EXISTENCE OF SOLUTIONS FOR GENERALIZED NONLINEAR MIXED VARIATIONAL-LIKE INEQUALITIES IN BANACH SPACES

YALI ZHAO, ZUNQUAN XIA, ZEQING LIU, AND SHIN MIN KANG

We introduce and study a new class of generalized nonlinear mixed variational-like inequalities in reflexive Banach spaces. By applying Ding's technique we prove several existence and uniqueness theorems of solutions for the generalized nonlinear mixed variational-like inequality, extend the auxiliary problem technique to suggest and analyze an iterative method to compute the approximate solutions of the generalized nonlinear mixed variational-like inequality, and establish the convergence criteria of the iterative method. The results presented in this paper improve, extend, and unify many known results in this area.

1. Introduction

Variational inequality theory has appeared as an effective and powerful tool to study and investigate a wide class of problems arising in pure and applied sciences including elasticity, optimization, economics, transportation, and structural analysis, see, for example, [1–8, 10–37] and the references therein. The classical variational inequality problem has been generalized in various directions for its own development and its applications. Mixed variational-like inequality is an important and useful generalization of the classical variational inequality. On the other hand, constructing effective numerical algorithm is an important and difficult problem in the variational inequality theory. Cohen [3] first extended the auxiliary variational inequality technique to suggest and analyze an innovative iterative method for solutions of mixed variational inequalities in reflexive Banach spaces. Ding [8] proved an existence-uniqueness theorem of solutions of a new class of nonlinear mixed variational-like inequalities in reflexive Banach spaces by making use of minimax inequality due to Ding and Tan [9]. Following the idea of Cohen [3], Ding suggested and analyzed a quite general iterative method to compute the approximate solutions and proved the convergence of the iterative method.

Motivated and inspired by the research work going on in this field, we will introduce and study a new class of generalized nonlinear mixed variational-like inequalities in reflexive Banach spaces, which includes many variational inequalities and variational-like

inequalities as special cases, for details, see [1–8, 10–37] and the references therein. Following the idea of Ding [8], we prove several existence and uniqueness theorems of solutions for the generalized nonlinear mixed variational-like inequality, extend the auxiliary problem technique to suggest and analyze an iterative method to compute the approximate solutions of the generalized nonlinear mixed variational-like inequality, and obtain the convergence criteria of the iterative method. The results presented in this paper improve, extend, and unify many known results in this area.

2. Preliminaries

Let D be a nonempty convex subset of a Banach space E with dual E^* and let $\langle u, v \rangle$ be the dual pairing between $u \in E^*$ and $v \in E$. Let $T, A, B, C : D \to E^*$, $M, N : E^* \times E^* \to E^*$, and $\eta : D \times D \to E$ be mappings. Let $b : D \times D \to R$ be a real-valued functional. Now we consider the following generalized nonlinear mixed variational-like inequality problem. For a given $w^* \in E^*$, find $u \in D$ such that

$$\langle N(Tu, Au) - M(Bu, Cu) + w^*, \eta(v, u) \rangle + b(u, v) - b(u, u) \geq 0, \quad \forall v \in D, \qquad (2.1)$$

where the functional b is nondifferential and satisfies the following conditions:
   (c1) b is linear in the first argument;
   (c2) for each $u \in D, b(u, \cdot)$ is a convex and lower semicontinuous functional on D;
   (c3) b is bounded, that is, there exists a constant $\nu > 0$ such that

$$b(u, v) \leq \nu \|u\| \, \|v\|, \quad \forall u, v \in D, \qquad (2.2)$$

   (c4) for all $u, v, w \in D$, $b(u, v) - b(u, w) \leq b(u, v - w)$.

Special cases. (A) If $M(Bu, Cu) = 0$ for all $u \in D$, and $-w^*$ is given, then the problem (2.1) reduces to the following variational-like inequality problem: for a given $w^* \in E^*$, find $u \in D$ such that

$$\langle N(Tu, Au) - w^*, \eta(v, u) \rangle + b(u, v) - b(u, u) \geq 0, \quad \forall v \in D, \qquad (2.3)$$

which was introduced and studied by Ding [8].
   (B) If $M(Bu, Cu) = M(Bu)$ for all $u \in D$, then the problem (2.1) is equivalent to the following problem: for a given $w^* \in E^*$, find $u \in D$ such that

$$\langle N(Tu, Au) - M(Bu) + w^*, \eta(v, u) \rangle + b(u, v) - b(u, u) \geq 0, \quad \forall v \in D. \qquad (2.4)$$

   (C) If $M(Bu) = w^*$ for all $u \in D$, then the problem (2.4) is equivalent to the following problem: find $u \in D$ such that

$$\langle N(Tu, Au), \eta(v, u) \rangle + b(u, v) - b(u, u) \geq 0, \quad \forall v \in D, \qquad (2.5)$$

which was introduced and considered by Liu and Tang [26].
   (D) If $\eta(v, u) = v - u$, $N(Tu, Au) = Tu - w^*$, and $b(u, v) = f(v)$ for all $u, v \in D$, where $f : D \to (-\infty, +\infty)$, then the problem (2.5) is equivalent to the following problem: find

$u \in D$ such that

$$\langle Tu - w^*, v - u \rangle + f(v) - f(u) \geq 0, \quad \forall v \in D, \tag{2.6}$$

which is originally considered by Cohen [3].

