B^{-1}}{\partial x^\mu} - B^{-1}\cdot\Gamma_\mu\right)\circ\pi\right\}_b^a\omega^b \\ &= \left\{\left[-B^{-1}\cdot\left(\frac{\partial B}{\partial x^\mu} + \Gamma_\mu\cdot B\right)B^{-1}\right]\circ\pi\right\}_b^a\omega^b.\end{aligned} \tag{4.72}$$

Thus (4.69), which entails (c), is equivalent to $\mathcal{L}_{X_\mu}((B^{-1}\circ\pi)_b^a\,\omega^b) = 0$, which is equivalent to $\mathrm{d}((B^{-1}\circ\pi)_b^a\,\omega^b) = 0$, due to $\omega^a(X_\mu) = \delta_\mu^a = 0$ and the second equality in (2.1a) (applied, e.g., for $Y = X_\nu$). Now the Poincaré's lemma (see [27, Section 6.3] or [8, pages 21, 106]) tells us that locally (on a contractible region in E) there are functions f^a on E such that the last equality is tantamount to $\mathrm{d}f^a = (B^{-1}\circ\pi)_b^a\,\omega^b$.

It remains to be proved that $f^a : E \to \mathbb{K}$ are first integrals of Δ^h, that is, $\operatorname{Ker}\mathrm{d}f^a \supset \Delta^h$ which means $(f^a)_*|_p(\Delta_p^h) = 0$, $p\in E$, or $(f^a)_*|_p(X_\mu) = 0$ as Δ^h is spanned by $\{X_\mu\}$. Using the global chart $(\mathbb{K}, \mathrm{id}_{\mathbb{K}})$ on $\mathbb{K}$, which induces the one-vector frame $\{\partial/\partial r\}$ for $r\in\mathbb{K}$ on

$\mathbb{K}$, we have (see (3.28))

$$\begin{aligned}(f^a)_*|_p(X_\mu) &= (f^a)_*|_p\Big(\frac{\partial}{\partial u^\mu}+\Gamma^b_\mu\frac{\partial}{\partial u^b}\Big)\Big|_p = \Big(\frac{\partial f^a}{\partial u^\mu}\Big|_p+\Gamma^b_\mu(p)\frac{\partial f^a}{\partial u^b}\Big|_p\Big)\frac{\mathrm{d}}{\mathrm{d}r}\Big|_{f^a(p)}\\ &\equiv 0 \quad \text{as } \mathrm{d}f^a=(B^{-1}\circ\pi)^a_b\,\omega^a=(B^{-1}\circ\pi)^a_b(\mathrm{d}u^a-\Gamma^b_\mu\circ\mathrm{d}u^\mu)\\ &\equiv \frac{\partial f^a}{\partial u^b}\mathrm{d}u^b+\frac{\partial f^a}{\partial u^\mu}\mathrm{d}u^\mu=\mathrm{d}f^a.\end{aligned} \tag{4.73}$$

□

4.5. Affine connections. In Section 4.4, we met a class of connections on a vector bundle whose local 2-index coefficients have the form (see (4.46))

$$\Gamma^a_\mu = -(\Gamma^a_{b\mu}\circ\pi)\cdot u^b + G^a_\mu\circ\pi \tag{4.74}$$

in the frame $\{X_I\}$ adapted to a vector bundle coordinate system $\{u^I\}$. From $\partial_b\Gamma^a_\mu=-\Gamma^a_{b\mu}$ and (3.32), one derives that the functions $\Gamma^a_{b\mu}$ in (4.74) transform according to (4.36), namely,

$$\Gamma^a_{b\mu}\longmapsto\widetilde{\Gamma}^a_{b\mu}=B^\nu_\mu\Big(B^a_d\Gamma^d_{c\nu}-\frac{\partial B^a_c}{\partial x^\nu}\Big)(B^{-1})^c_b, \tag{4.75}$$

when the vector bundle coordinates or adapted frames undergo the change (4.34) or (4.35), respectively. Thus, combining (3.32), (4.75), and (4.74), we see that (4.34) or (4.35) implies the transition

$$G^a_\mu\longmapsto\widetilde{G}^a_\mu=B^a_bG^b_\nu B^\nu_\mu. \tag{4.76}$$

Consequently, the functions $\Gamma^a_{b\mu}$ in (4.74) are 3-index coefficients of a linear connection, while G^a_μ in it are the components of a linear mapping $G:\mathscr{X}(M)\to\operatorname{End}(\operatorname{Sec}((E,\pi,M)^*))$ such that $G:F\mapsto G(F):\omega\mapsto(G(F))(\omega)$, for $F\in\mathscr{X}(M)$ and a section ω of the bundle $(E,\pi,M)^*$ dual to (E,π,M), and $(G(\partial/\partial x^\mu))(E^a)=G^a_\mu$. The invariant description of the connections with local 2-index coefficients of the type (4.74) is as follows.

Defintion 4.20. A connection on a vector bundle is termed *affine connection* if assigned its parallel transport P is an affine mapping along all paths $\gamma:[\sigma,\tau]\to M$ in the base space, that is,

$$P^\gamma(\rho X)=\rho P^\gamma(X)+(1-\rho)P^\gamma(\mathbf{0}), \tag{4.77a}$$

$$P^\gamma(X+Y)=P^\gamma(X)+P^\gamma(Y)-P^\gamma(\mathbf{0}), \tag{4.77b}$$

where $\rho\in\mathbb{K}$, $X,Y\in\pi^{-1}(\gamma(\sigma))$, and $\mathbf{0}$ is the zero vector in the fibre $\pi^{-1}(\gamma(\sigma))$, which is a $\mathbb{K}$-vector space.

Theorem 4.21. *Let (E,π,M) be a vector bundle, let $\{u^I\}$ be vector bundle coordinate system over an open set $U\subseteq E$, and let Δ^h be a connection on it with 2-index coefficients Γ^a_μ in the frame $\{X_I\}$ adapted to $\{u^I\}$. The connection Δ^h is an affine connection if and only if (4.74) holds for some functions $\Gamma^a_{b\mu},G^a_\mu:\pi(U)\to\mathbb{K}$.*

Proof (cf. the proof of Theorem 4.5). Take a C^1 path $\gamma : [\sigma,\tau] \to \pi(U)$ and consider the parallel transport equation (3.39a″), namely,

$$\frac{\mathrm{d}\overline{\gamma}_p^a(t)}{\mathrm{d}t} = \Gamma_\mu^a(\overline{\gamma}_p(t))\,\dot{\gamma}^\mu(t), \tag{4.78}$$

where $\overline{\gamma}_p : [\sigma,\tau] \to U$ is the horizontal lift of γ through $p \in \pi^{-1}(\gamma(\sigma))$, $\overline{\gamma}^a := u^a \circ \overline{\gamma}$, and $\dot{\gamma}^\mu(t) = \mathrm{d}(x^\mu \circ \gamma(t))/\mathrm{d}t = \mathrm{d}(u^\mu \circ \overline{\gamma}(t))/\mathrm{d}t$ as $u^\mu = x^\mu \circ \pi$ for some coordinates $\{x^\mu\}$ on $\pi(U)$.

Sufficiency. If (4.74) holds, (4.78) can be transformed into

$$\frac{\mathrm{d}\overline{\gamma}_p^a(t)}{\mathrm{d}t} = -\Gamma_{b\mu}^a(\gamma(t))\overline{\gamma}_p^b(t)\dot{\gamma}^\mu(t) + G_\mu^a(\gamma(t))\,\dot{\gamma}^\mu(t), \tag{4.79}$$

which is a system of r linear inhomogeneous first-order ordinary differential equations for the r functions $\overline{\gamma}_p^{n+1},\dots,\overline{\gamma}_p^{n+r}$. According to the general theorems of existence and uniqueness of the solutions of such systems [12], it has a unique solution

$$\overline{\gamma}_p^a(t) = Y_b^a(t)p^b + y^a(t), \tag{4.80}$$

satisfying the initial condition $\overline{\gamma}_p^a(\sigma) = u^a(p) =: p^a$, where $Y = [Y_b^a]$ is the fundamental solution of (4.27) (see (4.29)) and $y^a(t)$ is the solution of (4.79) with $y^a(t)$ for $\overline{\gamma}_p^a(t)$ satisfying the initial condition $y^a(\sigma) = 0$. The affinity of (3.13) in p, that is, (4.77), follows from (4.80) for $t = \tau$.

Necessity. Suppose (3.13) is affine in p for all paths $\gamma : [\sigma,\tau] \to \pi(U)$. Then $\overline{\gamma}_p(t) := P^{\gamma|[\sigma,t]}(p)$ is the horizontal lift of $\gamma \mid [\sigma,t]$ through p and (cf. (4.80)) $\overline{\gamma}_p^a(t) = A_b^a(\gamma(t))p^b + A^a(\gamma(t))$ for some C^1 functions $A_b^a, A^a : \pi(U) \to \mathbb{K}$. The substitution of this equation in (4.78) results into

$$\frac{\partial A_b^a(x)}{\partial x^\mu}\bigg|_{x=\gamma(t)=\pi(\overline{\gamma}_p(t))} \cdot \dot{\gamma}^\mu(t)p^b + \frac{\partial A^a(x)}{\partial x^\mu}\bigg|_{x=\gamma(t)=\pi(\overline{\gamma}_p(t))} \cdot \dot{\gamma}^\mu(t). = \Gamma_\mu^a(\overline{\gamma}_p(t))\dot{\gamma}^\mu(t). \tag{4.81}$$

Since $\gamma : [\sigma,\tau] \to M$, we get (4.74) from here, for $t = \sigma$, with $\Gamma_{b\mu}^a(x) = -\partial A_b^a(x)/\partial x^\mu$ and $G_\mu^a(x) = \partial A^a(x)/\partial x^\mu$ for $x \in \pi(U)$. □

PROPOSITION 4.22. *There is a bijective mapping α between the set of affine connections and the set of pairs (∇, G) of a linear connection ∇ and a linear mapping $G : \mathscr{X}(M) \to$* $\mathrm{End}(\mathrm{Sec}((E,\pi,M)^*))$.

