ON THE CONVERGENCE OF A NEWTON-LIKE METHOD IN $\mathbb{R}^n$ AND THE USE OF BERINDE'S EXIT CRITERION

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Berinde has shown that Newton's method for a scalar equation $f(x) = 0$ converges under some conditions involving only f and f' and not f'' when a generalized stopping inequality is valid. Later Sen et al. have extended Berinde's theorem to the case where the condition that $f'(x) \neq 0$ need not necessarily be true. In this paper we have extended Berinde's theorem to the class of n-dimensional equations, $F(x) = 0$, where $F : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. We have also assumed that $F'(x)$ has an inverse not necessarily at every point in the domain of definition of F.

1. Introduction

Let F be a nonlinear continuous operator mapping $D_0 \subset \mathbb{R}^n \to \mathbb{R}^n$. D_0 is an open convex subset of $\mathbb{R}^n$, the n-dimensional Euclidean space. We introduce componentwise partial ordering in $\mathbb{R}^n$.

Componentwise partial ordering in $\mathbb{R}^n$ is defined as follows. For $x, y \in \mathbb{R}^n$, $x \leq y$ if and only if $x_i \leq y_i$, $i = 1, 2, \dots, n$. Let $\langle a, b\rangle$ denote the order interval $\{x \in \mathbb{R}^n \mid a \leq x \leq b\}$.

We seek the solution of

$$F(x) = 0 \quad \text{in } \langle a, b\rangle \subset D_0. \tag{1.1}$$

In case the finite derivative $F'(x)$ has an inverse at the iteration points, Newton's method is given by the iteration

$$x_{m+1} = x_m - \left[F'(x_m)\right]^{-1} F(x_m), \quad x_0 \in \langle a, b\rangle,\ m \geq 0. \tag{1.2}$$

The importance of Newton's method lies in the fact that it offers a quadratic convergence. Nevertheless this quadratic convergence is achieved after making certain assumptions about F, F', F'' (Ortega and Rheinboldt [6]). In case $F(x) = 0$ is a scalar equation, Berinde (Berinde [1–4]) without making any assumption about the existence of $F''(x)$

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has achieved linear convergence under the condition that $F'(x) \neq 0$ in the interval under consideration. In the case of a scalar equation we have not assumed the conditions that $F'(x) \neq 0$ (Sen et al. [9]). It has been shown that the sequence $\{x_m\}$ given by

$$x_{m+1} = x_m - \frac{2F(x_m)}{F'(x_m) + M'_m}, \quad x_0 \text{ prechosen}, \ m = 0, 1, 2, \ldots, \tag{1.3}$$

is convergent. In the above expression, $M = \sup_{x\in[a,b]} |F'(x)|$, $M'_m = M \cdot \operatorname{Sign} F'(x_m)$.

In case $F : \mathbb{R}^n \to \mathbb{R}^n$ we proceed as follows. Let F be differentiable in $D_0 \subseteq \mathbb{R}^n$ and let $F'(x) = (a_{ij}(x))$.

Let

$$A = (C_{ij}), \quad \text{where } 0 \geq C_{ij} = -\sup_x |a_{ij}(x)|, \ i \neq j, \ C_{ii} \geq \sup_x \sum_{j=1}^{n} |a_{ij}(x)|. \tag{1.4}$$

We can show that under certain conditions $A + F'(x)$ has an inverse even if $F'(x)$ may not have an inverse. The Newton-like iterative sequence $\{x_m\}$ is given by

$$x_{m+1} = x_m - 2\left[A + F'(x_m)\right]^{-1} F(x_m), \quad x_0 \in \langle a, b\rangle, \ m = 0, 1, 2, \ldots. \tag{1.5}$$

We show that under certain assumptions $\{x_m\}$ given by (1.2) converges to a solution of $F(x) = 0$. The exit criterion has been established. Section 2 presents the mathematical preliminaries. Section 3 contains the convergence theorem, Section 4 an extension of Newton's method, Section 5 contains a numerical example, and Section 6 the discussion.

2. Preliminaries

By componentwise partial ordering in $\mathbb{R}^n$, we mean for $x, y \in \mathbb{R}^n$,

$$\begin{gathered} x < y \Longleftrightarrow x_i < y_i, \quad \forall i \\ x \leq y \Longleftrightarrow x_i \leq y_i, \quad \forall i, \text{ but } x \neq y, \\ x \leqq y \Longleftrightarrow x_i \leq y_i. \end{gathered} \tag{2.1}$$

Let $L(\mathbb{R}^n, \mathbb{R}^n)$ be the class of matrices which maps $\mathbb{R}^n \to \mathbb{R}^n$.