For the appropriate and suitable choices of mappings $M, N, \eta, b, T, A, B, C, D, w^*$, and the space E, we can obtain many known and new classes of variational inequalities and variational-like inequalities as special cases of the problem (2.1), see [1–4, 6, 7, 16–19, 25] and the references therein. In brief, the problem (2.1) is more general and a unifying one, which is also one of the main motivations of this paper.

Now, we recall the following definitions and some known results.

Definition 2.1. Let D be a nonempty subset of a Banach space E with the dual space E^*. Let $T : D \to E^*$ and $\eta : D \times D \to E$ be two mappings. The mapping T is called

(1) *η-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, \eta(u,v) \rangle \geq \alpha \|u - v\|^2, \quad \forall u, v \in D; \tag{2.7}$$

(2) *η-relaxed Lipschitz* if there exists a constant $\alpha \geq 0$ such that

$$\langle Tu - Tv, \eta(u,v) \rangle \leq -\alpha \|u - v\|^2, \quad \forall u, v \in D; \tag{2.8}$$

(3) *Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\|Tu - Tv\| \leq L\|u - v\|, \quad \forall u, v \in D; \tag{2.9}$$

(4) *η-generalized pseudocontractive* if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, \eta(u,v) \rangle \leq \alpha \|u - v\|^2, \quad \forall u, v \in D; \tag{2.10}$$

(5) *η-relaxed monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, \eta(u,v) \rangle \geq -\alpha \|u - v\|^2, \quad \forall u, v \in D. \tag{2.11}$$

Definition 2.2. Let D be a nonempty subset of a Banach space E with the dual space E^*, let $T, A : D \to E^*$, $N : E^* \times E^* \to E^*$, and $\eta : D \times D \to E$ be mappings, let $N(T,A)(u) = N(Tu, Au)$.

(1) N is said to be *Lipschitz continuous* in the first argument if there exists a constant $\alpha > 0$ such that

$$\|N(u,w) - N(v,w)\| \leq \alpha \|u - v\|, \quad \forall u, v, w \in E^*. \tag{2.12}$$

(2) N is said to be *η-cocoercive* in the first argument with respect to T if there exists a constant $\alpha > 0$ such that

$$\langle N(Tu,w) - N(Tv,w), \eta(u,v) \rangle \geq \alpha \|Tu - Tv\|^2, \quad \forall u, v \in D, w \in E^*. \tag{2.13}$$

(3) N is said to be *η-strongly monotone* in the first argument with respect to A if there exists a constant $\alpha > 0$ such that

$$\langle N(Au,w) - N(Av,w), \eta(u,v) \rangle \geq \alpha \|u - v\|^2, \quad \forall u,v \in D, w \in E^*. \tag{2.14}$$

(4) N is said to be *η-relaxed monotone* in the first argument with respect to A if there exists a constant $\alpha > 0$ such that

$$\langle N(Au,w) - N(Av,w), \eta(u,v) \rangle \geq -\alpha \|u - v\|^2, \quad \forall u,v \in D, w \in E^*. \tag{2.15}$$

(5) N is said to be *Lipschitz continuous* in the first argument with respect to A if there exists a constant $\alpha > 0$ such that

$$\|N(Au,w) - N(Av,w)\| \leq \alpha \|u - v\|, \quad \forall u,v \in D, w \in E^*. \tag{2.16}$$

(6) η is said to be *Lipschitz continuous* if there exists a constant $\delta \geq 0$ such that

$$\|\eta(u,v)\| \leq \delta \|u - v\|, \quad \forall u,v \in D. \tag{2.17}$$

Similarly, the Lipschitz continuity, η-strong monotonicity, relaxed monotonicity, cocoercivity, and generalized pseudocontractivity of N can be defined in the second argument.

(7) $N(Tu,Au)$ is said to be *η-strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle N(Tu,Au) - N(Tw,Aw), \eta(u,w) \rangle \geq \alpha \|u - w\|^2, \quad \forall u,w \in D. \tag{2.18}$$

(8) $N(Tu,Au)$ is said to be *η-generalized pseudocontractive* if there exists a constant $\alpha > 0$ such that

$$\langle N(Tu,Au) - N(Tw,Aw), \eta(u,w) \rangle \leq \alpha \|u - w\|^2, \quad \forall u,w \in D. \tag{2.19}$$

The following notion is a generalization of the corresponding notion introduced by Ding and Tarafdar [10].