Proof. If ${}^A\Delta^h$ is an affine connection with 2-index coefficients given by (4.74) (see Theorem 4.21), then (see the discussion after (4.74)) to it corresponds the pair $\alpha({}^A\Delta^h) := ({}^L\Delta^h, G)$ of a linear connection, with 3-index coefficients $\Gamma_{b\mu}^a$ and linear mapping $G : \mathscr{X}(M) \to \mathrm{End}(\mathrm{Sec}((E,\pi,M)^*))$, with components G_μ^a. Conversely, to a pair $({}^L\Delta^h, G)$, locally described via the 3-index coefficients $\Gamma_{b\mu}^a$ of ${}^L\Delta^h$ and components G_μ^a of G, there corresponds an affine connection ${}^A\Delta^h = \alpha^{-1}({}^L\Delta^h, G)$ with 2-index coefficients given by (4.74). □

In Section 4.4, it was demonstrated that covariant derivatives can be introduced for affine connections, not only for linear ones.

PROPOSITION 4.23. *The covariant derivative for an affine connection* ${}^{A}\Delta^h$ *coincides with the one for the linear connection* ${}^{L}\Delta^h$ *given via* $\alpha({}^{A}\Delta^h) = ({}^{L}\Delta^h, G)$ *with* α *defined in the proof of Proposition 4.22.*

Proof. Apply (4.44)–(4.51). □

If a linear connection ${}^{L}\Delta^h$ and an affine one ${}^{A}\Delta^h$ are connected by $\alpha({}^{A}\Delta^h) = ({}^{L}\Delta^h, G)$ for some G, then some of their characteristics coincide; for example, such are their fibre coefficients (see (3.37b), (4.74), and (4.25)) and all quantities expressed via their corresponding (identical) covariant derivatives. However, quantities, containing (depending on) partial derivatives relative to the basic coordinates u^μ, are generally different for those connections. For instance, if ${}^{A}R^a_{\mu\nu}$ and ${}^{L}R^a_{\mu\nu}$ are the fibre components of the curvatures of ${}^{A}\Delta^h$ and ${}^{L}\Delta^h$, respectively, then, by (3.37a) and (4.74), we have

$$\begin{aligned} {}^{A}R^a_{\mu\nu} &= -\left({}^{L}R^a_{b\mu\nu} \circ \pi\right) \cdot u^b - T^a_{\mu\nu} \circ \pi, \\ {}^{L}R^a_{\mu\nu} &= -\left({}^{L}R^a_{b\mu\nu} \circ \pi\right) \cdot u^b, \end{aligned} \tag{4.82}$$

where (see (4.39))

$$\begin{aligned} {}^{L}R^a_{b\mu\nu} &:= \frac{\partial}{\partial x^\mu}\left(\Gamma^a_{b\nu}\right) - \frac{\partial}{\partial x^\nu}\left(\Gamma^a_{b\mu}\right) - \Gamma^c_{b\mu}\Gamma^a_{c\nu} + \Gamma^c_{b\nu}\Gamma^a_{c\mu}, \\ T^a_{\mu\nu} &:= -\frac{\partial}{\partial x^\mu}\left(G^a_\nu\right) + \frac{\partial}{\partial x^\nu}\left(G^a_\mu\right) + \Gamma^a_{c\nu}G^c_\mu - \Gamma^a_{c\mu}G^c_\nu, \end{aligned} \tag{4.83}$$

and the functions $T^a_{\mu\nu}$ have a sense of components of the torsion of ${}^{L}\Delta^h$ relative to G [24, pages 42, 46].

Thus, in general, the affine connections and linear connections are essentially different. However, they imply identical theories of covariant derivatives.

If, for some reason, the linear mapping G is fixed, then the set of linear connections $\{{}^{L}\Delta^h\}$ can be identified with the subset $\{\alpha^{-1}({}^{L}\Delta^h, G)\}$ of the set of affine connections $\{{}^{A}\Delta^h\}$. We will exemplify this situation on the tangent bundle $(T(M), \pi_T, M)$ over a manifold M. Using the base indices $\mu, \nu, \ldots$, for the fibre ones $a, b, \ldots$, according to the rule $a \mapsto \mu = a - \dim M$ (see Section 4.2), we rewrite (4.74) as

$$\Gamma^\mu_\nu = -\left(\Gamma^\mu_{\lambda\nu} \circ \pi_T\right) \cdot u^\lambda_1 + G^\mu_\nu \circ \pi_T. \tag{4.84}$$

Now the affine connections on $(T(M), \pi_T, M)$ are the *generalized affine connections on M* [19, Chapter III, Section 3]. The choice of G via

$$G^\mu_\nu : M \to \delta^\mu_\nu, \tag{4.85}$$

which corresponds to the identical transformation of the spaces tangent to M, singles out the set of *affine connections on M*—see [19, Chapter III, Section 3] or [29, pages 103–105]—(known also as *Cartan connections on M* [24, pages 46]) whose 2-index coefficients have the form (see (4.84), (4.22a), and (4.85))

$$\Gamma^{\mu}_{\nu} = -(\Gamma^{\mu}_{\lambda\nu} \circ \pi_T) \cdot \mathrm{d}x^{\lambda} + \delta^{\mu}_{\nu}. \tag{4.86}$$

Combining this with Proposition 4.22, we derive the following.

Proposition 4.24 (cf. [19, Chapter III, Section 3, Theorem 3.3]). *There is a bijective correspondence between the sets of linear connections and of affine ones on a manifold.*

Often the terms "linear connection" and "affine connection" on a manifold are used as synonyms, due to the last result.

5. Morphisms of bundles with connections

(Some ideas in this section are borrowed from [30, Chapter I, Section 6].) A *morphism* between two bundles (E,π,M) and (E',π',M') is a pair of mappings (F,f) such that $F : E \to E'$, $f : M \to M'$, and $\pi' \circ F = f \circ \pi$. If (U,u) and (U',u') are charts in E and E', respectively, and $F(U) \subseteq U'$, we have the following local representation of (F,f):

$$\overline{F} = u' \circ F \circ u^{-1} : u(U) \longrightarrow u'(U'), \tag{5.1a}$$

$$\overline{f} = x' \circ f \circ x^{-1} : x(V) \longrightarrow x'(V'), \tag{5.1b}$$

where (V,x) and (V',x') are local charts, respectively, on M and M'. Further, we assume that $U' = F(U)$ and that the charts in the base and bundle spaces respect the fibre structure, $V = \pi(U)$ and $V' = \pi'(U')$ so that $V' = f(V)$, and that the basic coordinates are $u^{\mu} = x^{\mu} \circ \pi$ and $u'^{\mu'} = x'^{\mu'} \circ \pi'$. Here and henceforth the quantities referring to (E',π',M') will inherit the same notation as the similar ones with respect to (E,π,M) with exception of the prime symbol added to the latter ones; in particular, the primed indices $\lambda',\mu',\nu',\ldots$ and $a',b',c',\ldots$ run, respectively, over the ranges $1,\ldots,n' = \dim M'$ and $n'+1,\ldots,n'+r' = \dim E'$ with r' being the fibre dimension of (E',π',M'), that is, $r' = \dim((\pi')^{-1}(p'))$ for $p' \in M'$.

Using the local coordinates $\{x^{\mu}\}$ on M and $\{u^{\mu} = x^{\mu} \circ \pi, u^{a}\}$ on E, we rewrite (5.1) as (cf. (3.1))

$$\overline{F}^{I'} = u'^{I'} \circ F \circ u^{-1} : u(U) \longrightarrow \mathbb{K}, \tag{5.1a$'$}$$

$$\overline{f}^{\mu'} = x'^{\mu'} \circ f \circ x^{-1} : x(\pi(U)) \longrightarrow \mathbb{K}, \tag{5.1b$'$}$$

that is, one can simply write $u'^{I'} = \overline{F}^{I'}(u^1,\ldots,u^{n+r})$ and $x'^{\mu'} = \overline{f}^{\mu'}(x^1,\ldots,x^n)$. However, in what follows, the mappings

$$F^{\mu'} := u'^{\mu'} \circ F = x'^{\mu'} \circ \pi \circ F = x'^{\mu'} \circ f \circ \pi : U \longrightarrow \mathbb{K}, \qquad F^{a'} := u'^{a'} \circ F : U \longrightarrow \mathbb{K}, \tag{5.3a}$$

$$f^{\mu'} := x'^{\mu'} \circ f : \pi(U) \longrightarrow \mathbb{K}, \tag{5.3b}$$

will be employed. The reason is that the derivatives

$$F_{,J}^{I'} := \frac{\partial}{\partial u^J} F^{I'} : p \longmapsto \frac{\partial}{\partial u^J}\Big|_p F^{I'} = \frac{\partial(F^{I'} \circ u^{-1})}{\partial(u^J \circ u^{-1})}\Big|_{u(p)} = \frac{\partial \overline{F}^{I'}}{\partial(u^J \circ u^{-1})}\Big|_{u(p)}, \quad p \in U, \tag{5.4}$$

(note that $\{u^J \circ u^{-1}\}$ are Cartesian coordinates on $u(U) \subseteq \mathbb{K}^{n+r}$) are the elements of the matrix of the tangent mapping $F_* : T(E) \to T(E')$ in the charts (U,u) and (U',u'). Indeed, since this matrix, known as the Jacobi matrix of F, is defined by [40, Section 1.23(a)]

$$F_*(\partial_J|_p) = F_J^{I'}|_p(\partial'_{I'}|_{F(p)}), \quad p \in U, \tag{5.5}$$

we have

$$[F_J^{I'}]_{\text{in}\,(\{\partial_K\},\{\partial_{K'}\})} = \left[\frac{\partial F^{I'}}{\partial u^J}\right]_{\substack{I'=1,\dots,n'+r' \\ J=1,\dots,n+r}} = \begin{pmatrix} [F_{,\mu}^{\nu'}] & 0_{n'\times r} \\ [F_{,\mu}^{a'}] & [F_{,b}^{a'}] \end{pmatrix} = \begin{bmatrix} F_{,\mu}^{\nu'} & 0_{n'\times r} \\ F_{,\mu}^{a'} & F_{,b}^{a'} \end{bmatrix}. \tag{5.6}$$

