FINITE RANK INTERMEDIATE HANKEL OPERATORS AND THE BIG HANKEL OPERATOR

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Let L_a^2 be a Bergman space. We are interested in an intermediate Hankel operator H_ϕ^M from L_a^2 to a closed subspace M of L^2 which is invariant under the multiplication by the coordinate function z. It is well known that there do not exist any nonzero finite rank big Hankel operators, but we are studying same types in case H_ϕ^M is close to big Hankel operator. As a result, we give a necessary and sufficient condition about M that there does not exist a finite rank H_ϕ^M except $H_\phi^M = 0$.

Let D be the open unit disc in $\mathbb{C}$ and let dA be the normalized area measure on D. When $dA = r\,dr\,d\theta/\pi$, let $L^2 = L^2(D, dA)$ be the Lebesgue space on the open unit disc D and let $L_a^2 = L^2 \cap \mathrm{Hol}(D)$ be a Bergman space on D. When M is the closed subspace of L^2 and $zM \subseteq M$, M is called an invariant subspace. Suppose that $zL_a^2 \subseteq M$. P^M denotes the orthogonal projection from L^2 onto M. For ϕ in L^∞, the intermediate Hankel operator H_ϕ^M is defined by

$$H_\phi^M f = (I - P^M)(\phi f) \quad (f \in L_a^2). \tag{1}$$

When $M = L_a^2$, H_ϕ^M is called a big Hankel operator and when $M = (\bar{z}\bar{L_a^2})^\perp$, H_ϕ^M is called small Hankel operator. L^2 has the following orthogonal decomposition:

$$L^2 = \sum_{j=-\infty}^{\infty} \oplus \mathcal{L}^2 e^{ij\theta}, \tag{2}$$

where $\mathcal{L}^p = L^p([0,1), 2r\,dr)$ for $1 \le p \le \infty$. Set

$$\mathbf{H}^2 = \sum_{j=0}^{\infty} \oplus \mathcal{L}^2 e^{ij\theta}, \tag{3}$$

then $L_a^2 \subset \mathbf{H}^2 \subset (\bar{z}\bar{L_a^2})^\perp$.

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For an invariant subspace M, set

$$M_j = \left\{ f_j \in \mathscr{L}^2;\ f \in M,\ f(z) = \sum_{j=-\infty}^{\infty} f_j(r)e^{ij\theta} \right\}. \tag{4}$$

We call $\{M_j\}_{j=-\infty}^{\infty}$ the Fourier coefficients of M and then $rM_j \subseteq M_{j+1}$. If $M_j e^{ij\theta}$ belongs to M for any j, then M has the following decomposition:

$$M = \sum_{j=-\infty}^{\infty} \oplus M_j e^{ij\theta}. \tag{5}$$

When $M \subseteq \mathbf{H}^2$, H_ϕ^M is close to big Hankel operator. In this case, we give a necessary and sufficient condition about M that there does not exist a finite rank H_ϕ^M except $H_\phi^M = 0$.

The following lemma is proved in previous paper [1].

Lemma 1. *Suppose M is an invariant subspace which contains zL_a^2, and ϕ is a function in L^∞. H_ϕ^M is of finite rank $\le \ell$ if and only if ϕ belongs to $M^{\infty,\ell}$, where*

$$M^{\infty,\ell} = \left\{ \phi \in L^\infty;\ b\phi(z) \in M,\ b(z) = \sum_{j=0}^{\ell} b_j z^j \text{ and } b_j \in \mathbb{C} \right\}. \tag{6}$$

Note that we have proved in the previous paper [1, Theorem 5.4(1)] only when $k = 0$. We improve [1, Theorem 5.4]. That is the following theorem.

Theorem 2. *Suppose M is an invariant subspace between zL_a^2 and $e^{-ik\theta}\mathbf{H}^2$ where $k \ge 0$, and $\phi = \sum_{j=1+k}^{\infty} \phi_{-j}(r)e^{-ij\theta}$ is a function in L^∞. Then there does not exist any finite rank H_ϕ^M except for $H_\phi^M = 0$ if and only if $M_{-(k-j)} \cap r^{j+1}\mathscr{L}^\infty = \{0\}$ for any $j \ge 0$.*

Proof.

$$\int \sum_{j=1+k}^{\infty} \phi_{-j}(r)e^{-i(j-m)\theta}\frac{d\theta}{2\pi} = \phi_{-m}(r) \quad (1+k \le m \le \infty), \tag{7}$$

and so

$$\begin{aligned} |\phi_{-m}(r)| &\le \int \left| \sum_{j=1+k}^{\infty} \phi_{-j}(r)e^{-i(j-m)\theta} \right| \frac{d\theta}{2\pi} \\ &= \int |e^{-im\theta}| \left| \sum_{j=1+k}^{\infty} \phi_{-j}(r)e^{-i(j-m)\theta} \right| \frac{d\theta}{2\pi} = \int |\phi| \frac{d\theta}{2\pi} < \infty. \end{aligned} \tag{8}$$