Definition 2.3. Let D be a nonempty convex subset of a Banach space E. Let $T : D \to E^*$ and $\eta : D \times D \to E$ be mappings. T and η are said to have 0-*diagonally concave relation* with respect to $w^* \in E^*$, if the function $\varphi : D \times D \to (-\infty, +\infty]$ defined by

$$\varphi(u,v) = \langle Tu + w^*, \eta(u,v) \rangle \tag{2.20}$$

is 0-diagonally concave in v, that is, for any finite set $\{v_1, \ldots, v_m\} \subset D$ and for any $u = \sum_{i=1}^{m} \lambda_i v_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{m} \lambda_i = 1$, $\sum_{i=1}^{m} \lambda_i \varphi(u,v_i) \leq 0$.

Remark 2.4. Let D be a nonempty convex subset of a Banach space E with the dual space E^*. Let $T : D \to E^*$ and $\eta : D \times D \to E$ be mappings such that $\eta(u,u) = 0$ for all $u \in D$. Suppose that for given $u \in D$ and $w^* \in E^*$, the functional $v \mapsto \langle Tu + w^*, \eta(u,v) \rangle$ is concave. Then T and η have the 0-diagonally concave relation with respect to w^*.

LEMMA 2.5 [8]. *Let D be a nonempty convex subset of a topological vector space and let $\varphi : D \times D \to [-\infty, +\infty]$ be such that*

(a) *for each $x \in D$, $y \mapsto \varphi(x, y)$ is lower semicontinuous on each nonempty compact subset of D,*

(b) *for each nonempty finite set $\{x_1, \ldots, x_m\} \subset D$ and for each $y = \sum_{i=1}^{m} \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{m} \lambda_i = 1$, $\min_{1 \leq i \leq m} \varphi(x_i, y) \leq 0$,*

(c) *there exist a nonempty compact convex subset X_0 of D and a nonempty compact subset K of D such that for each $y \in D \setminus K$, there is an $x \in \operatorname{co}(X_0 \cup \{y\})$ with $\varphi(x, y) > 0$.*

Then there exists a $\tilde{y} \in K$ such that $\varphi(x, \tilde{y}) \leq 0$ for all $x \in D$.

3. Existence and uniqueness theorems

We now prove the following results.

THEOREM 3.1. *Let D be a nonempty convex subset of a reflexive Banach space E with dual space E^*. Let $T, A, B, C : D \to E^*$, $M, N : E^* \times E^* \to E^*$, $\eta : D \times D \to E$ be mappings, let $w^* \in E^*$ be given, and let $b : D \times D \to (-\infty, +\infty]$ be a functional such that*

(a) *the mapping $u \mapsto N(Tu, Au)$ is continuous from the weak topology on E to the strong topology on E^* such that N is η-strongly monotone in the first argument with respect to T with constant $\alpha > 0$, and is η-relaxed monotone in the second argument with respect to A with constant $\lambda > 0$,*

(b) *the mapping $u \mapsto M(Bu, Cu)$ is continuous from the weak topology on E to the strong topology on E^* such that M is η-generalized pseudocontractive in the first argument with respect to B with constant $\beta > 0$, and is η-relaxed Lipschitz in the second argument with respect to C with constant $\xi > 0$,*

(c) *η is Lipschitz continuous with constant $\delta \geq 0$ and for each $v \in D$, $u \mapsto \eta(u, v)$ is continuous from the weak topology to the weak topology and $\eta(v, u) = -\eta(u, v)$ for all $u, v \in D$,*

(d) *the mapping $u \mapsto N(Tu, Au) - M(Bu, Cu)$ and η have the 0-diagonally concave relation with respect to w^*,*

(e) *b satisfies conditions (c1)–(c4) with $v + \lambda + \beta < \alpha + \xi$.*

Then the generalized nonlinear mixed variational-like inequality (2.1) has a unique solution $\hat{u} \in D$.

Proof. We first prove that for each fixed $u^* \in D$, there exists a unique $\hat{w} \in D$ such that

$$\langle N(T\hat{w}, A\hat{w}) - M(B\hat{w}, C\hat{w}) + w^*, \eta(v, \hat{w}) \rangle + b(u^*, v) - b(u^*, \hat{w}) \geq 0, \quad \forall v \in D. \quad (3.1)$$

For any fixed $u^* \in D$, define a functional $\varphi : D \times D \to (-\infty, +\infty]$ by

$$\varphi(v, w) = \langle N(Tw, Aw) - M(Bw, Cw) + w^*, \eta(w, v) \rangle + b(u^*, w) - b(u^*, v), \quad \forall v, w \in D. \quad (3.2)$$

