ANALYTIC SOLUTION OF CERTAIN SECOND-ORDER FUNCTIONAL DIFFERENTIAL EQUATION

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We consider the existence of analytic solutions of a certain class of iterative second-order functional differential equation of the form $x''(x^{[r]}(z)) = c_0z^2 + c_1(x(z))^2 + (c_2x^{[2]}(z))^2 + \cdots + c_m(x^{[m]}(z))^2$, $m, r \geq 0$.

1. Introduction

In recent years, the study of the existence of analytic solutions of iterative functional differential equations has attracted several researchers, see [2–11] and references cited therein. In [3], the authors studied the existence of analytic solutions of iterative functional differential equation of the following form:

$$x''(z) = \left(x^{[m]}(z)\right)^2, \tag{1.1}$$

where m is a nonnegative integer. In the present paper, we propose to study a more general form of iterative functional differential equations than (1.1) as follows:

$$x''\left(x^{[r]}(z)\right) = c_0z^2 + c_1\left(x(z)\right)^2 + c_2\left(x^{[2]}(z)\right)^2 + \cdots + c_m\left(x^{[m]}(z)\right)^2, \tag{1.2}$$

where r and m are nonnegative integers, $c_0, c_1, c_2, \dots, c_m$ are complex numbers, $\sum_{j=0}^{m} |c_j| \neq 0$, and $x^{[j]}$ denotes the jth iterate of x. In order to obtain analytic solutions of (1.1), we first seek the analytic solutions $y(z)$ of the following companion equation:

$$\alpha^2 y''\left(\alpha^{r+1}z\right)y'\left(\alpha^r z\right) = \alpha y'\left(\alpha^{r+1}z\right)y''\left(\alpha^r z\right) + \left[y'\left(\alpha^r z\right)\right]^3\left[\sum_{j=0}^{m} c_j\left(y\left(\alpha^j z\right)\right)^2\right] \tag{1.3}$$

satisfying the initial value conditions

$$y(0) = \mu, \qquad y'(0) = \eta \neq 0, \tag{1.4}$$

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where μ, η are complex numbers, and α satisfies one of the following conditions:
(H1) $|\alpha| > 1$;
(H2) $0 < |\alpha| < 1$;
(H3) $|\alpha| = 1$, α is not a root of unity, and $\log(1/|\alpha^n - 1|) \le K\log n$, $n = 2,3,4,\ldots,$
for some positive constant K. Then we show that (1.2) has an analytic solution of the form

$$x(z) = y(\alpha y^{-1}(z)), \tag{1.5}$$

in a neighborhood of the number μ, where $y^{-1}(z)$ is the inverse function of $y(z)$. Finally, we make use of (1.5) to show how to derive an explicit power series solution of (1.2).

2. Preliminary lemmas

We first obtain the analytic solutions $y(z)$ of the companion equation (1.3). By (1.3), we have that

$$\begin{aligned}
\frac{\alpha^2 y''(\alpha^{r+1}z)y'(\alpha^r z) - \alpha y'(\alpha^{r+1}z)y''(\alpha^r z)}{[y'(\alpha^r z)]^2} &= y'(\alpha^r z)\sum_{j=0}^{m} c_j(y(\alpha^j z))^2, \quad \text{or}\\
\frac{1}{\alpha^{r-1}}\left(\frac{y'(\alpha^{r+1}z)}{y'(\alpha^r z)}\right)' &= y'(\alpha^r z)\sum_{j=0}^{m} c_j(y(\alpha^j z))^2, \quad \text{or}\\
\frac{1}{\alpha^{r-1}}\left[\frac{y'(\alpha^{r+1}z)}{y'(\alpha^r z)} - \frac{y'(0)}{y'(0)}\right] &= \int_0^z y'(\alpha^r t)\sum_{j=0}^{m} c_j(y(\alpha^j t))^2 dt, \quad \text{or}\\
\frac{1}{\alpha^{r-1}}\left[\frac{y'(\alpha^{r+1}z)}{y'(\alpha^r z)} - 1\right] &= \int_0^z y'(\alpha^r t)\sum_{j=0}^{m} c_j(y(\alpha^j t))^2 dt, \quad \text{or}\\
\frac{1}{\alpha^{r-1}}[y'(\alpha^{r+1}z) - y'(\alpha^r z)] &= y'(\alpha^r z)\int_0^z y'(\alpha^r t)\sum_{j=0}^{m} c_j(y(\alpha^j t))^2 dt.
\end{aligned} \tag{2.1}$$

Since $y(z)$ is an analytic function in a neighborhood of 0, $y(z)$ can be represented by a power series of the form

$$y(z) = \sum_{n=0}^{+\infty} b_n z^n, \tag{2.2}$$

and we can see easily that $b_0 = \mu$, $b_1 = \eta$, and $y'(z) = \sum_{n=0}^{+\infty}(n+1)b_{n+1}z^n$. We have

$$\begin{aligned}
&\frac{1}{\alpha^{r-1}}[y'(\alpha^{r+1}z) - y'(\alpha^r z)]\\
&\quad = \frac{1}{\alpha^{r-1}}\left[\sum_{n=0}^{+\infty}(n+1)b_{n+1}\alpha^{(r+1)n}z^n - \sum_{n=0}^{+\infty}(n+1)b_{n+1}\alpha^{rn}z^n\right]\\
&\quad = \frac{1}{\alpha^{r-1}}\left[\sum_{n=0}^{+\infty}(n+1)b_{n+1}(\alpha^n - 1)\alpha^{rn}z^n\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\alpha^{r-1}}\left[\sum_{n=1}^{+\infty}(n+1)b_{n+1}(\alpha^{n}-1)\alpha^{rn}z^{n}\right] \\
&= \frac{1}{\alpha^{r-1}}\left[\sum_{n=0}^{+\infty}(n+2)b_{n+2}(\alpha^{n+1}-1)\alpha^{r(n+1)}z^{n+1}\right] \\
&= \sum_{n=0}^{+\infty}(n+2)(\alpha^{n+1}-1)\alpha^{rn+1}b_{n+2}z^{n+1}.
\end{aligned}
\tag{2.3}
$$

Therefore,

$$
\frac{1}{\alpha^{r-1}}\left[y'(\alpha^{r+1}z)-y'(\alpha^{r}z)\right] = \sum_{n=0}^{+\infty}(n+2)(\alpha^{n+1}-1)\alpha^{rn+1}b_{n+2}z^{n+1}. \tag{2.4}
$$