Let $P \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $P = (\tilde{a}_{ij})$.

$$\begin{gathered} P > 0 \Longleftrightarrow \tilde{a}_{ij} > 0, \quad \forall i, j, \\ P \geqq 0 \Longleftrightarrow \tilde{a}_{ij} \geq 0, \quad \forall i, j, \\ P \geq 0 \Longleftrightarrow \tilde{a}_{ij} \geq 0, \quad \forall i, j, \text{ but } P \neq 0. \end{gathered} \tag{2.2}$$

Definition 2.1 (M-matrix, a Stieltjes matrix). A matrix $Q = (q_{ij}) \in L(\mathbb{R}^n, \mathbb{R}^n)$ is said to be an M-matrix if $q_{ij} \leq 0$, for all i, j, $i \neq j$. Q^{-1} exists and $Q^{-1} \geq 0$ (Ortega and Rheinboldt [6]). A symmetric M-matrix is called a Stieltjes matrix.

Definition 2.2 (spectral radius). A spectral radius of a matrix H is denoted by $\rho(H)$ and $\rho(H) = \sup_i |\lambda_i|$, where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of H.

THEOREM 2.3 (Ortega and Rheinboldt [6]). *$A = (C_{ij})$ is an M-matrix if and only if* (i) *the diagonal elements of A are positive, and* (ii) *the matrix $H = I - D^{-1}A$ where $D = \operatorname{diag}(a_{11},\dots,a_{nn})$ satisfies $\rho(H) < 1$.*

LEMMA 2.4. *The following results hold:*
(i) *$A = (C_{ij})$ is an M-matrix,*
(ii) *$[A + F'(x)]$ is an M-matrix, provided the condition*

$$\sup_x\left(\sum_{k=1}^{n}|a_{ik}(x)|\right) + a_{ii}(x) > \sum_{\substack{j=1\\ j\neq i}}^{n}\left(\sup_x|a_{ij}(x)| - a_{ij}(x)\right) \tag{2.3}$$

holds.

Proof. (i) Let D denote the diagonal matrix (C_{ii}). Then $D > 0$. Moreover $I - D^{-1}A$ is a matrix with zero diagonal elements and (i,j)th element $= -C_{ij}/C_{ii}$, $i \neq j$.

Therefore $\|I - D^{-1}A\|_{l_1} = \sup_i(\sum_{j,i\neq j}|C_{ij}|/C_{ii}) < 1$, where $\|\cdot\|_{l_1}$ denotes l_1-norm.

Thus $\rho(I - D^{-1}A)$, the spectral radius (Ortega and Rheinboldt [6]) of $(I - D^{-1}A)$, is less than 1. Hence A is an M-matrix.

(ii) We write $F'(x) = D_1(x) - B_1(x)$ where the diagonal matrix $D_1(x) = (a_{ii}(x))$ and the matrix $B_1(x) = (b_{ij}(x))$, where $b_{ii}(x) = 0$ and $b_{ij}(x) = -a_{ij}(x)$, for $i \neq j$.

Let $\widetilde{A} = A + F'(x) = D_2(x) - B_2(x)$, where the diagonal matrix is given by $D_2(x) = (C_{ii} + a_{ii}(x))$, $C_{ii} + a_{ii}(x) > 0$, and the matrix $B_2(x) = (b'_{ij}(x))$, where $b'_{ij}(x) = (-C_{ij} - a_{ij}(x))$, $b'_{ii} = 0$. Moreover, $b'_{ij}(x) = \sup_x|a_{ij}(x)| - a_{ij}(x) \geqq 0$.