Since b satisfies the conditions (c3) and (c4), it is easy to see that b also satisfies

$$|b(u, v) - b(u, w)| \leq v\|u\| \|v - w\|, \quad \forall u, v, w \in D, \quad (3.3)$$

and hence for each fixed $u \in D$, $b(u, \cdot)$ is continuous on D. Notice that the mappings $w \mapsto N(Tw, Aw)$ and $w \mapsto M(Bw, Cw)$ are continuous from the weak topology to the strong

topology. For each $w \in D$ and any sequence $\{w_n\} \subset D$ with $w_n \rightharpoonup w$ in the weak topology, we have $\lim_{n\to\infty} N(Tw_n, Aw_n) = N(Tw, Aw)$, $\lim_{n\to\infty} M(Bw_n, Cw_n) = M(Bw, Cw)$, and $\eta(w_n, v) \rightharpoonup \eta(w, v)$ as $n \to \infty$ in the weak topology and the sequence $\{\eta(w_n, v)\}$ is bounded. It follows that for each $v \in D$,

$$
\begin{aligned}
& \left| \langle N(Tw_n, Aw_n) - M(Bw_n, Cw_n) + w^*, \eta(w_n, v) \rangle \right. \\
& \qquad \left. - \langle N(Tw, Aw) - M(Bw, Cw) + w^*, \eta(w, v) \rangle \right| \\
& \quad \leq \left| \langle N(Tw_n, Aw_n) - N(Tw, Aw), \eta(w_n, v) \rangle \right| \\
& \qquad + \left| \langle M(Bw_n, Cw_n) - M(Bw, Cw), \eta(w_n, v) \rangle \right| \\
& \qquad + \left| N(Tw, Aw) - M(Bw, Cw) + w^*, \eta(w_n, v) - \eta(w, v) \rangle \right| \\
& \quad \leq \| N(Tw_n, Aw_n) - N(Tw, Aw) \| \, \| \eta(w_n, v) \| \\
& \qquad + \| M(Bw_n, Cw_n) - M(Bw, Cw) \| \, \| \eta(w_n, v) \| \\
& \qquad + \left| \langle N(Tw, Aw) - M(Bw, Cw) + w^*, \eta(w_n, v) - \eta(w, v) \rangle \right| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned}
\tag{3.4}
$$

It follows that for each $v \in D$, the functional $w \mapsto \langle N(Tw, Aw) - M(Bw, Cw) + w^*, \eta(w, v) \rangle$ is weakly continuous on D. Therefore, $w \mapsto \varphi(v, w)$ is weakly lower semicontinuous on D. We claim that φ satisfies the condition (b) of Lemma 2.5. If it is false, then there exists a finite set $\{v_1, \ldots, v_m\} \subset D$ and $w = \sum_{i=1}^m \lambda_i v_i$ with $\lambda_i \geq 0$, $\sum_{i=1}^m \lambda_i = 1$, such that $\varphi(v_i, w) > 0$ for all $i = 1, \ldots, m$, that is,

$$
\langle N(Tw, Aw) - M(Bw, Cw) + w^*, \eta(w, v_i) \rangle + b(u^*, w) - b(u^*, v_i) > 0, \quad i = 1, \ldots, m.
\tag{3.5}
$$

It follows that

$$
\sum_{i=1}^m \lambda_i \langle N(Tw, Aw) - M(Bw, Cw) + w^*, \eta(w, v_i) \rangle + b(u^*, w) - \sum_{i=1}^m b(u^*, v_i) > 0.
\tag{3.6}
$$

Note that $b(u, v)$ is convex in the second argument. It follows that

$$
\sum_{i=1}^m \lambda_i \langle N(Tw, Aw) - M(Bw, Cw) + w^*, \eta(w, v_i) \rangle > 0,
\tag{3.7}
$$

which contradicts the condition (d) of Theorem 3.1. Therefore the condition (b) of Lemma 2.5 holds. Let

$$
\begin{aligned}
r &= \frac{1}{\alpha - \lambda - \beta + \xi} \left[\delta \| N(Tu^*, Au^*) - M(Bu^*, Cu^*) \| + \delta \| w^* \| + \nu \| u^* \| \right], \\
K &= \{ w \in D : \| w - u^* \| \leq r \}.
\end{aligned}
\tag{3.8}
$$

Then K and $X_0 = \{u^*\}$ are both weakly compact convex subsets of D. By the assumptions (a)–(d), for each $w \in D \setminus K$, there exists a $u^* \in \mathrm{co}(X_0 \cup \{w\})$ such that

$$
\begin{aligned}
\varphi(u^*, w) &= \langle N(Tw, Aw) - M(Bw, Cw) + w^*, \eta(w, u^*) \rangle + b(u^*, w) - b(u^*, u^*) \\
&= \langle N(Tw, Aw) - N(Tu^*, Aw), \eta(w, u^*) \rangle \\
&\quad + \langle N(Tu^*, Aw) - N(Tu^*, Au^*), \eta(w, u^*) \rangle \\
&\quad - \langle M(Bw, Cw) - M(Bu^*, Cw), \eta(w, u^*) \rangle \\
&\quad - \langle M(Bu^*, Cw) - M(Bu^*, Cu^*), \eta(w, u^*) \rangle \\
&\quad + \langle N(Tu^*, Au^*) - M(Bu^*, Cu^*) + w^*, \eta(w, u^*) \rangle \\
&\quad + b(u^*, w) - b(u^*, u^*) \\
&\geq \|w - u^*\| [[(\alpha - \lambda - \beta + \xi)\|w - u^*\| - \delta\|w^*\| - \delta\| \\
&\qquad\qquad N(Tu^*, Au^*) - M(Bu^*, Cu^*)\| - \nu\|u^*\|] > 0.
\end{aligned}
\tag{3.9}
$$