By means of (2.2), we get that

$$
y^{2}(z) = \left(\sum_{n=0}^{+\infty}b_{n}z^{n}\right)^{2} = \sum_{n=0}^{+\infty}\left(\sum_{i=0}^{n}b_{i}b_{n-i}\right)z^{n}. \tag{2.5}
$$

Then

$$
y^{2}(\alpha^{j}z) = \sum_{n=0}^{+\infty}\left(\sum_{i=0}^{n}b_{i}b_{n-i}\right)\alpha^{jn}z^{n}, \quad j = 0,1,2,\dots,m. \tag{2.6}
$$

This implies

$$
\begin{aligned}
&\int_{0}^{z} y'(\alpha^{r}t)\sum_{j=0}^{m}c_{j}\left(y(\alpha^{j}t)\right)^{2}dt \\
&\quad = \int_{0}^{z}\left(\sum_{n=0}^{+\infty}(n+1)b_{n+1}\alpha^{rn}t^{n}\right)\left(\sum_{n=0}^{+\infty}\sum_{i=0}^{n}\sum_{j=0}^{m}c_{j}\alpha^{jn}b_{i}b_{n-i}t^{n}\right)dt \\
&\quad = \int_{0}^{z}\sum_{n=0}^{+\infty}\sum_{k=0}^{n}\sum_{i=0}^{n-k}\sum_{j=0}^{m}c_{j}\alpha^{(n-k)j+r}(k+1)b_{i}b_{k+1}b_{n-k-i}t^{n}dt \\
&\quad = \sum_{n=0}^{+\infty}\left(\frac{\sum_{k=0}^{n}\sum_{i=0}^{n-k}\sum_{j=0}^{m}c_{j}\alpha^{(n-k)j+kr}}{n+1}(k+1)b_{i}b_{k+1}b_{n-k-i}\right)z^{n+1}.
\end{aligned}
\tag{2.7}
$$

Therefore,

$$
\int_{0}^{z} y'(\alpha^{r}t)\sum_{j=0}^{m}c_{j}\left(y(\alpha^{j}t)\right)^{2}dt = \sum_{n=0}^{+\infty}\frac{\sum_{k=0}^{n}\sum_{i=0}^{n-k}\sum_{j=0}^{m}c_{j}\alpha^{(n-k)j+(k+1)r}}{n+1}(k+1)b_{i}b_{k+1}b_{n-k-i}z^{n+1}. \tag{2.8}
$$

Next, we will consider

$$
\begin{aligned}
& y'(\alpha^r z)\int_0^z y'(\alpha^r t)\sum_{j=0}^{m} c_j\left(y(\alpha^j t)\right)^2 dt \\
&\quad = \left(\sum_{n=0}^{+\infty}(n+1)b_{n+1}\alpha^{rn}z^n\right)\sum_{n=0}^{+\infty}\frac{\sum_{k=0}^{n}\sum_{i=0}^{n-k}\sum_{j=0}^{m}c_j\alpha^{(n-k)j+kr}}{n+1}(k+1)b_i b_{k+1}b_{n-k-i}z^{n+1} \\
&\quad = \sum_{n=0}^{+\infty}\frac{\sum_{s=0}^{n}\sum_{k=0}^{n-s}\sum_{i=0}^{n-k-s}\sum_{j=0}^{m}c_j\alpha^{(n-s-k)j+(k+s)r}}{n-s+1}(s+1)(k+1)b_i b_{s+1}b_{k+1}b_{n-s-k-i}z^{n+1}.
\end{aligned}
\tag{2.9}
$$

Therefore,

$$
\begin{aligned}
& y'(\alpha^r z)\int_0^z y'(\alpha^r t)\sum_{j=0}^{m} c_j\left(y(\alpha^j t)\right)^2 dt \\
&\quad = \sum_{n=0}^{+\infty}\frac{\sum_{s=0}^{n}\sum_{k=0}^{n-s}\sum_{i=0}^{n-k-s}\sum_{j=0}^{m}c_j\alpha^{(n-s-k)j+(k+s)r}}{n-s+1}(s+1)(k+1)b_i b_{s+1}b_{k+1}b_{n-s-k-i}z^{n+1}.
\end{aligned}
\tag{2.10}
$$

We see that (1.3) is equivalent to the integrodifferential equation (2.1). By (2.1), (2.4), and (2.10), we see that

$$
\begin{aligned}
(n+2)(\alpha^{n+1}-1)\alpha^{rn+1}b_{n+2} &= \frac{\sum_{s=0}^{n}\sum_{k=0}^{n-s}\sum_{i=0}^{n-k-s}\sum_{j=0}^{m}c_j\alpha^{(n-s-k)j+(k+s)r}}{n-s+1} \\
&\quad \times(s+1)(k+1)b_i b_{s+1}b_{k+1}b_{n-s-k-i}, \quad n=0,1,2\ldots.
\end{aligned}
\tag{2.11}
$$

Therefore,

$$
b_{n+2} = \frac{\sum_{s=0}^{n}\sum_{k=0}^{n-s}\sum_{i=0}^{n-k-s}\sum_{j=0}^{m}c_j\alpha^{(n-s-k)j+(k+s)r-rn-1}}{(n+2)(n-s+1)(\alpha^{n+1}-1)}(s+1)(k+1)b_i b_{s+1}b_{k+1}b_{n-s-k-i},
\tag{2.12}
$$

where $n = 0,1,2\ldots$. Next, we show that such a power series solution is majorized by a convergent power series. Now we begin with the following preparatory lemma, the proof of which can be found in [1, Chapter 6].

Lemma 2.1. *Assume that (H3) holds. Then there is a positive number δ such that $|\alpha^n - 1|^{-1} < (2n)^\delta$ for $n = 1,2,3,\ldots$. Furthermore, the sequence $\{d_n\}_{n=1}^{\infty}$ defined by $d_1 = 1$ and $d_n = (1/|\alpha^{n-1}-1|)\max_{n=n_1+n_2+\cdots+n_t, 0<n_1\le n_2\le\cdots\le n_t, t\ge 2}\{d_{n_1}d_{n_2}\cdots d_{n_t}\}$, $n = 2,3,4,\ldots$ satisfy $d_n \le (2^{5\delta+1})^{n-1}n^{-2\delta}$, $n = 1,2,3,\ldots$.*

Lemma 2.2. *Suppose that (H3) holds. Then, when $0 < |\eta| \le 1$, (1.3) has an analytic solution of the form (2.2) in a neighborhood of the origin.*