Let $I - D_2(x)^{-1}\widetilde{A} = (e_{ij}(x))$, where $e_{ii}(x) = 0$ and

$$e_{ij}(x) = \frac{-C_{ij} - a_{ij}(x)}{C_{ii} + a_{ii}(x)} \leq \frac{\sup_x|a_{ij}(x)| - a_{ij}(x)}{\sup_x\sum_{k=1}^{n}|a_{ik}(x)| + a_{ii}(x)}, \quad i \neq j. \tag{2.4}$$

Therefore,

$$\begin{aligned} ||I - D_2(x)^{-1}\widetilde{A}||_{l_1} &= \sup_i\sum_{\substack{j=1\\ j\neq i}}^{n}|e_{ij}(x)| = \sup_i\frac{\sum_{\substack{j\\ j\neq i}}|C_{ij} + a_{ij}(x)|}{|C_{ii} + a_{ii}(x)|} \\ &\leq \sup_i\frac{\sum_{\substack{j\\ j\neq i}}(\sup_x|a_{ij}(x)| - a_{ij}(x))}{\sup_x\sum_{k=1}^{n}|a_{ik}(x)| + a_{ii}(x)} < 1 \end{aligned} \tag{2.5}$$

provided that

$$\sup_x\sum_{k=1}^{n}|a_{ik}(x)| + a_{ii}(x) > \sum_{\substack{j=1\\ j\neq i}}^{n}(\sup|a_{ij}(x)| - a_{ij}(x)), \quad \forall i. \tag{2.6}$$

Thus $\rho(I - D_2(x)^{-1}\widetilde{A}) < 1$ provided the relation (2.6) holds for all i, j. □

3. Convergence

THEOREM 3.1. *Let the following conditions be fulfilled.*

(i) $F : D_0 \subset \mathbb{R}^n \to \mathbb{R}^n$.

(ii) D_0 *is an open convex subset* $\subseteq \mathbb{R}^n$.

(iii) F *is continuously differentiable in* D_0.

(iv) F *has a solution in the order-interval* $\langle a, b\rangle \subseteq D_0$.

(v) $x_0 \in \langle a, b\rangle$ *is an initial approximation to the solution.*

(vi) $A = (C_{ij})$ *is a matrix defined by (1.4).*

(vii) $[A + F'(x)]$ *is a Stieltjes matrix.*

(viii) $F'(x)$ *is symmetric for each* x *and* $F'(x) \geq 0$.

(ix) *The eigenspaces of* $[A + F'(x)]^{-1}$ *and of* $F'(x)$, *respectively, have nonempty intersection.*

(x) $\rho(F'(x)) < \rho([A + F'(x)]^{-1})^{-1}$, *where* ρ *stands for the spectral radius.*

(xi) $\rho(C(x)) < 1$ *and* $\rho(C(x))(1 - \rho(C(x))) < 1/2$, *where*

$$C(x) = [A + F'(x)]^{-1} F'(x + t(x^* - x)), \quad 0 < t < 1. \tag{3.1}$$

Then $\{x_m\}$ *given by*

$$x_{m+1} = x_m - 2[A + F'(x_m)]^{-1} F(x_m) \tag{3.2}$$

will converge to a solution x^* *of* $F(x) = 0$, *provided that* $\{x_m\} \subseteq \langle a, b\rangle$.

Proof. It follows from (3.2) and the application of mean-value theorem in $\mathbb{R}^n$ that

$$\begin{aligned} x_{m+1} - x^* &= x_m - x^* - 2[A + F'(x_m)]^{-1}(F(x_m) - F(x^*)) \\ &= \int_0^1 \left[I - 2[A + F'(x_m)]^{-1} F'(x_m + t(x^* - x_m))\right] \cdot (x_m - x^*)\, dt \end{aligned} \tag{3.3}$$

or

$$||x_{m+1} - x^*|| \leq \sup_{0<t<1} \left|\left|I - 2[A + F'(x_m)]^{-1} F'(x_m + t(x^* - x_m))\right|\right| ||x_m - x^*||. \tag{3.4}$$

Since $F'(x) \geq 0$, $F'(x_m + t(x^* - x_m)) \geq 0$.

Also, $[A + F'(x)]$ being a Stieltjes matrix is both symmetric and an M-matrix, $[A + F'(x)]^{-1} \geq 0$. Therefore,

$$C(x) = [A + F'(x)]^{-1} F'(x + t(x^* - x)) \geq 0. \tag{3.5}$$

Let $B(x) = [I - 2[A + F'(x)]^{-1} F'(x + t(x^* - x))]$.