Let connections Δ^h and Δ'^h on (E,π,M) and $(E,'\pi',M')$, respectively, be given. To the local coordinates $\{u^I\}$ and $\{u'^I\}$ correspond the adapted frames (see (3.27)–(3.30))

$$\begin{aligned} (X_\mu, X_a) &= (\partial_\nu, \partial_b) \cdot \begin{bmatrix} \delta_\mu^\nu & 0 \\ +\Gamma_\mu^b & \delta_a^b \end{bmatrix} = (\partial_\mu + \Gamma_\mu^b \partial_b, \partial_a), \\ (X'_{\mu'}, X'_{a'}) &= (\partial'_{\nu'}, \partial'_{b'}) \cdot \begin{bmatrix} \delta_{\mu'}^{\nu'} & 0 \\ +\Gamma_{\mu'}^{b'} & \delta_{a'}^{b'} \end{bmatrix} = (\partial'_{\mu'} + \Gamma_{\mu'}^{'b'} \partial_{b'}, \partial_{a'}), \end{aligned} \tag{5.7}$$

where $\partial_I := \partial/\partial u^I$, and adapted coframes

$$\begin{pmatrix} \omega^\mu \\ \omega^a \end{pmatrix} = \begin{bmatrix} \delta_\nu^\mu & 0 \\ -\Gamma_\nu^a & \delta_b^a \end{bmatrix} \cdots \begin{pmatrix} \mathrm{d}u^\nu \\ \mathrm{d}u^b \end{pmatrix} = \cdots, \qquad \begin{pmatrix} \omega'^{\mu'} \\ \omega'^{a'} \end{pmatrix} = \begin{bmatrix} \delta_{\nu'}^{\mu'} & 0 \\ -\Gamma_{\nu'}^{'a'} & \delta_{b'}^{a'} \end{bmatrix} \cdot \begin{pmatrix} \mathrm{d}u'^{\nu'} \\ \mathrm{d}u'^{b'} \end{pmatrix} = \cdots. \tag{5.8}$$

The symbols Γ_μ^a and $\Gamma_{\mu'}^{'a'}$ in (5.7) and (5.8) denote the 2-index coefficients of, respectively, Δ^h and Δ'^h in the respective adapted frames.

If $\{e_I\}$ and $\{e'_{I'}\}$ are arbitrary frames over U in $T(E)$ and over $U' = F(U)$ in $T(E')$, respectively, the (Jacobi) matrix of F_* in them is defined via (cf. (5.5))

$$F_*(e_I|_p) = (F_I^{I'}|_p)(e'_{I'}|_{F(p)}). \tag{5.9}$$

In particular, in the adapted frames (5.7), we have $F_*(X_I|_p) = (F_I^{I'}|_p)(X'_{I'}|_{F(p)})$ and therefore the Jacobi matrix of F_* in the adapted frames (5.7) is (the changes $e := \{e_I\} \mapsto \{B_I^J e_J\}$ and $e' := \{e'_{I'}\} \mapsto \{B_{I'}^{'J'} e'_{J'}\}$, with nondegenerate matrix-valued functions $B := [B_I^J]$ and

$B' := [B'^{J'}_{I'}]$, imply the transformation $F_{(e,e')} := [F^{I'}_J] \mapsto (B')^{-1} \cdot F_{(e,e')} \cdot B$ of the Jacobi matrix of F_*; from here, (5.10) follows immediately)

$$\begin{aligned}
[F^{I'}_J] = [F^{I'}_J]_{\text{in }(\{X_K\},\{X'_{K'}\})} &= \begin{bmatrix} F^{\mu'}_\nu & F^{\mu'}_a \\ F^{b'}_\nu & F^{b'}_a \end{bmatrix} \\
&= \begin{bmatrix} \delta^{\mu'}_{\lambda'} & 0 \\ -\Gamma'^{b'}_{\lambda'} \circ F & \delta^{b'}_{c'} \end{bmatrix} \cdot \begin{bmatrix} F^{\lambda'}_{,\varrho} & 0 \\ F^{c'}_{,\varrho} & F^{c'}_{,d} \end{bmatrix} \cdot \begin{bmatrix} \delta^{\varrho}_\nu & 0 \\ +\Gamma^d_\nu & \delta^d_a \end{bmatrix} \\
&= \begin{bmatrix} F^{\mu'}_{,\nu} & 0 \\ X_\nu(F^{b'}) - (\Gamma'^{b'}_{\lambda'} \circ F)F^{\lambda'}_{,\nu} & F^{b'}_{,a} \end{bmatrix}
\end{aligned} \tag{5.10}$$

with $F^{I'}_{,J} := \partial F^{I'}/\partial u^J$ defining the matrix of F_* in $(\{\partial_K\},\{\partial'_{K'}\})$ via (5.6). Thus, the general formula (5.9) now reads

$$F_*(X_\mu,X_a) = (X'_{\nu'},X'_{b'}) \cdot \begin{bmatrix} F^{\nu'}_\mu & 0 \\ F^{b'}_\mu & F^{b'}_a \end{bmatrix} = (F^{\nu'}_{,\mu}X'_{\nu'} + F^{b'}_\mu X'_{b'}, F^{b'}_{,a}X'_{b'}) \tag{5.11}$$

with

$$F^{b'}_\mu = X_\mu(F^{b'}) - (\Gamma'^{b'}_{\lambda'} \circ F) \cdot F^{\lambda'}_{,\mu}. \tag{5.12}$$

From (5.9), it is clear that the elements $F^{I'}_J|_p$ of the Jacobi matrix of F_* at $p \in U$ are elements of a $(1,1)$ (mixed) tensor from $T^*_p(E) \otimes T_{F(p)}(E')$; in particular, if the adapted frames are changed (see (3.31)), the block structure of (5.10) is preserved and the elements of its blocks are transformed as elements of their corresponding tensors (e.g., $F^{b'}_\nu(p)$ are elements of a tensor from the tensor space spanned by $\{\omega^\nu|_p \otimes X'_{b'}|_{F(p)}\}$.) An important corollary from (5.11) is

$$F_*(\Delta^h) \subseteq \Delta'^h \Longleftrightarrow F^{b'}_\mu = 0 \tag{5.13}$$

in any pair $(\{X_I\},\{X_{I'}\})$ of adapted frames. If it happens that $F_*(\Delta^h) = \Delta'^h$, we say that F *preserves the connections* Δ^h and Δ'^h, that is, F is a *connection preserving mapping*; in particular, if $(E',\pi',M') = (E,\pi,M)$ and $F_*(\Delta^h) = \Delta'^h$, the mapping F is called a *symmetry* of Δ^h.

If the bundles considered are vectorial ones, the fibre coordinates, morphisms, and connections which are compatible with the vector structure must be linear on the fibres, namely,

$$F^{a'} = (\mathscr{F}^{a'}_b \circ \pi)u^b, \qquad \Gamma^a_\mu = -(\Gamma^a_{b\mu} \circ \pi)u^b, \qquad \Gamma^{a'}_{\mu'} = -(\Gamma^{a'}_{b'\mu'} \circ \pi')u^{b'}, \tag{5.14}$$

where the functions $\mathscr{F}^{a'}_b : \pi(U) \to \mathbb{K}$ are of class C^1 and $\Gamma^a_{b\mu}$ (resp., $\Gamma^{a'}_{b'\mu'}$) are the 3-index coefficients of the linear connection Δ^h (resp., Δ'^h). Consequently, in a case of vector bundles, the Jacobi matrix (5.10) takes the form

$$\begin{bmatrix} F^{\mu'}_{\nu} & F^{\mu'}_{a} \\ F^{b'}_{\nu} & F^{b'}_{a} \end{bmatrix} = \begin{bmatrix} F^{\mu'}_{,\nu} & 0 \\ (F^{b'}_{c\nu} \circ \pi)u^{c} & \mathcal{F}^{b'}_{a} \circ \pi \end{bmatrix} \tag{5.15}$$

with

$$F^{b'}_{a\mu} := \partial_{\mu}(\mathcal{F}^{b'}_{a}) - \Gamma^{c}_{a\mu}\mathcal{F}^{b'}_{c} + (\Gamma'^{b'}_{c'\lambda'} \circ f) \cdot \mathcal{F}^{c'}_{a} \cdot f^{\lambda'}_{\mu}, \tag{5.16}$$

where we have used that $\pi' \circ F = f \circ \pi$ and $u'^{c'} \circ F = F^{c'}$, and we have set $f^{\lambda'}_{\mu} := \partial(x'^{\lambda'} \circ f)/\partial x^{\mu}$, so that $F^{\lambda'}_{,\mu} = f^{\lambda'}_{\mu} \circ \pi$. Therefore, (5.12) now reads

$$F^{b'}_{\mu} = (F^{b'}_{a\mu} \circ \pi) \cdot u^{a}. \tag{5.17}$$

If M and M' are manifolds and $f : M \to M'$ is of class C^1, the above general considerations are valid for the morphism (f_*, f) of the tangent bundles $(T(M), \pi_T, M)$ and $(T(M'), \pi'_T, M')$. A peculiarity of a tangent bundle is that the fibre dimension of the bundle equals to the dimension of its base. Due to that fact, the base indices $\lambda, \mu, \nu, \ldots = 1, \ldots, n$ is convenient to be used for the fibre ones $a, b, c, \ldots = n+1, \ldots, n+r$ according to the rule

$$a \longmapsto \mu = a - \dim M, \tag{5.18a}$$

which must be combined with a change of the notation for the fibre coordinates, like

$$u^{a} \longmapsto u^{\mu}_{1}, \tag{5.18b}$$

as otherwise the change (5.18a) will entail $u^a \mapsto u^\mu$, the result of which coincides with the notation for the basic coordinates. (The subscript 1 in (5.18b) indicates that u^{μ}_{1} are fibre coordinates in the *first*-order tangent bundle $(T(M), \pi_T, M)$ over M.) Since $f_*(\partial/\partial x^{\mu}|_z) = \partial(x'^{\mu'} \circ f)/\partial x^{\mu}|_z \partial/\partial x'^{\mu'}|_{f(z)}$ for $z \in \pi(U)$ [40, Section 1.23(a)], the Jacobi matrix of f relative to the charts $(\pi(U), x)$ and $(\pi'(U'), x') = (f(\pi(U)), x')$ has the elements

$$f^{\mu'}_{\nu} := \frac{\partial(x'^{\mu'} \circ f)}{\partial x^{\nu}} : \pi(U) \to \mathbb{K}. \tag{5.19}$$