Proof. For convenience, we let $M = \sum_{j=0}^{m} |c_j|$. By means of (2.12), it follows that for each $n = 0, 1, 2, \ldots$,

$$\begin{aligned} |b_{n+2}| &\le \frac{\sum_{s=0}^{n}\sum_{k=0}^{n-s}\sum_{i=0}^{n-k-s}\sum_{j=0}^{m}|c_j|\,|\alpha|^{(n-s-k)j+(k+s)r-rn-1}}{(n+2)(n-s+1)\,|(\alpha^{n+1}-1)|} \\ &\quad \times (s+1)(k+1)\,|b_i|\,|b_{s+1}|\,|b_{k+1}|\,|b_{n-s-k-i}| \\ &\le \frac{M}{|(\alpha^{n+1}-1)|}\sum_{s=0}^{n}\sum_{k=0}^{n-s}\sum_{i=0}^{n-k-s}|b_i|\,|b_{s+1}|\,|b_{k+1}|\,|b_{n-s-k-i}|. \end{aligned} \tag{2.13}$$

Therefore,

$$|b_{n+2}| \le \frac{M}{|(\alpha^{n+1}-1)|}\sum_{s=0}^{n}\sum_{k=0}^{n-s}\sum_{i=0}^{n-k-s}|b_i|\,|b_{s+1}|\,|b_{k+1}|\,|b_{n-s-k-i}|, \tag{2.14}$$

where $n = 0, 1, \ldots$. Let

$$Q(z,\omega) = \omega^4 - 2|\mu|\omega^3 + |\mu|^2\omega^2 - \frac{1}{M}(\omega - |\mu| - z) \tag{2.15}$$

for (z,ω) in a neighbor of $(0,|\mu|)$. We see that $Q(0,|\mu|) = |\mu|^4 - 2|\mu|^4 + |\mu|^4 - (1/M)(|\mu| - |\mu| - 0) = 0$ and $Q'_\omega(z,\omega) = 4\omega^3 - 6|\mu|\omega^2 + 2|\mu|^2\omega - 1/M$, so $Q'_\omega(0,|\mu|) = -1/M \neq 0$. Therefore, there exists a unique analytic function $G(z)$ in a neighborhood of 0 such that $G(0) = |\mu|$, $G'(0) = 1$ satisfy the equality $Q(z, G(z)) = 0$. It follows that

$$G(z) = \sum_{n=0}^{+\infty} C_n z^n, \tag{2.16}$$

where $C_0 = |\mu|$, $C_1 = 1$ in a neighborhood of 0. Next, we will show that

$$C_{n+2} = M\sum_{s=0}^{n}\sum_{k=0}^{n-s}\sum_{i=0}^{n-k-s} C_i C_{s+1} C_{k+1} C_{n-s-k-i}, \quad n = 0, 1, \ldots. \tag{2.17}$$

Suppose that (2.17) is true, by (2.16), we will get that

$$\begin{aligned} G^3(z) = G(z)G^2(z) &= \left(C_0 + \sum_{n=0}^{+\infty} C_{n+1}z^{n+1}\right)\left(\sum_{n=0}^{+\infty}\left(\sum_{i=0}^{n} C_i C_{n-i}\right)z^n\right) \\ &= C_0\sum_{n=0}^{+\infty}\left(\sum_{i=0}^{n} C_i C_{n-i})z^n + \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n}\sum_{i=0}^{n-k} C_i C_{k+1} C_{n-k-i}\right)z^{n+1}, \\ G^4(z) = G(z)G^3(z) &= \left(C_0 + \sum_{n=0}^{+\infty} C_{n+1}z^{n+1}\right) \\ &\quad \times \left[C_0\sum_{n=0}^{+\infty}\left(\sum_{i=0}^{n} C_i C_{n-i}\right)z^n + \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n}\sum_{i=0}^{n-k} C_i C_{k+1} C_{n-k-i}\right)z^{n+1}\right] \end{aligned}$$

$$\begin{aligned}
&= C_0^2 \sum_{n=0}^{+\infty} \left(\sum_{i=0}^{n} C_i C_{n-i} \right) z^n + 2C_0 \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} \sum_{i=0}^{n-k} C_i C_{k+1} C_{n-k-i} \right) z^{n+1} \\
&\quad + \sum_{n=0}^{+\infty} \left(M \sum_{s=0}^{n} \sum_{k=0}^{n-s} \sum_{i=0}^{n-k-s} C_i C_{s+1} C_{k+1} C_{n-s-k-i} \right) z^{n+2} \\
&= C_0^2 G^2(z) + 2C_0 [G^3(z) - C_0 G^2(z)] + \frac{1}{M} \sum_{n=0}^{+\infty} C_{n+2} z^{n+2} \\
&= 2C_0 G^3(z) - C_0^2 G^2(z) + \frac{1}{M} (G(z) - C_0 - C_1 z) \\
&= 2|\mu| G^3(z) - |\mu|^2 G^2(z) + \frac{1}{M} (G(z) - |\mu| - z),
\end{aligned} \tag{2.18}$$

that is,

$$G^4(z) = 2|\mu| G^3(z) - |\mu|^2 G^2(z) + \frac{1}{M} (G(z) - |\mu| - z), \tag{2.19}$$

or

$$G^4(z) - 2|\mu| G^3(z) + |\mu|^2 G^2(z) - \frac{1}{M} (G(z) - |\mu| - z) = 0. \tag{2.20}$$

Hence, $Q(z, G(z)) = 0$. Furthermore, we see that $Q(z, G(z)) = 0$ if and only if (2.17) is true. Therefore, we conclude that (2.17) holds. Now, we know that the power series (2.16) converges in a neighborhood of 0. Therefore, there exists a positive constant P such that

$$C_n < P^n \tag{2.21}$$

for $n = 1, 2, 3, \ldots$. In the following lemma, we show that $|b_n| \le C_n d_n$, $n = 1, 2, \ldots$, where the sequence $\{d_n\}_{n=1}^{\infty}$ is defined as in Lemma 2.1. Indeed, $|b_1| = |\eta| \le 1 = C_1 d_1$, so it suffices to prove that $|b_{n+1}| \le C_{n+1} d_{n+1}$, $n = 1, 2, \ldots$. Let $P(n)$ denote the statement that $|b_{n+1}| \le C_{n+1} d_{n+1}$. From (2.14) and (2.17), we obtain

$$\begin{aligned}
|b_2| &\le \left(\sum_{j=0}^{m} |c_j| \right) |\alpha - 1|^{-1} |b_0| |b_1| |b_1| |b_0| \\
&\le M |\alpha - 1|^{-1} C_0 C_1 d_1 C_1 d_1 C_0 \\
&= (M C_0 C_1 C_1 C_0) (|\alpha - 1|^{-1} d_1 d_1) \\
&= C_2 |\alpha - 1|^{-1} \max_{\substack{n_1 + n_2 = 2 \\ 0 < n_1 \le n_2}} \{ d_{n_1} d_{n_2} \} = C_2 d_2.
\end{aligned} \tag{2.22}$$