Since $C(x) \geq 0$, by the Perron-Frobenius theorem (Varga [12]), for a given x, $C(x)$ has an eigenvalue $\lambda(x) = \rho(C(x))$, with ρ being the spectral radius. Therefore, for a given x,

$B(x)$ has an eigenvalue $\mu(x) = 1 - 2\lambda'(x)$, where $\lambda'(x)$ is an eigenvalue of $C(x)$. If we use Euclidean norm of a vector and then matrix norm induced by the vector norm,

$$\|B\| = \left[\rho(B^TB)\right]^{1/2}, \quad \rho(B^TB) = \sup|\mu^*\mu| = \sup_{\lambda'}|(1-2\lambda')^*(1-2\lambda')| < 1 \tag{3.6}$$

if $|1 - 4\operatorname{Re}\lambda' + 4|\lambda'|^2| < 1$ for all λ', or, if $|\lambda'|^2 < \operatorname{Re}\lambda' < 1/2 + |\lambda'|^2$.

Thus $\|B\| < 1$ if

$$\rho(C(x))^2 < \rho(C(x)) < \frac{1}{2} + \rho(C(x))^2. \tag{3.7}$$

The left-hand inequality in (3.7) yields $\rho(C(x)) < 1$, and the right-hand inequality in (3.7) yields $\rho(C(x))(1 - \rho(C(x))) < 1/2$ always true.

Thus by condition (xi) $\|B\| < 1$.

If

$$\|B\| = \alpha < 1, \qquad ||x_{m+1} - x^*|| \le \alpha||x_m - x^*||. \tag{3.8}$$

□

Remark 3.2. The sequence $\{x_m\}$ given by (3.2) is convergent if conditions (ii)–(iv) of Theorem 3.1 are satisfied, if $[A+F'(x)]$ is an M-matrix and $\|I-2[A+F'(x_m)]^{-1}F'(x)\|<1$.

The following theorem determines the stopping inequality.

Theorem 3.3. *Let the conditions of Theorem 3.1 be true. Then the numerical computation of the sequence $\{x_m\}$ given by (3.2) is stopped when the following inequality is valid:*

$$||x_m - x^*|| \le ||[I + A^{-1}F'(x_m)]|| \|A\| \cdot \left(\sup_{0<t<1}||F'(x_m + t(x^* - x_m))||\right)^{-1} ||x_{m+1} - x_m||. \tag{3.9}$$

Proof. Equation (3.2) can be written as

$$\begin{aligned} ||x_{m+1} - x_m|| &= 2\left|\left|[A + F'(x_m)]^{-1}[F(x_m) - F(x^*)]\right|\right| \\ &= \sup_{0<t<1}\left|\left|\int_0^1 [A + F'(x_m)]^{-1}F'(x_m + t(x^* - x_m)) \cdot (x^* - x_m)\,dt\right|\right| \\ &\le \left|\left|[A + F'(x_m)]^{-1}\right|\right| \sup_{0<t<1}||F'(x_m + t(x^* - x_m))|| \cdot ||x^* - x_m||. \end{aligned} \tag{3.10}$$

Since A and $A + F'(x_m)$ are both M-matrices, $[I + A^{-1}F'(x_m)]^{-1}$ exists, and therefore $\|([I + A^{-1}F'(x_m)]^{-1})^{-1}\| \le \|I + A^{-1}F'(x_m)\|$.

Thus,

$$
\begin{aligned}
\|x^* - x_m\| &\leq \left\| \left([A + F'(x_m)]^{-1} \right)^{-1} \right\| \left(\sup_{0<t<1} \|F'(x_m + t(x^* - x_m))\| \right)^{-1} \|x^* - x_m\| \\
&\leq \|I + A^{-1}F'(x_m)\| \|A\| \left(\sup_{0<t<1} \|F'(x_m + t(x^* - x_m))\| \right)^{-1} \cdot \|x_{m+1} - x_m\|.
\end{aligned}
\tag{3.11}
$$

□

Note 3.4. The inequality (3.11) may be termed as "the exit criterion" because if $\|x_{m+1} - x_m\| < \varepsilon$, ε a small positive quantity, then $\|x^* - x_m\| \leq C_m\varepsilon$, where

$$
C_m = \|[I + A^{-1}F'(x_m)]\| \|A\| \left(\sup_{0<t<1} \|F'(x_m + t(x^* - x_m))\| \right)^{-1}. \tag{3.12}
$$

4. The extension of Newton's method

However $\langle a, b\rangle$ is generally not an invariant set with respect to iterations (3.2); that is, it is possible to obtain a certain p such that $x_p \notin \langle a, b\rangle$. In case $x_p < a$ or $x_p > b$, the mapping $F(x)$ is extended throughout $\mathbb{R}^n$ in the light of Berinde's extension, and the sequence $\{x_m\}$ given by (3.2) is extended throughout $\mathbb{R}^n$.