Combining this with the definition of the vector fibre coordinates u^{μ}_{1}, $u^{\mu}_{1}(p^{\nu}(\partial/\partial x^{\nu})|_{\pi(p)}) = p^{\mu}$, we see that (5.3), with f_* for F, reads

$$\begin{aligned} u'^{\mu'} &= f^{\mu'}_{*}(u^{1}, \ldots, u^{n}) = x'^{\mu'} \circ f \circ \pi, \\ u'^{\mu'}_{1} &= f'^{\mu'}_{*}(u^{1}, \ldots, u^{n}, u^{1}_{1}, \ldots, u^{n}_{1}) = (f^{\mu'}_{\nu} \circ \pi) \cdot u^{\nu}_{1}, \end{aligned} \tag{5.20a}$$

$$x'^{\mu'} = f^{\mu'}(x^{1}, \ldots, x^{n}) = x^{\mu'} \circ f. \tag{5.20b}$$

Therefore, the derivatives in (5.6) and (5.10)–(5.12) should be replaced according to $(u^\mu = x^\mu \circ \pi)$

$$F^{\nu'}_{,\mu} \longmapsto \frac{\partial f^{\mu'}_*}{\partial x^\mu} = f^{\nu'}_\mu \circ \pi, \qquad F^{a'}_{,\nu} \longmapsto \frac{\partial f^{\mu'}_*}{\partial x^\nu} = \left(\frac{\partial f^{\mu'}_\lambda}{\partial x^\nu} \circ \pi\right) u^\lambda_1, \qquad F^{a'}_{,b} \longmapsto \frac{\partial f^{\mu'}_*}{\partial u^\nu_1} = f^{\mu'}_\nu \circ \pi. \tag{5.21}$$

If Δ^h and $\Delta^{'h}$ are linear connections on M and M', respectively, their 2- and 3-index coefficients are connected through (cf. (5.14))

$$\Gamma^\lambda_\nu = -(\Gamma^\lambda_{\mu\nu} \circ \pi) \cdot u^\lambda_1, \qquad \Gamma'^{\lambda'}_{\nu'} = -(\Gamma'^{\lambda'}_{\mu'\nu'} \circ \pi') \cdot u^{\lambda'}_1. \tag{5.22}$$

Thus the Jacobi matrix of $(f_*)_* =: f_{**}$ in the pair of frames $(\{X_\mu = (\partial/\partial x^\mu) + \Gamma^\lambda_\mu(\partial/\partial u^\lambda_1), X^1_\nu = \partial/\partial u^\nu_1\}, \{X'_{\mu'}, X'^1_{\nu'}\})$ is (cf. (5.15) and (5.16))

$$\begin{bmatrix} f^{\mu'}_\nu \circ \pi & 0 \\ -(f^{\varrho'}_{\lambda\nu} \circ \pi) \cdot u^\lambda_1 & f^{\varrho'}_\tau \circ \pi, \end{bmatrix}, \tag{5.23}$$

where

$$f^{\lambda'}_{\mu\nu} := f^{\lambda'}_{*\mu\nu} := \partial_\nu(f^{\lambda'}_\mu) - f^{\lambda'}_\sigma \Gamma^\sigma_{\mu\nu} + (\Gamma'^{\lambda'}_{\sigma'\tau'} \circ f) f^{\sigma'}_\mu f^{\tau'}_\nu. \tag{5.24}$$

The quantities (5.24) are components of a $T(M')$-valued 2-form on M, that is, of an element in $T(M') \otimes \Lambda^2(M)$. (Moreover, if we consider $f^{\mu'}_\nu$, defined via (5.19), as components of an element in $T_{f(p)}(M') \otimes \Lambda^1_p(M)$, then (5.24) are the components of the *mixed* covariant derivative (along $\partial/\partial x^\nu$) of $f^{\mu'}_\mu (\partial'_{\mu'}|_{f(\cdot)}) \otimes \mathrm{d}u^\mu$ relative to the connection $\Delta^h \times \Delta^{'h}$ on $M \times M$.)

6. General (co)frames

Until now two special kinds of local (co)frames in the (co)tangent bundle to the bundle space of a bundle were employed, namely, the natural holonomic ones, induced by some local coordinates, and the adapted (co)frames determined by local coordinates and a connection on the bundle. The present section is devoted to (re)formulation of some important results and formulae in arbitrary (co)frames, which in particular can be natural or adapted (if a connection is presented) ones.

Let (E, π, M) be a C^2 bundle and $\{e_I\}$ a (local) frame in $T(E)$. The components C^K_{IJ} of the anholonomicity object of $\{e_I\}$ are defined by (3.19) and a change

$$\{e_I\} \longmapsto \{\bar{e}_I = B^J_I e_J\} \tag{6.1}$$

with a nondegenerate matrix-valued function $B = [B^J_I]^{n+r}_{I,J=1}$ entails (see (2.9))

$$C^K_{IJ} \longmapsto \overline{C}^K_{IJ} = (B^{-1})^K_L (B^M_I e_M(B^L_J) - B^M_J e_M(B^L_I) + B^M_I B^N_J C^L_{MN}). \tag{6.2}$$

Let a connection Δ^h on (E,π,M) be given. If $\{e_I\}$ is a specialized frame for Δ^h (see Section 3.2), then the set $\{C^K_{IJ}\}$ is naturally divided into the six groups (3.20). The value of that division is in its invariance with respect to the class of specialized frames, which means that, if $\{\bar{e}_I\}$ is also a specialized frame, then the transformed components of the elements of each group are functions only in the elements of the nontransformed components of the same group—see (3.24), (3.21), and (2.9). By means of (6.1), one can prove that, if such a division holds in a frame $\{e_I\}$, then it holds in $\{\bar{e}_I\}$ if and only if the matrix-valued function B is of the form (3.16). In particular, we cannot talk about fibre coefficients of Δ^h and of fibre components of the curvature of Δ^h in frames more general than the specialized ones as in that case the transformation (6.1), with $\{e_I\}$ (resp., $\{\bar{e}_I\}$) being a specialized (resp., nonspecialized) frame, will mix, for instance, the fibre coefficients and the curvature's fibre components of Δ^h in $\{\bar{e}_I\}$—see (6.2).

It is a simple, but important, fact that the specialized frames are (up to renumbering) the most general ones which respect the splitting of $T(E)$ into vertical and horizontal components. Suppose $\{e_I\}$ is a specialized frame. Then the general element of the set of all specialized frames is (see (3.4a) and (3.16))

$$(\bar{e}_\mu,\bar{e}_a) = (e_\nu,e_b)\cdot\begin{bmatrix} A^\nu_\mu & 0 \\ 0 & A^b_a \end{bmatrix} = (A^\nu_\mu e_\nu, A^b_a e_b), \tag{6.3a}$$

where $[A^\nu_\mu]^n_{\mu,\nu=1}$ and $[A^b_a]^{n+r}_{a,b=n+1}$ are nondegenerate matrix-valued functions on E, which are constant on the fibres of (E,π,M), that is, we can set $A^\nu_\mu = B^\nu_\mu\circ\pi$ and $A^b_a = B^b_a\circ\pi$ for some nondegenerate matrix-valued functions $[B^\nu_\mu]$ and $[B^b_a]$ on M. Respectively, the general specialized coframe dual to $\{\bar{e}_I\}$ is (see (3.4b) and (3.16))

$$\begin{pmatrix} \bar{e}^\mu \\ \bar{e}^a \end{pmatrix} = \begin{bmatrix} [A^\lambda_\rho]^{-1} & 0 \\ 0 & [A^c_d]^{-1} \end{bmatrix}\cdot\begin{pmatrix} e^\nu \\ e^b \end{pmatrix} = \begin{bmatrix} ([A^\lambda_\rho]^{-1})^\mu_\nu e^\nu \\ ([A^c_d]^{-1})^a_b e^b \end{bmatrix}, \tag{6.3b}$$

where $\{e^I\}$ is the specialized coframe dual to $\{e_I\}$, $e^I(e_J) = \delta^I_J$.

Since $\pi_*|_{\Delta^h} : \{X\in\Delta^h\}\to\mathscr{X}(M)$ is an isomorphism, any basis $\{\varepsilon_\mu\}$ *for* Δ^h defines a basis $\{E_\mu\}$ of $\mathscr{X}(M)$ such that

$$E_\mu = \pi_*|_{\Delta^h}(\varepsilon_\mu), \tag{6.4}$$

and vice versa, a basis $\{E_\mu\}$ for $\mathscr{X}(M)$ induces a basis $\{\varepsilon_\mu\}$ for Δ^h via

$$\varepsilon_\mu = (\pi_*|_{\Delta^h})^{-1}(E_\mu). \tag{6.5}$$

Similarly, there is a bijection $\{\varepsilon^\mu\}\mapsto\{E^\mu\}$ between the "horizontal" coframes $\{\varepsilon^\mu\}$ and the coframes $\{E^\mu\}$ dual to the frames in $T(M)$ ($E^\mu\in\Lambda^1(M)$, $E^\mu(E_\nu)=\delta^\mu_\nu$). Thus a "horizontal" change

$$\varepsilon_\mu \longmapsto \bar{\varepsilon}_\mu = (B^\nu_\mu\circ\pi)\varepsilon_\nu, \tag{6.6}$$

which is independent of a "vertical" one given by

$$\varepsilon_a \longmapsto \bar{\varepsilon}_a = (B^b_a\circ\pi)\varepsilon_b \tag{6.7}$$

with $\{\varepsilon_a\}$ being a basis for Δ^v, is equivalent to the transformation

$$E_\mu \longmapsto \overline{E}_\mu = B_\mu^\nu E_\nu \tag{6.8}$$

of the basis $\{E_\mu\}$ for $\mathscr{X}(M)$, related via (6.4) to the basis $\{\varepsilon_\mu\}$ for Δ^h. Here $[B_\mu^\nu]$ and $[B_a^b]$ are nondegenerate matrix-valued functions on M.