Thus, $P(2)$ is true. Next, suppose that $P(1), P(2), \ldots, P(n)$ are true, that is, $|b_{s+1}| \le C_{s+1} d_{s+1}$, for all $s = 1, 2, \ldots, n$. By (2.14) and (2.17), we get that

$$|b_{n+2}| \le \frac{M}{|(\alpha^{n+1} - 1)|} \sum_{s=0}^{n} \sum_{k=0}^{n-s} \sum_{i=0}^{n-k-s} |b_i| |b_{k+1}| |b_{s+1}| |b_{n-s-k-i}|$$

$$
\begin{aligned}
&= \frac{M}{|(\alpha^{n+1}-1)|}\sum_{s=0}^{n}\sum_{k=0}^{n-s}\Bigg(|b_0||b_{k+1}||b_{s+1}||b_{n-s-k}| \\
&\qquad + |b_{n-s-k}||b_{k+1}||b_{s+1}||b_0| \\
&\qquad + \sum_{i=1}^{n-k-s-1}|b_i||b_{k+1}||b_{s+1}||b_{n-s-k-i}|\Bigg) \\
&= \frac{M}{|(\alpha^{n+1}-1)|}\sum_{s=0}^{n}\sum_{k=0}^{n-s}\Bigg(2|b_0||b_{k+1}||b_{s+1}||b_{n-s-k}| \\
&\qquad + \sum_{i=1}^{n-k-s-1}|b_i||b_{k+1}||b_{s+1}||b_{n-s-k-i}|\Bigg) \\
&= \frac{M}{|(\alpha^{n+1}-1)|}\Bigg(\sum_{s=0}^{n}\sum_{k=0}^{n-s}2|b_0||b_{k+1}||b_{s+1}||b_{n-s-k}| \\
&\qquad + \sum_{s=0}^{n}\sum_{k=0}^{n-s}\sum_{i=1}^{n-k-s-1}|b_i||b_{k+1}||b_{s+1}||b_{n-s-k-i}|\Bigg) \\
&= \frac{M}{|(\alpha^{n+1}-1)|}\Bigg[\sum_{s=0}^{n}\Bigg(2|b_0||b_{n-s+1}||b_{s+1}||b_0| \\
&\qquad + \sum_{k=0}^{n-s-1}2|b_0||b_{k+1}||b_{s+1}||b_{n-s-k}|\Bigg) \\
&\qquad + \sum_{s=0}^{n}\sum_{k=0}^{n-s}\sum_{i=1}^{n-k-s-1}|b_i||b_{k+1}||b_{s+1}||b_{n-s-k-i}|\Bigg] \\
&= \frac{M}{|(\alpha^{n+1}-1)|}\Bigg[\sum_{s=0}^{n}2|b_0||b_{n-s+1}||b_{s+1}||b_0| \\
&\qquad + \sum_{s=0}^{n}\sum_{k=0}^{n-s-1}2|b_0||b_{k+1}||b_{s+1}||b_{n-s-k}| \\
&\qquad + \sum_{s=0}^{n}\sum_{k=0}^{n-s}\sum_{i=1}^{n-k-s-1}|b_i||b_{k+1}||b_{s+1}||b_{n-s-k-i}|\Bigg] \\
&\leq \frac{M}{|(\alpha^{n+1}-1)|}\Bigg[\sum_{s=0}^{n}2C_0^2C_{n-s+1}d_{n-s+1}C_{s+1}d_{s+1} \\
&\qquad + \sum_{s=0}^{n}\sum_{k=0}^{n-s-1}2C_0C_{k+1}d_{k+1}C_{s+1}d_{s+1}C_{n-s-k}d_{n-s-k} \\
&\qquad + \sum_{s=0}^{n}\sum_{k=0}^{n-s}\sum_{i=1}^{n-k-s-1}C_id_iC_{k+1}d_{k+1}C_{s+1}d_{s+1}C_{n-s-k-i}d_{n-s-k-i}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{M}{|(\alpha^{n+1}-1)|}\Bigg[\sum_{s=0}^{n} 2C_0^2 C_{n-s+1}C_{s+1}d_{n-s+1}d_{s+1} \\
&\qquad + \sum_{s=0}^{n}\sum_{k=0}^{n-s-1} 2C_0C_{k+1}C_{s+1}C_{n-s-k}d_{k+1}d_{s+1}d_{n-s-k} \\
&\qquad + \sum_{s=0}^{n}\sum_{k=0}^{n-s}\sum_{i=1}^{n-k-s-1} C_iC_{k+1}C_{s+1}C_{n-s-k-i}d_id_{k+1}d_{s+1}d_{n-s-k-i}\Bigg] \\
&\le \frac{M}{|(\alpha^{n+1}-1)|}\max_{\substack{n_1+n_2+\cdots+n_t=n+2\\ 0<n_1\le n_2\le\cdots\le n_t,\,t\ge 2}}\{d_{n_1}d_{n_2}\cdots d_{n_t}\} \\
&\quad\times\Bigg[\sum_{s=0}^{n} 2C_0^2C_{n-s+1}C_{s+1} + \sum_{s=0}^{n}\sum_{k=0}^{n-s-1} 2C_0C_{k+1}C_{s+1}C_{n-s-k} \\
&\quad + \sum_{s=0}^{n}\sum_{k=0}^{n-s}\sum_{i=1}^{n-k-s-1} C_iC_{k+1}C_{s+1}C_{n-s-k-i}\Bigg] \\
&= d_{n+2}C_{n+2}.
\end{aligned}
\tag{2.23}
$$

Therefore, $P(n+1)$ is true, we conclude that $|b_n| \le C_n d_n$, for all $n = 1,2,3,\ldots$. In view of (2.21) and Lemma 2.1, we see that

$$
|b_n| \le P^n\left(2^{5\delta+1}\right)^{n-1} n^{-2\delta}, \quad n = 1,2,3,\ldots. \tag{2.24}
$$

Therefore,

$$
\begin{aligned}
\limsup |b_n|^{1/n} &\le \limsup P\left(2^{5\delta+1}\right)^{(n-1)/n} n^{-2\delta/n} \\
&= \lim P\left(2^{5\delta+1}\right)^{(n-1)/n} n^{-2\delta/n} = P2^{5\delta+1}.
\end{aligned}
\tag{2.25}
$$

Thus, $1/\limsup|b_n|^{1/n} \ge 1/P2^{5\delta+1}$, which shows that power series (2.2) converges for $|z| < 1/P2^{5\delta+1}$. The proof is complete. □

Lemma 2.3. *Suppose that (H1) holds. Then for any $r \ge m$, (1.3) has an analytic solution of the form (2.2) in a neighborhood of* 0.