Theorem 4.1. *Let the following conditions be fulfilled:*

(i) $F(a) \leq 0$,

(ii) *$F(x)$ is differentiable at a and A is an M-matrix (Ortega and Rheinboldt [6]),*

(iii) $F(b) \geq 0$,

(iv) *$F(x)$ is differentiable at b.*

Then if x_p goes out of $\langle a, b\rangle$, x_{p+1} will lie in $\langle a, b\rangle$.

Extend $F(x)$ throughout $\mathbb{R}^n$ as follows:

$$
\widetilde{F}(x) = \begin{cases} A(x-a) + F(a) & x \leq a, \\ F(x) & x \in \langle a, b\rangle, \\ A(x-b) + F(b) & x \geq b. \end{cases} \tag{4.1}
$$

Proof. If some iteration x_p does not lie in $\langle a, b\rangle$ we have either $x_p < a$ or $x_p > b$. In the first case applying (3.2) after extension to $\widetilde{F}(x)$ we get

$$
\begin{aligned}
x_{p+1} &= x_p - 2[A + \widetilde{F}'(x_p)]^{-1}\widetilde{F}(x_p) = x_p - 2[2A]^{-1}[A(x_p - a) + F(a)] \\
&= x_p - (x_p - a) - A^{-1}F(a) = a - A^{-1}F(a) > a,
\end{aligned}
\tag{4.2}
$$

since A is an M-matrix and $F(a) \leq 0$. Therefore, $x_{p+1} \in \langle a, b\rangle$.

If $x_p > b$, repeating the same steps as above we get

$$
x_{p+1} = x_p - 2[2A]^{-1}[A(x_p - b) + F(b)] = x_p - (x_p - b) - A^{-1}F(b) = b - A^{-1}F(b). \tag{4.3}
$$

Since A is an M-matrix and $F(b) \geq 0$, $A^{-1}F(b) \geq 0$. Hence $x_{p+1} \in \langle a, b\rangle$.

Thus beginning from a step $p_0 \geq 0$, we necessarily have $x_m \in [a,b]$. If Theorems 3.1 and 4.1 are valid, $x_m \subseteq \langle a,b\rangle$ for $m \geq p_0$, and the convergence of $\{x_m\}$ to a solution x^* in $\langle a,b\rangle$ is guaranteed. Furthermore, the error estimate (3.8) and the exit criterion or the stopping inequality (3.9) are both valid. □

5. Numerical example

Let $z = [x,y]^T$, $D_0 = \langle -(\pi/2),\pi\rangle \times \langle 0,1\rangle$.

$$F(z) = \begin{Bmatrix} f_1(x,y) \\ f_2(x,y) \end{Bmatrix} = \begin{cases} \left(x-\dfrac{\pi}{2}\right)^3 + \left(\left(x-\dfrac{\pi}{2}\right)\sin\left(x-\dfrac{\pi}{2}\right)\right)y - 0.752 \\ \pi^2 y + \pi^2 y^3 - \left(x-\dfrac{\pi}{2}\right)\cos\left(x-\dfrac{\pi}{2}\right) + \sin\left(x-\dfrac{\pi}{2}\right) - \dfrac{5\pi^2}{8} - 0.152. \end{cases} \tag{5.1}$$

We are interested in solving $F(z) = 0$ for $z \in D_0$.

Initial approximation $z_0 = (x_0, y_0)^T$.

$$\begin{aligned} \frac{\partial f_1}{\partial x} &= 3\left(x-\frac{\pi}{2}\right)^2 + \left(\sin\left(x-\frac{\pi}{2}\right)\right)y + \left(\left(x-\frac{\pi}{2}\right)\cos\left(x-\frac{\pi}{2}\right)\right)y, \\ \frac{\partial f_1}{\partial y} &= \left(x-\frac{\pi}{2}\right)\sin\left(x-\frac{\pi}{2}\right), \\ \frac{\partial f_2}{\partial x} &= \left(x-\frac{\pi}{2}\right)\sin\left(x-\frac{\pi}{2}\right), \\ \frac{\partial f_2}{\partial y} &= \pi^2 + 3\pi^2 y^2. \end{aligned} \tag{5.2}$$

$F'(z)$ is symmetric.