As $\pi_*(\varepsilon_a) = 0 \in \mathscr{X}(M)$, the "vertical" transformations (6.7) do not admit interpretation analogous to the "horizontal" ones (6.6). However, in a case of a *vector* bundle (E,π,M), they are tantamount to changes of frames in the bundle space E, that is, of the bases for $\operatorname{Sec}(E,\pi,M)$. Indeed, if ν is the mapping defined by (4.5), the sections

$$E_a = \nu^{-1}(\varepsilon_a) \tag{6.9}$$

form a basis for $\operatorname{Sec}(E,\pi,M)$ as the vertical vector fields ε_a form a basis for Δ^v. Conversely, any basis $\{E_a\}$ for the sections of (E,π,M) induces a basis $\{\varepsilon_a\}$ for Δ^v such that

$$\varepsilon_a = \nu(E_a). \tag{6.10}$$

As ν and ν^{-1} are linear, the change (6.7) is equivalent to the transformation

$$E_a \longmapsto \overline{E}_a = B_a^b E_b \tag{6.11}$$

of the frame $\{E_a\}$ in E related to $\{\varepsilon_a\}$ via (6.9). In this way, we have proved the following result.

Proposition 6.1. *There is a bijective correspondence between the set of specialized frames $\{\varepsilon_I\} = \{\varepsilon_\mu,\varepsilon_a\}$ on a vector bundle (E,π,M) and the set of pairs $(\{E_\mu\},\{E_a\})$ of frames $\{E_\mu\}$ on $T(M)$ over M and $\{E_a\}$ on E over M. (It should be mentioned the evident fact that a frame $\{E_\mu\}$ in $T(M)$ over M is also a basis for the module $\mathfrak{X}(M)$ of vector fields over M and hence is a basis for the set* $\operatorname{Sec}(T(M),\pi_T,M)$ *of section of the bundle tangent to M, due to $\mathfrak{X}(M) = \operatorname{Sec}(T(M),\pi_T,M)$. Similarly, a frame $\{E_a\}$ on E over M is a basis for the set* $\operatorname{Sec}(E,\pi,M)$ *of sections of the vector bundle (E,π,M).)*

Since conceptually the frames in the tangent bundle space $T(M)$ and in the bundle space E are easier to be understood and in some cases have a direct physical interpretation, one often works with the pair $(\{E_\mu = \pi_*|_{\Delta^h}(\varepsilon_\mu)\},\{E_a = \nu^{-1}(\varepsilon_a)\})$ of frames instead with a specialized frame $\{\varepsilon_I\} = \{\varepsilon_\mu,\varepsilon_a\}$; for instance $\{E_\mu\}$ and $\{E_a\}$ can be completely arbitrary frames in $T(M)$ and E, respectively, while the specialized frames represent only a particular class of frames in $T(E)$.

One can *mutatis mutandis* localize the above considerations when M is replaced with an open subset U_M in M and E is replaced with $U = \pi^{-1}(U_M)$. Such a localization is important when the bases/frames considered are connected with some local coordinates or when they should be smooth. (Recall, not every manifold admits a *global* nowhere vanishing C^m, $m \geq 0$, vector field (see [34] or [33, Section 4.24]); e.g., such are the even-dimensional spheres $\mathbb{S}^{2k}$, $k \in \mathbb{N}$, in Euclidean space.)

Let us turn now our attention to frames adapted to local coordinate system $\{u^I\}$ on an open set $U \subseteq E$ for a given connection Δ^h on a general C^1 bundle (E,π,M) (see (3.27)–(3.30)). Since in their definition the local coordinates u^I enter only via the vector fields

$\partial_I := \partial/\partial u^I \in \mathscr{X}(E)$, we can generalize this definition by replacing $\{\partial_I\}$ with an arbitrary frame $\{e_I\}$ defined in $T(E)$ over an open set $U \subseteq E$ and such that $\{e_a|_p\}$ is a *basis for the space* $T_p(\pi^{-1}(\pi(p)))$ *tangent to the fibre through* $p \in U$. So, using $\{e_I\}$ for $\{\partial_I\}$, we have

$$(e_\mu^U, e_a^U) = (D_\mu^\nu e_\nu + D_\mu^a e_a, D_a^b e_b) = (e_\nu, e_b) \cdot \begin{pmatrix} [D_\mu^\nu] & 0 \\ [D_\mu^b] & [D_a^b] \end{pmatrix}, \tag{6.12}$$

where $\{e_I^U\}$ is a *specialized* frame in $T(U)$, $[D_\mu^\nu]$ and $[D_a^b]$ are nondegenerate matrix-valued functions on U, and $D_\mu^a : U \to \mathbb{K}$.

Defintion 6.2. The specialized frame $\{X_I\}$ over U in $T(U)$, obtained from (6.12) via an admissible transformation (3.4a) with matrix $A = \begin{pmatrix} [D_\nu^\mu]^{-1} & 0 \\ 0 & [D_b^a]^{-1} \end{pmatrix}$, is called *adapted to the frame* $\{e_I\}$ *for* Δ^h. (Recall, here and below the adapted frames are defined only with respect to frames $\{e_I\} = \{e_\mu, e_a\}$ such that $\{e_a\}$ is a basis for the vertical distribution Δ^v over U, i.e., $\{e_a|_p\}$ is a basis for Δ_p^v for all $p \in U$. Since Δ^v is integrable, the relation $e_a \in \Delta^v$ for all $a = n+1, \dots, n+r$ implies $[e_a, e_b]_- \in \Delta^v$ for all $a, b = n+1, \dots, n+r$.)

Exercise 6.3. Using (3.4) and (3.16), verify that the adapted frame $\{X_I\}$ and coframe $\{\omega^I\}$ dual to it are independent of the particular specialized frame $\{e_I^U\}$ entering into their definitions via (6.12). The equalities (6.13a) and (6.20) derived below are indirect proof of that fact too.

According to (3.4), the adapted frame $\{X_I\} = \{X_\mu, X_a\}$ and coframe $\{\omega^I\} = \{\omega^\mu, \omega^a\}$ the dual to it are given by the equations

$$(X_\mu, X_a) = (e_\nu, e_b) \cdot \begin{bmatrix} \delta_\mu^\nu & 0 \\ +\Gamma_\mu^b & \delta_a^b \end{bmatrix} = (e_\mu + \Gamma_\mu^b e_b, e_a), \tag{6.13a}$$

$$\begin{pmatrix} \omega^\mu \\ \omega^a \end{pmatrix} = \begin{bmatrix} \delta_\nu^\mu & 0 \\ -\Gamma_\nu^a & \delta_b^a \end{bmatrix} \cdot \begin{pmatrix} e^\nu \\ e^b \end{pmatrix} = \begin{pmatrix} e^\mu \\ e^a - \Gamma_\nu^a e^\nu \end{pmatrix}, \tag{6.13b}$$

where $\{e^I\}$ is the coframe dual to $\{e_I\}$, $e^I(e_J) = \delta_J^I$, and the functions $\Gamma_\mu^a : U \to \mathbb{K}$, called *(2-index) coefficients of* Δ^h in $\{X_I\}$, are defined by

$$[\Gamma_\mu^a] := +[D_\nu^a] \cdot [D_\mu^\nu]^{-1}. \tag{6.14}$$

Proposition 6.4. *A change* $\{e_I\} \mapsto \{\tilde{e}_I\}$ *with*

$$(\tilde{e}_\mu, \tilde{e}_a) = (e_\nu, e_b) \cdot \begin{pmatrix} [A_\mu^\nu] & 0 \\ [A_\mu^b] & [A_a^b] \end{pmatrix} = (A_\mu^\nu e_\nu + A_\mu^b e_b, A_a^b e_b), \tag{6.15}$$

where $[A^\nu_\mu]$ *and* $[A^b_a]$ *are nondegenerate matrix-valued functions on* U*, which are constant on the fibres of* (E,π,M)*, and* $A^b_\mu : U \to \mathbb{K}$*, entails the transformations*

$$(X_\mu, X_a) \longmapsto (\widetilde{X}_\mu, \widetilde{X}_a) = (\widetilde{e}_\mu + \widetilde{\Gamma}^b_\mu \widetilde{e}_b, \widetilde{e}_a) = (A^\nu_\mu X_\nu, A^b_a X_b) = (X_\nu, X_b) \cdot \begin{bmatrix} A^\nu_\mu & 0 \\ 0 & A^b_a \end{bmatrix}, \tag{6.16}$$

$$\Gamma^a_\mu \longmapsto \widetilde{\Gamma}^a_\mu = \left(\left[A^c_d\right]^{-1}\right)^a_b \left(\Gamma^b_\nu A^\nu_\mu - A^b_\mu\right) \tag{6.17}$$

of the frame $\{X_I\}$ *adapted to* $\{e_I\}$ *and of the coefficients* Γ^a_μ *of* Δ^h *in* $\{X_I\}$*, that is,* $\{\widetilde{X}_I\}$ *is the frame adapted to* $\{\widetilde{e}_I\}$ *and* $\widetilde{\Gamma}^a_\mu$ *are the coefficients of* Δ^h *in* $\{\widetilde{X}_I\}$*.*

Proof. Apply (6.12)–(6.14). □

Note 6.5. If $\{e_I\}$ and $\{\widetilde{e}_I\}$ are adapted, then $A^b_\mu = 0$. If $\{Y_I\}$ is a specialized frame, it is adapted to any frame $\{e_\mu = A^\nu_\mu Y_\nu, e_a = A^b_a Y_b\}$ and hence any specialized frame can be considered as an adapted one; in particular, any specialized frame is a frame adapted to itself. Obviously (see (6.14)), the coefficients of a connection identically vanish in a given specialized frame considered as an adapted one. This leads to the concept of a *normal frame*, which will be studied on this context in a forthcoming paper. Besides, from the above observation follows that the set of adapted frames coincides with the one of specialized frames.