Hence $\phi_{-m}(r) \in \mathscr{L}^\infty$ for $1+k \le m \le \infty$. If $r(H_\phi^M) \le \ell(<\infty)$ by Lemma 1 then there exist complex numbers $b_0,\dots,b_\ell$ such that $b_\ell = 1$, $b = \sum_{j=0}^{\ell} b_j z^j$:

$$b\phi = \sum_{n=-\infty}^{\ell-(1+k)} \left(\sum_{m=0}^{\ell} b_m r^m \phi_{n-m}(r) \right) e^{in\theta} \in M. \tag{9}$$

Since $M \subseteq e^{-ik\theta}\mathbf{H}^2$,

$$b\phi = \sum_{n=-k}^{\ell-(1+k)} \left(\sum_{m=0}^{\ell} b_m r^m \phi_{n-m}(r) \right) e^{in\theta} \in M \quad (-(\ell+k) \le n-m \le \ell-(1+k)). \tag{10}$$

Since $M = \sum_{j=-\infty}^{\infty} \oplus M_j e^{ij\theta}$, by (10),

$$\sum_{m=0}^{\ell} b_m r^m \phi_{n-m}(r) \in M_n \quad (-k \le n \le \ell-(1+k)). \tag{11}$$

As $n = \ell-(1+k)$,

$$\sum_{m=0}^{\ell} b_m r^m \phi_{n-m}(r) = r^\ell \phi_{-(1+k)}(r) \in M_{\ell-(1+k)}. \tag{12}$$

If $\phi_{-(1+k)}(r) \neq 0$, then $M_{\ell-(1+k)} \cap r^\ell \mathscr{L}^\infty \neq \{0\}$. So we assume $\phi_{-(1+k)}(r) = 0$. As $n = \ell - (2+k)$,

$$\sum_{m=0}^{\ell} b_m r^m \phi_{n-m}(r) = r^\ell \phi_{-(2+k)}(r) \in M_{\ell-(2+k)}. \tag{13}$$

If $\phi_{-(2+k)}(r) \neq 0$, then $M_{\ell-(2+k)} \cap r^\ell \mathscr{L}^\infty \neq \{0\}$ and so $M_{\ell-(2+k)} \cap r^{\ell-1} \mathscr{L}^\infty \neq \{0\}$. So we assume $\phi_{-(2+k)}(r) = 0$. Repeating the same way from $n = \ell-(3+k)$ to $n = \ell-(\ell-1+k)$, we can get $\phi_{-(3+k)}(r) = \cdots = \phi_{-(\ell-1+k)}(r) = 0$. As $n = -k$,

$$\sum_{m=0}^{\ell} b_m r^m \phi_{n-m}(r) = r^\ell \phi_{-(\ell+k)}(r) \in M_{-k}. \tag{14}$$

If $\phi_{-(\ell+k)}(r) \neq 0$, then $M_{-k} \cap r^\ell \mathscr{L}^\infty \neq \{0\}$ and so $M_{-k} \cap r\mathscr{L}^\infty \neq \{0\}$. If $\phi_{-(\ell+k)}(r) = 0$, then $\phi_{-(1+k)}(r) = \phi_{-(2+k)}(r) = \cdots = \phi_{-(\ell+k)}(r) = 0$ and $\phi = 0$ by (10). This result contradicts $H_\phi^M \neq 0$, and so $M_{j-k} \cap r^{j+1}\mathscr{L}^\infty \neq \{0\}$ for $j \ge 0$.

If $r^{j+1} f \in M_{j-k} \cap r^{j+1}\mathscr{L}^\infty (f \in \mathscr{L}^\infty)$, then put $\phi = f e^{-i(k+1)\theta} \in L^\infty$. If $f \neq 0$, then $\phi \notin M$ and

$$z^{j+1}\phi = r^{j+1} f e^{i(j-k)\theta} \in M_{j-k} e^{i(j-k)\theta}. \tag{15}$$

Since $M = \sum_{j=-\infty}^{\infty} \oplus M_j e^{ij\theta}$, $M_{j-k} e^{i(j-k)\theta} \subseteq M$ and so $z^{j+1}\phi \in M$. Lemma 1 gives a contradiction. □

References

[1] T. Nakazi and T. Osawa, *Finite-rank intermediate Hankel operators on the Bergman space*, International Journal of Mathematics and Mathematical Sciences **25** (2001), no. 1, 19–31.

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