Proof. For $r \ge m$, $0 \le k+s \le n$, we have $s+1 \le n+1$, and $k+1 \le n-s+1$, it follows that $(s+1)/(n+1) \le 1$ and $(k+1)/(n-s+1) \le 1$. Next, we have

$$
\begin{aligned}
&(k+s+1)r + j(n-s-k) - rn \\
&\qquad = (k+s)r + r - (k+s)j + jn - rn \\
&\qquad = (k+s)(r-j) - n(r-j) + r \\
&\qquad = (k+s-n)(r-j) + r, \quad \text{so}
\end{aligned}
\tag{2.26}
$$

$$
(k+s+1)r + j(n-s-k) - rn = (k+s-n)(r-j) + r.
$$

Since $|\alpha|>1$, $|\alpha|^{(k+s+1)r+j(n-s-k)-rn}=|\alpha|^{(k+s-n)(r-j)+r}=|\alpha|^{(k+s-n)(r-j)}|\alpha|^{r}\le|\alpha|^{r}$ and the sequence

$$\left\{\frac{|\alpha|^{r-1}\sum_{j=0}^{m}|c_j|}{|\alpha|^{n+1}-1}\right\}_{n=1}^{\infty} \tag{2.27}$$

converges to 0, this sequence is bounded, namely, there exists $M>0$ such that

$$\frac{|\alpha|^{r-1}\sum_{j=0}^{m}|c_j|}{|\alpha|^{n+1}-1}\le M,\quad \forall n=1,2,3,\dots. \tag{2.28}$$

Therefore,

$$\begin{aligned}&\left|\frac{(s+1)(k+1)\sum_{j=0}^{m}|c_j|\,|\alpha|^{(k+s+1)r+j(n-s-k)-rn-1}}{(n+2)(n-s+1)(\alpha^{n+1}-1)}\right|\\&\quad\le\frac{|\alpha|^{r-1}\sum_{j=0}^{m}|c_j|}{|\alpha^{n+1}-1|}\le\frac{|\alpha|^{r-1}\sum_{j=0}^{m}|c_j|}{|\alpha|^{n+1}-1}\le M,\quad \forall n=1,2,3,\dots.\end{aligned} \tag{2.29}$$

In view of (2.10), we get that

$$|b_{n+2}|\le M\sum_{s=0}^{n}\sum_{k=0}^{n-s}\sum_{i=0}^{n-k-s}|b_i|b_{s+1}|b_{k+1}|\,|b_{n-s-k-i}|,\quad \forall n=0,1,2,\dots. \tag{2.30}$$

We define a sequence $\{D_n\}_{n=0}^{\infty}$ by $D_0=|\mu|$, $D_1=|\eta|$ and

$$D_{n+2}=M\sum_{s=0}^{n}\sum_{k=0}^{n-s}\sum_{i=0}^{n-k-s}D_iD_{s+1}D_{k+1}D_{n-s-k-i},\quad \forall n=0,1,2,\dots. \tag{2.31}$$

Next, we will show that $|b_{n+1}|\le D_{n+1}$, $n=1,2,3,\dots$. By definition of D_n, we have $|b_0|\le D_0$, $|b_1|\le D_1$ and we let $P(n)$ denote the statement that $|b_{n+1}|\le D_{n+1}$. Then

$$\begin{aligned}|b_2|&\le M\sum_{s=0}^{0}\sum_{k=0}^{0-s}\sum_{i=0}^{0-k-s}|b_i|b_{s+1}|b_{k+1}|\,|b_{0-s-k-i}|\\&=M|b_0|\,|b_1|\,|b_1|\,|b_0|=M|b_0|^2|b_1|^2=D_2.\end{aligned} \tag{2.32}$$

Therefore, $P(1)$ is true. Next, suppose that $P(1),P(2),\dots,P(n)$ are true, so $|b_{t+1}|\le D_{t+1}$, for $t=1,2,3,\dots,n$. Therefore,

$$\begin{aligned}|b_{n+2}|&\le M\sum_{s=0}^{n}\sum_{k=0}^{n-s}\sum_{i=0}^{n-k-s}|b_i|b_{s+1}|b_{k+1}|\,|b_{n-s-k-i}|\\&\le M\sum_{s=0}^{n}\sum_{k=0}^{n-s}\sum_{i=0}^{n-k-s}D_iD_{s+1}D_{k+1}D_{n-s-k-i}=D_{n+2}.\end{aligned} \tag{2.33}$$

Hence, $P(n+1)$ is true, so we can conclude that $|b_n| \le D_n$, for $n = 0,1,2,\ldots$. Now, if we define

$$G(z) = \sum_{n=0}^{+\infty} D_n z^n, \tag{2.34}$$

then

$$\begin{aligned}
G^3(z) = G(z)G^2(z) &= \left(D_0 + \sum_{n=0}^{+\infty} D_{n+1} z^{n+1}\right)\left(\sum_{n=0}^{+\infty}\left(\sum_{i=0}^{n} D_i D_{n-i}\right) z^n\right) \\
&= D_0 \sum_{n=0}^{+\infty}\left(\sum_{i=0}^{n} D_i D_{n-i}\right) z^n + \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n}\sum_{i=0}^{n-k} D_i D_{k+1} D_{n-k-i}\right) z^{n+1}, \\
G^4(z) = G(z)G^3(z) &= \left(D_0 + \sum_{n=0}^{+\infty} D_{n+1} z^{n+1}\right) \\
&\quad \times \left[D_0 \sum_{n=0}^{+\infty}\left(\sum_{i=0}^{n} D_i D_{n-i}\right) z^n + \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n}\sum_{i=0}^{n-k} D_i D_{k+1} D_{n-k-i}\right) z^{n+1}\right] \\
[5pt] &= D_0^2 \sum_{n=0}^{+\infty}\left(\sum_{i=0}^{n} D_i D_{n-i}\right) z^n + 2D_0 \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n}\sum_{i=0}^{n-k} D_i D_{k+1} D_{n-k-i}\right) z^{n+1} \\
&\quad + \sum_{n=0}^{+\infty}\left(M \sum_{s=0}^{n}\sum_{k=0}^{n-s}\sum_{i=0}^{n-k-s} D_i D_{s+1} D_{k+1} D_{n-s-k-i}\right) z^{n+2} \\
&= D_0^2 G^2(z) + 2D_0\left[G^3(z) - D_0 G^2(z)\right] + \frac{1}{M}\sum_{n=0}^{+\infty} D_{n+2} z^{n+2} \\
&= 2D_0 G^3(z) - D_0^2 G^2(z) + \frac{1}{M}\left(G(z) - D_0 - D_1 z\right) \\
&= 2|\mu| G^3(z) - |\mu|^2 G^2(z) + \frac{1}{M}\left(G(z) - |\mu| - |\eta| z\right).
\end{aligned} \tag{2.35}$$