Exercise 6.6. Verify that the formulae dual to (6.15) and (6.16) are (see (3.4b) and (3.5b))

$$\begin{aligned} \begin{pmatrix} \widetilde{e}^\mu \\ \widetilde{e}^a \end{pmatrix} &= \begin{pmatrix} [A^\varrho_\tau]^{-1} & 0 \\ -[A^c_d]^{-1}[A^c_\tau][A^\varrho_\tau]^{-1} & [A^c_d]^{-1} \end{pmatrix} \cdot \begin{pmatrix} e^\nu \\ e^b \end{pmatrix} \\ &= \begin{pmatrix} \left([A^\varrho_\tau]^{-1}\right)^\mu_\nu e^\nu \\ \left([A^c_d]^{-1}\right)^a_b e^b - \left([A^c_d]^{-1}[A^c_\tau][A^\varrho_\tau]^{-1}\right)^a_\nu e^\nu \end{pmatrix} \\ \begin{pmatrix} \omega^\mu \\ \omega^a \end{pmatrix} \longmapsto \begin{pmatrix} \widetilde{\omega}^\mu \\ \widetilde{\omega}^a \end{pmatrix} &= \begin{pmatrix} \left([A^\varrho_\tau]^{-1}\right)^\mu_\nu e^\nu \\ \left([A^c_d]^{-1}\right)^a_b e^b \end{pmatrix}. \end{aligned} \tag{6.18}$$

Example 6.7. If $\{e_I\}$ and $\{\widetilde{e}_I\}$ are the frames generated by local coordinates $\{u^I\}$ and $\{\widetilde{u}^I\}$, namely, $e_I = \partial/\partial u^I$ and $\widetilde{e}_I = \partial/\partial\widetilde{u}^I$, the changes (6.16) and (6.17) reduce to (3.31) and (3.32), respectively. The choice $e_I = \partial/\partial u^I$ also reduces Definition 6.2 to Definition 3.6.

A result similar to Proposition 3.9 is valid too provided in its formulation equation (3.22) is replaced with (6.17).

If e_μ has an expansion $e_\mu = e^\nu_\mu(\partial/\partial u^\nu) + e^b_\mu(\partial/\partial u^b)$ in the domain U of $\{u^I\} = \{u^\mu = x^\mu \circ \pi, u^a\}$, where $e^b_\mu : U \to \mathbb{K}$ and $e^\nu_\mu = x^\nu_\mu \circ \pi$ for some $x^\nu_\mu : \pi(U) \to \mathbb{K}$ such that $\det[x^\nu_\mu] \neq 0, \infty$, and we define a frame $\{x_\mu\}$ in $T(\pi(U)) \subseteq T(M)$ by $\{x_\mu := x^\nu_\mu(\partial/\partial x^\nu)\}$, then

$$\pi_*(X_\mu) = x_\mu, \tag{6.19}$$

by virtue of (3.33) and (3.35). Thus, we have (cf. (3.34))

$$X_\mu = (\pi_*|_{\Delta^h})^{-1}(x_\mu) = (\pi_*|_{\Delta^h})^{-1} \circ \pi_*(e_\mu), \tag{6.20}$$

which can be used in an equivalent definition of a frame $\{X_I\}$ adapted to $\{e_I\}$ (with $\{e_a\}$ being a basis for Δ^v): X_μ should be defined by (6.20) and $X_a = e_a$. If one accepts such a definition of an adapted frame, the 2-index coefficients of a connection should be defined via (6.13a), not by (6.14), and the proofs of some results, like (6.16) and (6.17), should be modified.

Proposition 6.8. *If $\{X_I\}$ is a frame adapted to a frame $\{e_I\}$, with $\{e_a\}$ being a basis for Δ^v, for a C^1 connection Δ^h, then (cf. (3.36))*

$$[X_\mu, X_\nu]_- = R^a_{\mu\nu} X_a + S^\lambda_{\mu\nu} X_\lambda, \tag{6.21a}$$

$$[X_\mu, X_b]_- = {}^\circ\Gamma^a_{b\mu} X_a + C^\lambda_{\mu b} X_\lambda, \tag{6.21b}$$

$$[X_a, X_b]_- = C^d_{ab} X_d, \tag{6.21c}$$

where (cf. (3.37))

$$\begin{aligned} R^a_{\mu\nu} :=\ & X_\mu(\Gamma^a_\nu) - X_\nu(\Gamma^a_\mu) - C^a_{\mu\nu} - \Gamma^b_\mu C^a_{\nu b} + \Gamma^b_\nu C^a_{\mu b} \\ & + \Gamma^a_\lambda(-C^\lambda_{\mu\nu} + \Gamma^b_\mu C^\lambda_{\nu b} - \Gamma^b_\nu C^\lambda_{\mu b}) + \Gamma^b_\mu \Gamma^d_\nu C^a_{bd}, \end{aligned} \tag{6.22a}$$

$$\begin{aligned} S^\lambda_{\mu\nu} &:= C^\lambda_{\mu\nu} + \Gamma^b_\mu C^\lambda_{\nu b} - \Gamma^b_\nu C^\lambda_{\mu b}, \\ {}^\circ\Gamma^a_{b\mu} &:= -X_b(\Gamma^a_\mu) - C^a_{\mu b} + \Gamma^d_\mu C^a_{db} - \Gamma^a_\lambda C^\lambda_{\mu b}, \end{aligned} \tag{6.22b}$$

$$[e_I, e_J]_- =: C^K_{IJ} e_K = C^a_{IJ} e_a + C^\lambda_{IJ} e_\lambda. \tag{6.22c}$$

Proof. Insert (6.13a) into the corresponding commutators, use the definition (6.22c) of the components of the anholonomicity object of $\{e_I\}$, and apply (6.13a). Notice, as $\{e_a\}$ is a basis for the integrable distribution Δ^v, we have $[e_a, e_b]_- \in \Delta^v$ and consequently $C^\lambda_{ab} \equiv 0$. □

The functions $R^a_{\mu\nu}$ are the *fibre components of the curvature* of Δ^h and ${}^\circ\Gamma^a_{b\mu}$ are the *fibre coefficients* of Δ^h in the adapted frame $\{X_I\}$; if $e_I = \partial/\partial u^I$ for some bundle coordinates $\{u^I\}$ on E, they reduce to (3.37a) and (3.37b), respectively. From (6.21), we immediately derive the following.

Corollary 6.9. *A connection Δ^h is integrable if and only if in some (and hence any) adapted frame:*

$$R^a_{\mu\nu} = 0. \tag{6.23}$$

Corollary 6.10. *An adapted frame $\{X_I\}$ is (locally) holonomic if and only if*

$$R^a_{\mu\nu} = {}^\circ\Gamma^a_{b\mu} = S^\lambda_{\mu\nu} = C^d_{ab} = C^\lambda_{\mu b} = 0. \tag{6.24}$$

If the initial frame $\{e_I\}$ is changed into (6.15), then the transformation laws of the quantities (6.22) follow from (6.21) and (6.16); in particular, the curvature components transform according to the tensor equation (3.24b).

Let us now pay attention to the case when (E,π,M) is a *vector* bundle endowed with a connection Δ^h.

According to the abovesaid in this section, any *adapted* frame $\{X_I\} = \{X_\mu, X_a\}$ in $T(E)$ is equivalent to a pair of frames in $T(M)$ and E according to

$$\{X_\mu, X_a\} \longleftrightarrow (\{E_\mu = \pi_*|_{\Delta^h}(X_\mu)\}, \{E_a = \nu^{-1}(X_a)\}). \tag{6.25}$$

Therefore, the vertical and horizontal lifts are given by (cf. Lemma 4.1, (4.11a), and (4.14))

$$\operatorname{Sec}(E,\pi,M) \ni Y = Y^a E_a \xrightarrow{\nu} \nu(Y) := Y^\nu = (Y^a \circ \pi) X_a \in \Delta^\nu, \tag{6.26a}$$

$$\mathscr{X}(M) \ni F = F^\mu E_\mu \xrightarrow{h} h(F) := F^h = (F^\mu \circ \pi) X_\mu \in \Delta^h. \tag{6.26b}$$

Thus, we have the linear isomorphism

$$\begin{aligned} (h,\nu) &: \mathscr{X}(M) \times \operatorname{Sec}(E,\pi,M) \longrightarrow \mathscr{X}(E), \\ (h,\nu) &: (F,Y) \longmapsto (F^h, Y^\nu), \end{aligned} \tag{6.27}$$

which explains why the covariant derivatives (see Definition 4.9) represent an equivalent description of the linear connections in vector bundles. Since any vector field $\xi = (\xi^I \circ \pi) X_I \in \mathscr{X}(E)$ has a unique decomposition $\xi = \xi^h \oplus \xi^\nu$, with $\xi^h = (\xi^\mu \circ \pi) X_\mu$ and $\xi^\nu = (\xi^a \circ \pi) X_a$, we have

$$(h,\nu)^{-1}(\xi) = (\pi_*|_{\Delta^h}(\xi^h), \nu^{-1}(\xi^\nu)) = (\xi^\mu E_\mu, \xi^a E_a). \tag{6.28}$$

Suppose $\{X_I\}$ and $\{\widetilde{X}_I\}$ are two adapted frames. Then they are connected via (cf. (6.3a) and (6.16))

$$\widetilde{X}_\mu = (B^\nu_\mu \circ \pi) X_\nu, \qquad \widetilde{X}_a = (B^b_a \circ \pi) X_b, \tag{6.29}$$

where $[B^\nu_\mu]$ and $[B^b_a]$ are some nondegenerate matrix-valued functions on M. The pairs of frames corresponding to them, in accordance with (6.25), are related via

$$\widetilde{E}_\mu = B^\nu_\mu E_\nu, \qquad \widetilde{E}_a = B^b_a E_b, \tag{6.30}$$

and *vice versa*.