$F'(z) \geq 0$ for all $x \in \langle -\pi,\pi\rangle$ and $y \in \langle 0,1\rangle$ except at $x = -\pi$, $y = \pi$.

$$\begin{aligned} F(a) = F(x)\big|_{\substack{x=-\pi/2 \\ y=0}} &= \begin{cases} -\pi^3 - 0.752 \\ -\dfrac{5\pi^2}{8} - \pi - 0.152 \end{cases} \leq 0, \\ F(b) = F(x)\big|_{\substack{x=\pi \\ y=1}} &= \begin{cases} \dfrac{\pi^3}{8} + \dfrac{\pi}{2} - 0.752 \\ \dfrac{3\pi^2}{8} + 1 - 0.152 \end{cases} \geq 0. \end{aligned} \tag{5.3}$$

$F'(z)\big|_{\substack{x=\pi/2 \\ y=0}}$ does not have an inverse.

We choose A as

$$A = \begin{pmatrix} 2 & -1.5708 \\ -1.5708 & 22 \end{pmatrix} \tag{5.4}$$

and $\in = 10^{-11}$, the desired accuracy is achieved in 17 iterations.

Starting from $x_0 = 1.5708$ and $y_0 = 0$, we obtain $x_1 = 2.67715530056448$ and $y_1 = 0.45117812654687$; and $x_{17} = 2.35205300236830$ and $y_{17} = 0.50014720328245$.

6. Discussion

(i) Convergence of Newton's method as proposed by Kantorovich (see [11]) is based on majorization principle which ensures that all the members of the sequence $\{x_m\}$ will lie in a small neighborhood of the initial approximation x_0. Hence majorization principle has not been used. But in order to ensure that $\{x_m\}$ does not go beyond $\langle a,b\rangle$, barring a finite number of members, an extended formula of the mapping F is taken.

(ii) Here the condition that $F'(x) \neq 0$ has been relaxed and the extended method is called Newton-like method.

(iii) The convergence is linear.

(iv) The numerical equation under consideration being nonlinear has more than one solution, the x-component of one solution being $x^* = 1.087961617$. In our case, the initial point taken is close to a point where the Jacobian becomes singular and the purpose is to show that the sequence of iterations (1.5) with the initial point mentioned above still converges to a solution of the given equation.

(v) For other modifications of Newton's method please see (Ortega and Rheinboldt [6], Keller [5], Sen [7, 8], Sen and Guhathakurta [10]).

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References

[1] V. Berinde, *An extension of the Newton-Raphson method for solving non-linear equations*, Proceedings of the International Scientific Conference MICROCAD-SYSTEMS, Technical University, Kösice, 1993, pp. 63–64.

[2] ———, *On some exit criteria for the Newton method*, Novi Sad Journal of Mathematics **27** (1997), no. 1, 19–26.

[3] ———, *On the convergence of the Newton method*, Transactions of the Technical University of Kosice (1997), no. 1, 68–77.

[4] ———, *On the extended Newton's method*, Advances in Difference Equations (S. Elaydi, I. Györi, and G. Ladas, eds.), Gordon and Breach, Amsterdam, 1997, pp. 81–88.

[5] H. B. Keller, *Newton's method under mild differentiability conditions*, Journal of Computer and System Sciences **4** (1970), no. 1, 15–28.

[6] J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, 1970.

[7] R. Sen, *A modification of the Newton-Kantorovitch method*, Mathematica (Cluj) **8 (31)** (1966), 155–161.

[8] ———, *On a variant of Newton's method under relaxed differentiability conditions*, Bulletin Mathématique de la Société des Sciences Mathématiques de la R. S. Roumanie **22(70)** (1978), no. 1, 87–93.

[9] R. Sen, A. Biswas, R. Patra, and S. Mukherjee, *An extension on Berinde's criterion for the convergence of a Newton-like method*, to appear in Bulletin of the Calcutta Mathematical Society.

[10] R. Sen and P. Guhathakurta, *On a variant of Ball Newton method under relaxed differentiability condition*, Soochow Journal of Mathematics **19** (1993), no. 2, 199–211.

[11] M. M. Vainberg, *Variational Methods for the Study of Nonlinear Operators*, Holden-day, California, 1964, with a chapter on Newton's method by L. V. Kantorovich and G. P. Akilov.
[12] R. S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, New Jersey, 1962.

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