Thus,

$$G^4(z) - 2|\mu| G^3(z) + |\mu|^2 G^2(z) - \frac{1}{M}\left(G(z) - |\mu| - |\eta| z\right) = 0. \tag{2.36}$$

Let

$$R(z,\omega) = \omega^4 - 2|\mu|\omega^3 + |\mu|^2\omega^2 - \frac{1}{M}\left(\omega - |\mu| - |\eta| z\right) \tag{2.37}$$

for (z,ω) in a neighborhood of $(0,|\mu|)$, so we see that $R(0,\mu) = |\mu|^4 - 2|\mu|^4 + |\mu|^4 - (1/M)(|\mu| - |\mu| - |\eta|0) = 0$ and $R'_\omega(z,\omega) = 4\omega^3 - 6|\mu|\omega^2 + 2|\mu|^2\omega - 1/M$, then $R'_\omega(0, |\mu|) = -1/M \ne 0$. Therefore, there exists a unique function $\omega(z)$ which is analytic in a neighborhood of 0 such that $\omega(0) = |\mu|$, $\omega'(0) = |\eta|$ and satisfies $R(z,\omega(z)) = 0$. According to (2.34) and (2.36), we have $G(z) = \omega(z)$. It follows that the power series (2.34) converges in a neighborhood of 0, which implies that the power series (2.2) is also convergent in a neighborhood of 0. The proof is complete. □

Lemma 2.4. *Suppose that (H2) holds. Then for either* $0 < r \le m$ *and* $c_0 = 0, c_1 = 0, \dots, c_{r-1} = 0$, *or* $r = 0$, *(1.3) has an analytic solution of the from (2.2) in a neighborhood of* 0.

Proof. By assumption, we get that

$$\left\{ \frac{|\alpha|^{-1}\sum_{j=0}^{m}|c_j|}{1-|\alpha|^{n+1}} \right\}_{n=1}^{+\infty} \tag{2.38}$$

converges to $|\alpha|^{-1}\sum_{j=0}^{m}|c_j|$, so it is a bounded sequence which implies that there exists $M > 0$ such that

$$\frac{|\alpha|^{-1}\sum_{j=0}^{m}|c_j|}{1-|\alpha|^{n+1}} \le M, \quad \forall n = 1,2,3,\dots. \tag{2.39}$$

There are two cases to consider as follows.

Case 1. $r = 0$. As $0 \le k+s \le n$, we have

$$\begin{aligned} &\left| \frac{(s+1)(k+1)\sum_{j=0}^{m} c_j \alpha^{(n-s-k)j+(k+s+1)r-rn-1}}{(n+2)(n-s+1)(\alpha^{n+1}-1)} \right| \\ &\quad = \left| \frac{(s+1)(k+1)\sum_{j=0}^{m} c_j \alpha^{(n-s-k)j-1}}{(n+2)(n-s+1)(\alpha^{n+1}-1)} \right| \le \frac{|\alpha|^{-1}\sum_{j=0}^{m}|c_j|}{1-|\alpha|^{n+1}}. \end{aligned} \tag{2.40}$$

Case 2. $0 < r \le m$ and $c_0 = 0, c_1 = 0, \dots, c_{r-1} = 0$. We see that $|\alpha|^{(r-j)(k+s-n)} \le 1$, where $r \le j \le m$. Then,

$$\begin{aligned} |\alpha|^{(n-s-k)j+(k+s+1)r-rn-1} &= |\alpha|^{(r-j)(k+s-n)+r-1} \\ &= |\alpha|^{(r-j)(k+s-n)}|\alpha|^{r}|\alpha|^{-1} \le |\alpha|^{-1}. \end{aligned} \tag{2.41}$$

Thus,

$$|\alpha|^{(n-s-k)j+(k+s+1)r-rn-1} \le |\alpha|^{-1}. \tag{2.42}$$

Next, we consider

$$\begin{aligned} &\left| \frac{(s+1)(k+1)\sum_{j=0}^{m} c_j \alpha^{(n-s-k)j+(k+s+1)r-rn-1}}{(n+2)(n-s+1)(\alpha^{n+1}-1)} \right| \\ &\quad \le \frac{\sum_{j=0}^{m}|c_j||\alpha|^{(n-s-k)j+(k+s+1)r-rn-1}}{1-|\alpha|^{n+1}} \le \frac{|\alpha|^{-1}\sum_{j=0}^{m}|c_j|}{1-|\alpha|^{n+1}}. \end{aligned} \tag{2.43}$$

Therefore, by both cases, we have

$$\left|\frac{(s+1)(k+1)\sum_{j=0}^{m} c_j\alpha^{(n-s-k)j+(k+s+1)r-rn-1}}{(n+2)(n-s+1)\left(\alpha^{n+1}-1\right)}\right| \leq \frac{|\alpha|^{-1}\sum_{j=0}^{m}|c_j|}{1-|\alpha|^{n+1}}. \tag{2.44}$$

It follows that

$$\left|\frac{(s+1)(k+1)\sum_{j=0}^{m} c_j\alpha^{(n-s-k)j+(k+s+1)r-rn-1}}{(n+2)(n-s+1)\left(\alpha^{n+1}-1\right)}\right| \leq M \quad \text{for } n=1,2,3,\dots. \tag{2.45}$$

The conclusion of Lemma 2.4 now follows easily from the same argument as in the proof of Lemma 2.3. □

3. Main results

We now state the main result of this paper. Consider the following three hypotheses:

(i) (H3) holds;

(ii) (H1) holds, and $r \geq m$;

(iii) (H2) holds, and either $0 < r \leq m$ and $c_0 = 0, c_1 = 0, \dots, c_{r-1} = 0$, or $r = 0$.