Proposition 6.11. *Let Δ^h be a linear connection on a vector bundle (E,π,M) and let $\{X_\mu\}$ be the frame adapted for Δ^h to a frame $\{e_I\}$ such that $\{e_a\}$ is a basis for Δ^ν and*

$$\begin{aligned} (e_\mu, e_a)|_U &= (\partial_\nu, \partial_b) \cdot \begin{bmatrix} B^\nu_\mu \circ \pi & 0 \\ (B^b_{c\mu} \circ \pi) \cdot E^c & B^b_a \circ \pi \end{bmatrix} \\ &= ((B^\nu_\mu \circ \pi)\partial_\nu + ((B^b_{c\mu} \circ \pi) \cdot E^c)\partial_b, (B^b_a \circ \pi)\partial_b), \end{aligned} \tag{6.31}$$

where $\partial_I := \partial/\partial u^I$ for some local bundle coordinates $\{u^I\} = \{u^\mu = x^\mu \circ \pi, u^b = E^b\}$ on $U \subseteq E$, $[B^\nu_\mu]$ and $[B^b_a]$ are nondegenerate matrix-valued functions on U, $B^b_{c\mu} : U \to \mathbb{K}$, and $\{E^a\}$ is the coframe dual to $\{E_a = \nu^{-1}(X_a)\}$. Then the 2-index coefficients Γ^a_μ of Δ^h in $\{X_I\}$ have the representation (cf. (4.25))

$$\Gamma^a_\mu = -(\Gamma^a_{b\mu} \circ \pi) \cdot E^b \tag{6.32}$$

on U for some functions $\Gamma^a_{b\mu} : U \to \mathbb{K}$, called 3-index coefficients of Δ^h in $\{X_I\}$.

Remark 6.12. The representation (6.32) is not valid for frames more general than the ones given by (6.31). Precisely, (6.32) is valid if and only if (6.31) holds for some local coordinates $\{u^I\}$ on U—see (6.17).

Proof. Writing (6.17) for the transformation $\{\partial_I\} \mapsto \{e_I\}$, with $\{e_I\}$ given by (6.31), we get (6.32) with

$$\Gamma^a_{b\mu} = \left([B^e_d]^{-1}\right)^a_c \left({}^\partial\Gamma^c_{b\nu} B^\nu_\mu + B^c_{b\mu}\right), \tag{6.33}$$

where ${}^\partial\Gamma^a_{b\nu}$ are the 3-index coefficients of Δ^h in the frame adapted to the coordinates $\{u^I\}$ (see (4.25)). □

Let $\{X_I\}$ and $\{\widetilde{X}_I\}$ be frames adapted to $\{e_I\}$ and $\{\widetilde{e}_I\}$, respectively, with (cf. (6.31))

$$(\widetilde{e}_\mu, \widetilde{e}_a) = (e_\nu, e_b) \cdot \begin{bmatrix} B^\nu_\mu \circ \pi & 0 \\ (B^b_{c\mu} \circ \pi) \cdot E^c & B^b_a \circ \pi \end{bmatrix}, \tag{6.34}$$

in which Δ^h admits 3-index coefficients. Then, due to (6.17) and (6.32), the 3-index coefficients $\Gamma^a_{b\mu}$ and $\widetilde{\Gamma}^a_{b\mu}$ of Δ^h in, respectively, $\{X_I\}$ and $\{\widetilde{X}_I\}$ are connected by (cf. (4.36))

$$\widetilde{\Gamma}^a_{b\mu} = \left([B^e_f]^{-1}\right)^a_c \left(\Gamma^c_{d\nu} B^\nu_\mu + B^c_{d\mu}\right) B^d_b. \tag{6.35}$$

Exercise 6.13. Prove that the transformation $\{e_I\} \mapsto \{\widetilde{e}_I\}$, with $\{\widetilde{e}_I\}$ given by (6.34), is the most general one that preserves the existence of 3-index coefficients of Δ^h provided they exist in $\{e_I\}$.

Introducing the matrices $\Gamma_\mu := [\Gamma^a_{b\mu}]^{n+r}_{a,b=n+1}$, $\widetilde{\Gamma}_\mu := [\widetilde{\Gamma}^a_{b\mu}]^{n+r}_{a,b=n+1}$, $B := [B^a_b]$, and $B_\mu := [B^a_{b\mu}]$, we rewrite (6.35) as (cf. (4.37))

$$\widetilde{\Gamma}_\mu = B^{-1} \cdot \left(\Gamma_\nu B^\nu_\mu + B_\mu\right) \cdot B. \tag{6.35$'$}$$

A little below (see the text after (6.37)), we will prove that the compatibility of the developed formalism with the theory of covariant derivatives requires further restrictions on the general transformed frames (6.15) to the ones given by (6.34) with

$$B_\mu = \widetilde{E}_\mu(B) \cdot B^{-1} = B^\nu_\mu E_\nu(B) \cdot B^{-1}, \tag{6.36}$$

where $\widetilde{E}_\mu := \pi_*|_{\Delta^h}(\widetilde{X}_\mu) = \pi_*|_{\Delta^h}((B^\nu_\mu \circ \pi)X_\nu) = B^\nu_\mu E_\nu$. In this case, (6.35′) reduces to (cf. (4.37))

$$\widetilde{\Gamma}_\mu = B^\nu_\mu B^{-1} \cdot (\Gamma_\nu \cdot B + E_\nu(B)) = B^\nu_\mu (B^{-1} \cdot \Gamma_\nu - E_\nu(B^{-1})) \cdot B. \tag{6.37}$$

At last, a few words on the covariant derivative operators ∇ are in order. Without lost of generality, we define such an operator (4.48) via (4.54). Suppose $\{E_\mu\}$ is a basis for $\mathscr{X}(M)$ and $\{E_a\}$ is a one for $\operatorname{Sec}^1(E,\pi,M)$. Define the *components* $\Gamma^a_{b\mu} : M \to \mathbb{K}$ of ∇ in the pair of frames $(\{E_\mu\},\{E_a\})$ by (cf. (4.55))

$$\nabla_{E_\mu}(E_b) = \Gamma^a_{b\mu} E_a. \tag{6.38}$$

Then (4.54) imply (cf. (4.51))

$$\nabla_F Y = F^\mu (E_\mu(Y^a) + \Gamma^a_{b\mu} Y^b) E_a \tag{6.39}$$

for $F = F^\mu E_\mu \in \mathscr{X}(M)$ and $Y = Y^a E_a \in \operatorname{Sec}^1(E,\pi,M)$. A change $(\{E_\mu\},\{E_a\}) \mapsto (\{\widetilde{E}_\mu\}, \{\widetilde{E}_a\})$, given via (6.30), entails

$$\Gamma^a_{b\mu} \longmapsto \widetilde{\Gamma}^a_{b\mu} = B^\nu_\mu ([B^e_f]^{-1})^a_c (\Gamma^c_{d\nu} B^d_b + E_\nu(B^c_b)), \tag{6.40}$$

as a result of (6.38). In a more compact matrix form, the last result reads

$$\widetilde{\Gamma}_\mu = B^\nu_\mu B^{-1} \cdot (\Gamma_\nu \cdot B + E_\nu(B)) \tag{6.41}$$

with $\Gamma_\mu := [\Gamma^a_{b\mu}]$, $\widetilde{\Gamma}_\mu := [\widetilde{\Gamma}^a_{b\mu}]$, and $B := [B^a_b]$.

Thus, if we identify the 3-index coefficients of Δ^h, defined by (6.32), with the components of ∇, defined by (6.38), (such an identification is justified by the definition of ∇ via the parallel transport assigned to Δ^h (see Proposition 4.12) or via a projection, generated by Δ^h, of a suitable Lie derivative on $\mathfrak{X}(E)$ (see Definition 4.9)) then the quantities (6.35′) and (6.41) must coincide, which immediately leads to the equality (6.36). Therefore,

$$(e_\mu, e_a) \longmapsto (\widetilde{e}_\mu, \widetilde{e}_a) = (e_\nu, e_b) \cdot \left.\begin{bmatrix} B^\nu_\mu \circ \pi & 0 \\ ((B^\nu_\mu E_\nu(B^b_d)(B^{-1})^d_c) \circ \pi) E^c & B^b_a \circ \pi \end{bmatrix}\right|_{B=[B^b_a]} \tag{6.42}$$

is the most general transformation between frames in $T(E)$ such that the frames adapted to them are compatible with the linear connection and the covariant derivative corresponding to it. In particular, such are all frames $\{\partial/\partial u^I\}$ in $T(E)$ induced by vector bundle coordinates $\{u^I\}$ on E—see (4.34) and (3.1)–(3.3); the rest members of the class of frames mentioned are obtained from them via (6.42) with $e_I = \partial/\partial u^I$ and nondegenerate matrix-valued functions $[B^\nu_\mu]$ and B.