Hence the condition (c) of Lemma 2.5 is also satisfied. By Lemma 2.5, there exists a $\widehat{w} \in K$ such that $\varphi(v, \widehat{w}) \leq 0$ for all $v \in D$, that is,

$$
\langle N(T\widehat{w}, A\widehat{w}) - M(B\widehat{w}, C\widehat{w}) + w^*, \eta(\widehat{w}, v) \rangle + b(u^*, \widehat{w}) - b(u^*, v) \leq 0, \quad \forall v \in D.
\tag{3.10}
$$

Since $\eta(\widehat{w}, v) = -\eta(v, \widehat{w})$, we see that

$$
\langle N(T\widehat{w}, A\widehat{w}) - M(B\widehat{w}, C\widehat{w}) + w^*, \eta(v, \widehat{w}) \rangle + b(u^*, v) - b(u^*, \widehat{w}) \geq 0, \quad \forall v \in D.
\tag{3.11}
$$

Now we prove that $\widehat{w}$ is a unique solution of the problem (3.1). Suppose that w_1, w_2 are arbitrary two solutions of the problem (3.1). It follows that for all $v \in D$,

$$
\langle N(Tw_1, Aw_1) - M(Bw_1, Cw_1) + w^*, \eta(v, w_1) \rangle + b(u^*, v) - b(u^*, w_1) \geq 0,
\tag{3.12}
$$
$$
\langle N(Tw_2, Aw_2) - M(Bw_2, Cw_2) + w^*, \eta(v, w_2) \rangle + b(u^*, v) - b(u^*, w_2) \geq 0.
\tag{3.13}
$$

Taking $v = w_2$ in (3.12) and $v = w_1$ in (3.13), and adding these inequalities, we obtain that

$$
\langle N(Tw_1, Aw_1) - M(Bw_1, Cw_1) - N(Tw_2, Aw_2) + M(Bw_2, Cw_2), \eta(w_2, w_1) \rangle \geq 0.
\tag{3.14}
$$

By the η-strong monotonicity of N in the first argument and η-relaxed strong monotonicity of N in the second argument, the η-generalized pseudocontractivity of M in the first argument and η-relaxed Lipschitz continuity of M in the second argument, we get that

$$
(\alpha - \lambda - \beta + \xi)\|w_1 - w_2\|^2 \leq 0.
\tag{3.15}
$$

Since $\beta + \lambda + v < \alpha + \xi$, it follows that $w_1 = w_2$ and $\hat{w}$ is the unique solution of the problem (3.1). Thus we have proved that for each $u \in D$, there exists a unique solution $w(u)$ satisfying (3.1). Define a mapping $F : D \to D$ by $u \mapsto w(u) = Fu$. We prove that the mapping F is a contraction mapping. Indeed, for any $u_1, u_2 \in D$, there exist $w_1 = F(u_1)$, $w_2 = F(u_2)$ such that for all $v \in D$,

$$\langle N(Tw_1, Aw_1) - M(Bw_1, Cw_1) + w^*, \eta(v, w_1) \rangle + b(u_1, v) - b(u_1, w_1) \geq 0, \qquad (3.16)$$

$$\langle N(Tw_2, Aw_2) - M(Bw_2, Cw_2) + w^*, \eta(v, w_2) \rangle + b(u_2, v) - b(u_2, w_2) \geq 0. \qquad (3.17)$$

Taking $v = w_2$ in (3.16) and $v = w_1$ in (3.17) and adding these inequalities, and considering the assumption on b and η, we have

$$\|w_1 - w_2\| \leq \frac{v}{\alpha - \beta - \lambda + \xi} \|u_1 - u_2\|, \qquad (3.18)$$

which implies that F is a contraction mapping. Hence there exists a unique point $\hat{u} \in D$ such that $\hat{u} = F(\hat{u})$, that is,

$$\langle N(T\hat{u}, A\hat{u}) - M(B\hat{u}, C\hat{u}) + w^*, \eta(v, \hat{u}) \rangle + b(\hat{u}, v) - b(\hat{u}, \hat{u}) \geq 0, \quad \forall v \in D, \qquad (3.19)$$

and $\hat{u} \in D$ is the unique solution of the generalized nonlinear mixed variational-like inequality problem (2.1). This completes the proof. $\qquad\qquad\square$

As in the proof of Theorem 3.1, we have the following result.