Theorem 3.1. *Suppose one of the conditions (i), (ii), or (iii) is fulfilled. Then, for any μ, (1.2) has an analytic solution $x(z)$ in a neighborhood of μ satisfying the initial conditions $x(\mu) = \mu$, $x'(\mu) = \alpha$. This solution has the form $x(z) = y(\alpha y^{-1}(z))$, where $y(z)$ is an analytic solution of the initial value problem (1.3)-(1.4).*

Proof. In view of Lemmas 2.2–2.4, we may find a sequence $\{b_n\}_{n=2}^{\infty}$ such that the function $y(z)$ of the form (2.2) is an analytic solution of (1.3) in a neighborhood of 0. Since $y'(0) = \eta \neq 0$, the function $y^{-1}(z)$ is analytic in a neighborhood of the $y(0) = \mu$. If we define $x(z)$ by means of (1.5), then

$$\begin{aligned} x''\left(x^{[r]}(z)\right) &= x''\left(y\left(\alpha^r y^{-1}(z)\right)\right) \\ &= \frac{\alpha^2 y''\left(\alpha^{r+1}y^{-1}(z)\right)y'\left(\alpha^r y^{-1}(z)\right) - \alpha y'\left(\alpha^{r+1}y^{-1}(z)\right)y''\left(\alpha^r y^{-1}(z)\right)}{\left[y'\left(\alpha^r y^{-1}(z)\right)\right]^3} \\ &= \sum_{j=0}^{m} c_j\left(y\left(\alpha^j y^{-1}(z)\right)\right)^2, \quad \text{by (1.3)}, \\ &= \sum_{j=0}^{m} c_j\left(x^{[j]}(z)\right)^2, \quad \text{as required.} \end{aligned} \tag{3.1}$$

The proof is complete. □

We now show how to explicitly construct an analytic solution of (1.2). Since $x(z) = y(\alpha y^{-1}(z))$, $x(\mu) = y(\alpha y^{-1}(\mu)) = y(0) = \mu$. By Theorem 3.1, $x(z)$ is an analytic function in a neighborhood of μ. Thus $x(z)$ can be written in a neighborhood of μ as

$$x(z) = \mu + x'(\mu)(z-\mu) + \frac{x''(\mu)(z-\mu)^2}{2!} + \frac{x'''(\mu)(z-\mu)^3}{3!} + \cdots. \tag{3.2}$$

Next, we will determine the derivatives $x^{(n)}(\mu)$, $n=1,2,\dots$. We have $x(z)=y(\alpha y^{-1}(z))$, so that $x'(z)=\alpha y'(\alpha y^{-1}(z))/y'(y^{-1}(z))$. That is, $x'(\mu)=\alpha y'(\alpha y^{-1}(\mu))/y'(y^{-1}(\mu))=\alpha y'(0)/y'(0)=\alpha$. Hence $x'(\mu)=\alpha$. By means of (1.2), we get that

$$x''(\mu)=x''\left(x^{[r]}(\mu)\right)=\sum_{j=0}^{m}c_j\left(x^{[j]}(\mu)\right)^2=\mu^2\sum_{j=0}^{m}c_j; \tag{3.3}$$

hence $x''(\mu)=\mu^2\sum_{j=0}^{m}c_j$. Next, we have

$$\begin{aligned}\left(x''\left(x^{[r]}(z)\right)\right)'&=x'''\left(x^{[r]}(z)\right)\left(x^{[r]}(z)\right)'\\&=x'''\left(x^{[r]}(z)\right)x'\left(x^{[r-1]}(z)\right)x'\left(x^{[r-2]}(z)\right)\cdots x'\left(x(z)\right)x'(z).\end{aligned} \tag{3.4}$$

Therefore, the derivative of $(x''(x^{[r]}(z)))$ at $z=\mu$ is

$$\begin{gathered}x'''\left(x^{[r]}(\mu)\right)x'\left(x^{[r-1]}(\mu)\right)x'\left(x^{[r-2]}(\mu)\right)\cdots x'\left(x(\mu)\right)x'(\mu)=x'''(\mu)\left[x'(\mu)\right]^r=x'''(\mu)\alpha^r,\\ \begin{aligned}\left(\sum_{j=0}^{m}c_j\left(x^{[j]}(z)\right)^2\right)'&=\sum_{j=0}^{m}c_j\left(\left(x^{[j]}(z)\right)^2\right)'=2\sum_{j=0}^{m}c_jx^{[j]}(z)\left(x^{[j]}(z)\right)'\\&=2\sum_{j=0}^{m}c_jx^{[j]}(z)x'\left(x^{[j-1]}(z)\right)x'\left(x^{[j-2]}(z)\right)\cdots x'\left(x(z)\right)x'(z).\end{aligned}\end{gathered} \tag{3.5}$$

Hence, the first derivative of $(\sum_{j=0}^{m}c_j(x^{[j]}(z))^2)$ at $z=\mu$ is $2\mu\sum_{j=0}^{m}c_j\alpha^j$. Next, by taking the first derivative of (1.2) at $z=\mu$, we get that

$$x'''(\mu)\alpha^r=2\mu\sum_{j=0}^{m}c_j\alpha^j. \tag{3.6}$$

Thus,

$$x'''(\mu)=2\mu\sum_{j=0}^{m}c_j\alpha^{j-r}. \tag{3.7}$$

In general, we can show by induction that

$$\begin{aligned}\left(x''\left(x^{[r]}(z)\right)\right)^{(n+1)}&=\left(\left(x^{[r]}(z)\right)'\right)^{n+1}x^{(n+3)}\left(x^{[r]}(z)\right)\\&\quad+\sum_{k=1}^{n}\left[P_{k,n+1}\left(\left(x^{[r]}(z)\right)',\left(x^{[r]}(z)\right)'',\dots,\left(x^{[r]}(z)\right)^{(n+1)}\right)\right]x^{(k+2)}\left(x^{[r]}(z)\right),\end{aligned} \tag{3.8}$$

for $n=1,2,\dots$, and

$$\left(x^{[j]}(z)\right)^{(l)}=Q_{jl}\left(x_{10}(z),\dots,x_{1,j-1}(z);\dots;x_{l0}(z),\dots,x_{l,j-1}(z)\right), \tag{3.9}$$

respectively, where $x_{ij}(z)=x^{(i)}(x^{[j]}(z))$, P_{jk} and Q_{jl} are polynomials with nonnegative coefficients. Next, we have

$$\begin{aligned}\left(\sum_{j=0}^{m}c_j\left(\left(x^{[j]}(z)\right)^2\right)\right)^{(n+1)}&=\sum_{j=0}^{m}c_j\left(\left(x^{[j]}(z)\right)^2\right)^{(n+1)}\\&=2\sum_{j=0}^{m}c_j\left(x^{[j]}(z)\right)\left(\left(x^{[j]}(z)\right)'\right)^{(n)}\\&=2\sum_{j=0}^{m}c_j\left(\sum_{k=0}^{n}C_k^n\left(x^{[j]}(z)\right)^{(k)}\left(x^{[j]}(z)\right)^{(n-k+1)}\right)\\&=2\sum_{j=0}^{m}\sum_{k=0}^{n}c_jC_k^n\left(x^{[j]}(z)\right)^{(k)}\left(x^{[j]}(z)\right)^{(n-k+1)},\quad n=1,2,\ldots.\end{aligned}\tag{3.10}$$