If $\{X_I\}$ (resp., $\{\widetilde{X}_I\}$) is the frame adapted to a frame $\{e_I\}$ (resp., $\{\widetilde{e}_I\}$), then the change $\{e_I\} \mapsto \{\widetilde{e}_I\}$, given by (6.42), entails $\{X_I\} \mapsto \{\widetilde{X}_I\}$ with $\{\widetilde{X}_I\}$ given by (6.29) (see (6.15) and (6.16)). Since the last transformation is tantamount to the change

$$(\{E_\mu\},\{E_a\}) \longmapsto (\{\widetilde{E}_\mu\},\{\widetilde{E}_a\}) \tag{6.43}$$

of the basis of $\mathscr{X}(M)\times \mathrm{Sec}(E,\pi,M)$ corresponding to $\{X_I\}$ via the isomorphism (6.27) (see (6.25), (6.29), and (6.30)), we can say that the transition (6.43) induces the change (6.40) of the 3-index coefficients of the connection Δ^h. Exactly the same is the situation one meets in the literature [19, 29, 40] when covariant derivatives are considered (and identified with connections).

Regardless that the change (6.42) of the frames in $T(E)$ looks quite special, it is the most general one that, through (6.16) and (6.25), is equivalent to an arbitrary change (6.43) of a basis in $\mathscr{X}(M)\times \mathrm{Sec}(E,\pi,M)$, that is, of a pair of frames in $T(M)$ and E.

We would like to remark that, in the general case, (4.59) also holds with $F = F^\mu E_\mu$, $G = G^\mu E_\mu$, and

$$(R(E_\mu,E_\nu))(E_b) = R^a_{b\mu\nu}E_a, \tag{6.44}$$

so that

$$R^a_{b\mu\nu} = E_\mu(\Gamma^a_{b\nu}) - E_\nu(\Gamma^a_{b\mu}) - \Gamma^c_{b\mu}\Gamma^a_{c\nu} + \Gamma^c_{b\nu}\Gamma^a_{c\mu} - \Gamma^a_{b\lambda}C^\lambda_{\mu\nu}, \tag{6.45}$$

where the functions $C^\lambda_{\mu\nu}$ define the anholonomicity object of $\{E_\mu\}$ via $[E_\mu,E_\nu]_- =: C^\lambda_{\mu\nu}E_\lambda$.

The above results, concerning linear connections on vector bundles, can be generalized for affine connections on vector bundles. For instance, the analogue of Proposition 6.11 reads.

Proposition 6.14. *Let Δ^h be an affine connection on a vector bundle (E,π,M) and let $\{X_\mu\}$ be the frame adapted for Δ^h to a frame $\{e_I\}$ such that $\{e_a\}$ is a basis for Δ^v and*

$$\begin{aligned}(e_\mu,e_a)\big|_U &= (\partial_\nu,\partial_b)\cdot\begin{bmatrix} B^\nu_\mu\circ\pi & 0\\ (B^b_{c\mu}\circ\pi)\cdot E^c & B^b_a\circ\pi\end{bmatrix}\\ &= \big((B^\nu_\mu\circ\pi)\partial_\nu + ((B^b_{c\mu}\circ\pi)\cdot E^c)\partial_b, (B^b_a\circ\pi)\partial_b\big),\end{aligned} \tag{6.46}$$

where $\partial_I := \partial/\partial u^I$ for some local bundle coordinate system $\{u^I\} = \{u^\mu = x^\mu\circ\pi, u^b = E^b\}$ on $U\subseteq E$, $[B^\nu_\mu]$ and $[B^b_a]$ are nondegenerate matrix-valued functions on U, $B^b_{c\mu}: U\to\mathbb{K}$, and $\{E^a\}$ is the coframe dual to $\{E_a = \nu^{-1}(X_a)\}$. Then the 2-index coefficients Γ^a_μ of Δ^h in $\{X_I\}$ have the representation (cf. (4.74))

$$\Gamma^a_\mu = -(\Gamma^a_{b\mu}\circ\pi)\cdot E^b + G^a_\mu\circ\pi, \tag{6.47}$$

on U for some functions $\Gamma^a_{b\mu}, G^a_\mu : U\to\mathbb{K}$.

Remark 6.15. The representation (6.47) is not valid for frames more general than the ones given by (6.46). Precisely, (6.47) is valid if and only if (6.46) holds for some local coordinate system $\{u^I\}$ on U—see (6.17).

Proof. Writing (6.17) for the transformation $\{\partial_I\}\mapsto\{e_I\}$, with $\{e_I\}$ given by (6.46), we get (6.47) with

$$\Gamma^a_{b\mu} = ([B^e_d]^{-1})^a_c({}^\partial\Gamma^c_{b\nu}B^\nu_\mu + B^c_{b\mu}), \qquad G^a_\mu = ([B^e_d]^{-1})^a_b\,{}^\partial G^b_\nu B^\nu_\mu, \tag{6.48}$$

where ${}^{\partial}\Gamma^a_{b\nu}$ and ${}^{\partial}G^b_\nu$ are defined via the 2-index coefficients ${}^{\partial}\Gamma^a_\mu$ of Δ^h in the frame adapted to the coordinates $\{u^I\}$ via ${}^{\partial}\Gamma^a_\mu = -({}^{\partial}\Gamma^a_{b\mu} \circ \pi) \cdot E^b + {}^{\partial}G^a_\mu \circ \pi$ (see Theorem 4.21). □

Let $\{X_I\}$ and $\{\widetilde{X}_I\}$ be frames adapted to $\{e_I\}$ and $\{\widetilde{e}_I\}$, respectively, with (cf. (6.46))

$$(\widetilde{e}_\mu, \widetilde{e}_a) = (e_\nu, e_b) \cdot \begin{bmatrix} B^\nu_\mu \circ \pi & 0 \\ (B^b_{c\mu} \circ \pi) \cdot E^c & B^b_a \circ \pi \end{bmatrix}, \tag{6.49}$$

in which (6.47) holds for Δ^h. Then, due to (6.17) and (6.47), the pairs $(\Gamma^a_{b\mu}, G^a_\mu)$ and $(\widetilde{\Gamma}^a_{b\mu}, \widetilde{G}^a_\mu)$ for Δ^h in, respectively, $\{X_I\}$ and $\{\widetilde{X}_I\}$ are connected by (cf. (4.75) and (4.76))

$$\widetilde{\Gamma}^a_{b\mu} = \left(\left[B^e_f\right]^{-1}\right)^a_c \left(\Gamma^c_{d\nu} B^\nu_\mu + B^c_{d\mu}\right) B^d_b, \tag{6.50a}$$

$$\widetilde{G}^a_\mu = \left(\left[B^e_f\right]^{-1}\right)^a_b G^b_\nu B^\nu_\mu. \tag{6.50b}$$

Exercise 6.16. Prove that the transformation $\{e_I\} \mapsto \{\widetilde{e}_I\}$, with $\{\widetilde{e}_I\}$ given by (6.49), is the most general one that preserves the existence of relation (6.47) for Δ^h provided it holds in $\{e_I\}$.

Further, one can repeat *mutatis mutandis* the text after Exercise 6.13 to the paragraph containing (6.43) including.

7. Conclusion

In this paper, we have presented a short (and partial) review of (one of the approaches to) the connection theory on bundles whose base and bundle spaces are (C^2) differentiable manifolds. Special attention was paid to connections, in particular linear ones, on vector bundles, which find wide applications in physics [7, 24]. However, many other approaches, generalizations, alternative descriptions, particular methods, and so forth were not mentioned at all. In particular, these include connections on more general (e.g., topological) bundles, connections on principal bundles (which are important in the gauge field theories), holonomy groups, flat connections, Riemannian connections, and so forth etc. The surveys [1, 23] contain essential information on these and many other items. Consistent and self-contained exposition of such problems can be found in [3, 9–11, 19, 20, 22, 25, 28, 29].

If additional geometric structures are added to the theory considered in Section 3, there will become important connections compatible with these structures. In this way many theories of particular connections arise; we have demonstrated that on the example of linear connections on vector bundles (Section 4). Here are two more such cases.

If a free right action $R : g \mapsto R_g : E \to E$, $g \in G$, of a Lie group G on the bundle space E of a bundle (E, π, M) is given and $\pi : E \to M = E/G$ is the canonical projection, we have a principal bundle (E, π, M, G). The connections that respect the right action R are the most important ones on principal bundles. Such a connection Δ^h is defined by Definition 3.1 to which the condition

$$(R_g)_*(\Delta^h_p) = \Delta^h_{R_g(p)}, \quad g \in G,\ p \in E, \tag{7.1}$$

is added and is called a principal connection. Alternatively, one can require the parallel transport P generated by Δ^h to commute with R, namely,

$$R_g \circ P^\gamma = P^\gamma \circ R_g, \quad g \in G,\ \gamma : [\sigma,\tau] \to M. \tag{7.2}$$

The theory of connections satisfying (7.1) is very well developed; see, for example, [9–11, 19].

Suppose a real bundle (E,π,M) is endowed with a bundle metric g, that is, $g : x \mapsto g_x$, $x \in M$, with $g_x : \pi^{-1}(x) \times \pi^{-1}(x) \to \mathbb{R}$ being bilinear and nondegenerate mapping for all $x \in M$. The equality

$$g_{\gamma(\sigma)} = g_{\gamma(\tau)} \circ \left(P^\gamma \times P^\gamma\right), \quad \gamma : [\sigma,\tau,] \longrightarrow M, \tag{7.3}$$

which expresses the preservation of the g-scalar products by the parallel transport P assigned to a connection Δ^h, specifies the class of g-compatible (metric-compatible) connections on (E,π,M). Such are the Riemannian connections on a Riemannian manifold M, which are g-compatible connections on the tangent bundle $(T(M),\pi_T,M)$; see, for instance, [19, 29].

The consideration of arbitrary (co)frames in Section 6 may seem slightly artificial as the general theory can be developed without them. However, this is not the generic case when one starts to apply the connection theory for investigation of particular problems. It may happen that some problem has solutions in general (co)frames but it does not possess solutions when (co)frames generated directly by local coordinates are involved. For example [16], local coordinates (holonomic frames) normal at a given point for a covariant derivative operator (linear connection) ∇ on a manifold exist if ∇ is torsionless at that point, but anholonomic frames normal at a given point for ∇ always exist.

Acknowledgment

This work was partially supported by the National Science Fund of Bulgaria under Grant no. F 1515/2005.

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