THEOREM 3.2. *Let D, E, E^*, η, w^* be as in Theorem 3.1. Let $T, A, B, C : D \to E^*$, $M, N :$ $E^* \times E^* \to E^*$ be mappings, and let $b : D \times D \to (-\infty, +\infty]$ be a functional such that*

  (a) *the mapping $u \mapsto N(Tu, Au)$ is continuous from the weak topology on E to the strong topology on E^* and η-strongly monotone with constant $\alpha_1 > 0$,*
  (b) *the mapping $u \to M(Bu, Cu)$ is continuous from the weak topology on E to the strong topology on E^* and η-generalized pseudocontractive with constant $\alpha_2 > 0$,*
  (c) *η satisfies the condition (c) of Theorem 3.1, and the condition (d) of Theorem 3.1 still holds,*
  (d) *b satisfies conditions (c1)–(c4) with $v + \alpha_2 < \alpha_1$.*

Then the generalized nonlinear mixed variational-like inequality (2.1) has a unique solution $\bar{u} \in D$.

Remark 3.3. Theorems 3.1 and 3.2 improve and generalize Dien [4, Theorem 2], Ding [6–8, Theorems 3.1 and 3.2], Parida et al. [31, Theorems 3.1 and 3.9], and Yao [36, Theorem 3.1].

4. Iterative method and convergence

In this section, by using the auxiliary principle technique, we will give a general iterative method of approximate solutions of the generalized nonlinear mixed variational-like inequality problem (2.1) and convergence analysis of the iterative method.

Let D be a nonempty convex subset of a reflexive Banach space E with dual space E^*. Let $N : E^* \times E^* \to E^*$, $T, A, B, C : D \to E^*$ be mappings. Suppose that $\eta : D \times D \to E$ is a mapping such that $\eta(v, u) = \eta(v, z) + \eta(z, u)$ for all $v, u, z \in D$. We consider an auxiliary differentiable convex functional $K : D \to (-\infty, +\infty]$ and a positive number $\rho > 0$. For given $w^* \in D$ and $u^* \in D$, we introduce the following auxiliary minimizing problem:

$$
\begin{aligned}
\min_{w \in D} \{ & K(w) + \rho \langle N(Tu^*, Au^*) - M(Bu^*, Cu^*) + w^*, \eta(w, u^*) \rangle \\
& - \langle K'(u^*), w \rangle + \rho b(u^*, w) \}.
\end{aligned}
\tag{4.1}
$$

If the function $w \mapsto \langle N(Tu^*, Au^*) - M(Bu^*, Cu^*) + w^*, \eta(w, u^*) \rangle$ is convex, then the solution w of the auxiliary problem (4.1) can be characterized by the following auxiliary variational inequality problem:

$$
\begin{aligned}
\langle K'&(w) - K'(u^*), v - w \rangle \\
& \geq -\rho \langle N(Tu^*, Au^*) - M(Bu^*, Cu^*) + w^*, \eta(v, u) \rangle \\
& \quad + \rho b(u^*, w) - \rho b(u^*, v), \quad \forall v \in D.
\end{aligned}
\tag{4.2}
$$

We note that if $w = u^*$, then u^* is a solution of the generalized nonlinear mixed variational-like inequality problem (2.1). Based on these observations, we suggest the following iterative method for computing the approximate solutions of the problem (2.1).

Algorithm 4.1. (a) At $n = 0$, start with some initial u_0.

(b) At step n, solve the auxiliary minimizing problem (4.1) or the auxiliary variational inequality problem (4.2) with $u^* = u_n$. Let u_{n+1} denote the solution of problem (4.1) or (4.2).

(c) If, for given $\varepsilon > 0$, $\|u_{n+1} - u_n\| \leq \varepsilon$, stop. Otherwise, repeat (b).

We now consider the convergence of Algorithm 4.1.

THEOREM 4.2. *Let D be a nonempty convex subset of a reflexive Banach space E with dual space E^*. Let $M, N : E^* \times E^* \to E^*$, $T, A, B, C : D \to E^*$ and $\eta : D \times D \to E$ be mappings. Let $w^* \in E^*$ be given let $b : D \times D \to (-\infty, +\infty]$ be a functional, and let $K : E \to (-\infty, +\infty]$ be a differentiable convex functional such that*

(a) *the mapping $w \mapsto N(Tw, Aw)$ is continuous from the weak topology to the strong topology, N is η-cocoercive in the first argument with respect to T with constant $\alpha > 0$, and η-strongly monotone in the second argument with respect to A with constant $\xi > 0$, and is Lipschitz continuous in the first and second arguments with constants σ_1 and σ_2, respectively,*

(b) *the mapping $w \mapsto M(Bw, Cw)$ is continuous from the weak topology to the strong topology, M is η-generalized pseudocontractive in the first argument with respect to B with constant $\beta > 0$, and η-relaxed Lipschitz in the second argument with respect to C with constants $\sigma > 0$ and $\beta > \sigma$, and is Lipschitz continuous in the first and second arguments with constants $\sigma_3 > 0$ and $\sigma_4 > 0$, respectively,*