For convenience, we denote the following notations:

$$\beta_{jl}=Q_{jl}\left(x'(\mu),\ldots,x'(\mu);\ldots;x^{(j)}(\mu),\ldots,x^{(j)}(\mu)\right),\tag{3.11}$$

where the number of repeats of $x^{(t)}(\mu)$ is l, for $t=1,2,\ldots,j$. Then, we see that $\beta_{lj}=(x^{[j]}(\mu))^{(l)}$. By differentiating (1.1) for $n+1$ times at $z=\mu$, we get

$$\begin{aligned}&\left(\left(x^{[r]}(\mu)\right)'\right)^{n+1}x^{(n+3)}\left(x^{[r]}(\mu)\right)\\&\qquad+\sum_{k=1}^{n}\left[P_{k,n+1}\left(\left(x^{[r]}(\mu)\right)',\left(x^{[r]}(\mu)\right)'',\ldots,\left(x^{[r]}(\mu)\right)^{(n+1)}\right)\right]x^{(k+2)}\left(x^{[r]}(\mu)\right)\\&\quad=2\sum_{j=0}^{m}\sum_{k=0}^{n}c_jC_k^n\left(x^{[j]}(\mu)\right)^{(k)}\left(x^{[j]}(\mu)\right)^{(n-k+1)}.\end{aligned}\tag{3.12}$$

Thus,

$$\alpha^{r(n+1)}x^{(n+3)}(\mu)+\sum_{k=1}^{n}\left[P_{k,n+1}\left(\beta_{1r},\beta_{2r},\ldots,\beta_{n+1,r}\right)\right]x^{(k+2)}(\mu)=2\sum_{j=0}^{m}\sum_{k=0}^{n}c_jC_k^n\beta_{kj}\beta_{n-k+1,j}.\tag{3.13}$$

This shows that

$$x^{(n+3)}(\mu)=\frac{2\sum_{j=0}^{m}\sum_{k=0}^{n}c_jC_k^n\beta_{kj}\beta_{n-k+1,j}-\sum_{k=1}^{n}\left[P_{k,n+1}\left(\beta_{1r},\beta_{2r},\ldots,\beta_{n+1,r}\right)\right]x^{(k+2)}(\mu)}{\alpha^{r(n+1)}},\tag{3.14}$$

where $n = 1, 2, \ldots$. By means of this formula, it is then easy to write out the explicit form of our solution $x(z)$ as follows:

$$\begin{aligned} x(z) = \mu + \alpha(z-\mu) + \frac{\mu^2}{2!}\sum_{j=0}^{m} c_j (z-\mu)^2 + \frac{2\mu}{3!}\sum_{j=0}^{m} c_j \alpha^{j-r}(z-\mu)^3 \\ + \sum_{n=1}^{+\infty} \frac{1}{(n+3)!} x^{(n+3)}(\mu)(z-\mu)^{n+3}. \end{aligned} \tag{3.15}$$

Example 3.2. The following example shows how to construct an analytic solution by using the previous argument. Consider the following functional equation:

$$x''(x(z)) = x^2(z) + \left(x^{[2]}(z)\right)^2. \tag{3.16}$$

This is just (1.2) with the choice of $r = 1$, $m = 2$, $c_0 = 1$, $c_1 = 1$, and $c_2 = 1$. We can easily see that (3.16) satisfies condition (iii) of Theorem 3.1; hence, for any complex numbers μ and α such that $0 < |\alpha| < 1$, (3.16) has an analytic solution $x(z)$ in a neighborhood of μ which satisfies $x(\mu) = \mu$ and $x'(\mu) = \alpha$. This analytic solution has the form as in (3.2) in case $r = 1$, $m = 2$, $c_0 = 1$, $c_1 = 1$, and $c_2 = 1$. We already know that $x(\mu) = \mu$ and $x'(\mu) = \alpha$. We will find $x^{(n)}(\mu)$, $n \geq 2$. For $n = 2$, it follows from (3.16) that

$$x''(\mu) = x''(x(\mu)) = x^2(\mu) + \left(x^{[2]}(\mu)\right)^2 = 2\mu^2. \tag{3.17}$$

For $n = 3$, it follows from (3.16) that

$$x''(x(z))' = \left(x^2(z)\right)' + \left(\left(x^{[2]}(z)\right)^2\right)'. \tag{3.18}$$

Thus,

$$\begin{aligned} x'''(x(z))x'(z) &= 2x(z)x'(z) + 2x^{[2]}(z)x'(x(z))x'(z) \\ &= 2x'(z)\left[x(z) + x^{[2]}(z)x'(x(z))\right]. \end{aligned} \tag{3.19}$$

By putting $z = \mu$, we obtain

$$x'''(\mu)\alpha = 2\alpha[\mu + \mu\alpha], \tag{3.20}$$

which gives

$$x'''(\mu) = 2(1+\alpha)\mu. \tag{3.21}$$

Similarly, for $n = 4$, we obtain

$$x^{(4)}(\mu) = 2\left(1 + 2\mu^3 + \alpha^2\right) + \frac{4(1+\mu)\mu^3}{\alpha^2}. \tag{3.22}$$

By continuing this process, we obtain an analytic solution of (3.16) as

$$x(z) = \mu + \alpha(z-\mu) + \mu^2(z-\mu)^2 + \frac{(1+\alpha)}{3}(z-\mu)^3 + \left(\frac{1+2\mu^3+\alpha^2}{12} + \frac{(1+\mu)\mu^3}{6\alpha^2}\right)(z-\mu)^4 + \cdots. \tag{3.23}$$

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