(c) *$A, B, C : D \to E^*$ are Lipschitz continuous with constants $\sigma_A, \sigma_B, \sigma_C$, respectively,*

(d) *η is Lipschitz continuous with constant $\delta \geq 0$ such that*

(1) for each $v \in D$, $u \mapsto \eta(u,v)$ is continuous from the weak topology to the weak topology;

(2) for all $u,v,z \in D$, $\eta(u,v) = \eta(u,z) + \eta(z,v)$;

(3) for each fixed $u,w \in D$, the function $v \mapsto \langle N(Tu,Au) - M(Bu,Cu) + w^*, \eta(w,v) \rangle$ is concave,

(e) b satisfies conditions (c1)–(c4) and $\beta + \nu < \xi + \sigma$,

(f) the derivative K' of K is strongly convex with constant $\mu > 0$ and the mapping $w \mapsto K'(w)$ is continuous from the weak topology to the strong topology.

Then

(A) the generalized nonlinear mixed variational-like inequality problem has a unique solution (2.1) $\hat{u} \in D$;

(B) for each $\rho > 0$, there exists a unique solution $u_{n+1} \in D$ of Problems (4.1) or (4.2) with u_n substituted for u^*;

(C) if

$$0 < \rho < \frac{2\alpha\mu(\xi + \sigma - \beta - \nu)}{\sigma_1^2 \delta^2 (\xi + \sigma - \beta - \nu) + \alpha(\sigma_2 \delta\sigma_A + \nu + \sigma_3 \delta\sigma_B + \sigma_4 \delta\sigma_C)^2}, \tag{4.3}$$

then the sequence $\{u_n\}$ defined by Algorithm 4.1 strongly converges to $\hat{u}$.

Proof. (A) In view of conditions (a) and (b), we get that the mapping $u \mapsto N(Tu,Au)$ is η-strongly monotone with constant $\xi > 0$ and the mapping $u \mapsto M(Bu,Cu)$ is η-generalized pseudocontractive with constant $\beta - \sigma$, respectively. Hence the conditions (a) and (b) of Theorem 3.2 are satisfied. By (3) of (d), for each $u \in D$, the function $v \mapsto \langle (Tu,Au) - M(Bu,Cu) + w^*, \eta(u,v) \rangle$ is concave, it follows from Remark 2.4 that condition (d) of Theorem 3.1 holds. Noting that condition (2) of (d) implies that $\eta(u,u) = 0$ and $\eta(u,v) = -\eta(v,u)$ for all $u,v \in D$, it is easy to check that all conditions of Theorem 3.2 are satisfied. By Theorem 3.2, the conclusion (A) holds.

(B) For each fixed $\rho > 0$ and $u_n \in D$, define a functional $\varphi : D \times D \rightarrow [-\infty, +\infty]$ by

$$\begin{aligned}
\varphi(v,w) = {} & \langle K'(u_n) - K'(w), v - w \rangle \\
& - \rho \langle N(Tu_n, Au_n) - M(Bu_n, Cu_n) + w^*, \eta(v,w) \rangle \\
& + \rho b(u_n, w) - \rho b(u_n, v).
\end{aligned} \tag{4.4}$$

Since $w \mapsto K'(w)$ is continuous from the weak topology to the strong topology, it is easy to show that the function $w \mapsto \langle K'(w), w \rangle$ is weakly continuous on D and for each $v \in D$, $w \mapsto \varphi(v,w)$ is weak lower semicontinuous. By using similar argument as in the proof of Theorem 3.1, it is easy to show that φ satisfied all conditions of Lemma 2.5, and hence there exists $\hat{w} \in D$ such that

$$\begin{aligned}
\langle K'(\hat{w}) & - K'(u_n), v - \hat{w} \rangle \\
& \geq -\rho \langle N(Tu_n, Au_n) - M(Bu_n, Cu_n) + w^*, \eta(v, \hat{w}) \rangle \\
& \quad + \rho b(u_n, \hat{w}) - \rho b(u_n, v), \quad \forall v \in D.
\end{aligned} \tag{4.5}$$

It is easy to show that $\hat{w}$ is unique, and hence $u_{n+1} = \hat{w}$ is a unique solution of the auxiliary variational inequality problem (4.2).

(C) Let $\hat{u}$ be the unique solution of the generalized nonlinear mixed variational-like inequality problem (2.1). Define a functional $\Lambda : D \to (-\infty, +\infty]$ by

$$\Lambda(u) = K(\hat{u}) - K(u) - \langle K'(u), \hat{u} - u \rangle, \quad \forall u \in D. \tag{4.6}$$

Since K' is strongly convex, it follows that

$$\Lambda(u) \geq \frac{\mu}{2} \|u - \hat{u}\|^2. \tag{4.7}$$

Note that $\eta(v, u) = -\eta(u, v)$ for all $v, u \in D$. It follows from (4.5) with $u_{n+1} = \hat{w}$ and $v = \hat{u}$ that

$$
\begin{aligned}
&\Lambda(u_n) - \Lambda(u_{n+1}) \\
&\quad = K(u_{n+1}) - K(u_n) - \langle K'(u_n), u_{n+1} - u_n \rangle + \langle K'(u_{n+1}) - K'(u_n), \hat{u} - u_{n+1} \rangle \\
&\quad \geq \frac{\mu}{2} \|u_{n+1} - u_n\|^2 + \rho \langle N(Tu_n, Au_n) - M(Bu_n, Cu_n) + w^*, \eta(u_{n+1}, \hat{u}) \rangle \\
&\qquad + \rho b(u_n, u_{n+1}) - \rho b(u_n, \hat{u}) \\
&\quad = \frac{\mu}{2} \|u_{n+1} - u_n\|^2 \\
&\qquad + \rho \langle N(Tu_n, Au_n) - M(Bu_n, Cu_n) - (N(T\hat{u}, A\hat{u}) - M(B\hat{u}, C\hat{u})), \eta(u_{n+1}, \hat{u}) \rangle \\
&\qquad + \rho \langle N(T\hat{u}, A\hat{u}) - M(B\hat{u}, C\hat{u}) + w^*, \eta(u_{n+1}, \hat{u}) \rangle \\
&\qquad + \rho b(u_n, u_{n+1}) - \rho b(u_n, \hat{u}).
\end{aligned}
\tag{4.8}
$$

Since $\hat{u}$ is a solution of the generalized nonlinear mixed variational-like inequality problem (2.1) and $u_{n+1} \in D$, it follows that

$$\langle N(T\hat{u}, A\hat{u}) - M(B\hat{u}, C\hat{u}) + w^*, \eta(u_{n+1}, \hat{u}) \rangle + b(\hat{u}, u_{n+1}) - b(\hat{u}, \hat{u}) \geq 0, \tag{4.9}$$

which implies that

$$\Lambda(u_n) - \Lambda(u_{n+1}) \geq \frac{\mu}{2} \|u_n - u_{n+1}\|^2 + Q, \tag{4.10}$$

where

$$
\begin{aligned}
Q = {}&\rho \langle N(Tu_n, Au_n) - M(Bu_n, Cu_n) - (N(T\hat{u}, A\hat{u}) - M(B\hat{u}, C\hat{u})), \eta(u_{n+1}, \hat{u}) \rangle \\
&- \rho [b(u_n - \hat{u}, \hat{u} - u_n) + b(u_n - \hat{u}, u_n - u_{n+1})].
\end{aligned}
\tag{4.11}
$$

By the conditions (a)–(f), we deduce that

$$
\begin{aligned}
Q &= \rho\langle N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, u_n)\rangle \\
&\quad - \rho\langle M(Bu_n, Cu_n) - M(B\hat{u}, C\hat{u}), \eta(u_{n+1}, u_n)\rangle \\
&\quad + \rho\langle N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_n, \hat{u})\rangle \\
&\quad - \rho\langle M(Bu_n, Cu_n) - M(B\hat{u}, C\hat{u}), \eta(u_n, \hat{u})\rangle \\
&\quad - \rho[b(u_n - \hat{u}, \hat{u} - u_n) + b(u_n - \hat{u}, u_n - u_{n+1})] \\
&\geq \rho\langle N(Tu_n, Au_n) - N(T\hat{u}, Au_n), \eta(u_n, \hat{u})\rangle \\
&\quad + \rho\langle N(T\hat{u}, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_n, \hat{u})\rangle \\
&\quad + \rho\langle N(Tu_n, Au_n) - N(T\hat{u}, Au_n), \eta(u_{n+1}, u_n)\rangle \\
&\quad + \rho\langle N(T\hat{u}, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, u_n)\rangle \\
&\quad - \rho\langle M(Bu_n, Cu_n) - M(B\hat{u}, Cu_n), \eta(u_n, \hat{u})\rangle \\
&\quad - \rho\langle M(B\hat{u}, Cu_n) - M(B\hat{u}, C\hat{u}), \eta(u_n, \hat{u})\rangle \\
&\quad - \rho\langle M(Bu_n, Cu_n) - M(B\hat{u}, Cu_n), \eta(u_{n+1}, u_n)\rangle \\
&\quad - \rho\langle M(B\hat{u}, Cu_n) - M(B\hat{u}, C\hat{u}), \eta(u_{n+1}, u_n)\rangle \\
&\quad - \rho\nu[\|u_n - \hat{u}\|^2 + \|u_n - \hat{u}\|\|u_n - u_{n+1}\|] \\
&\geq \rho[\alpha\|Tu_n - T\hat{u}\|^2 - \sigma_1\delta\|Tu_n - T\hat{u}\|\|u_{n+1} - u_n\|] \\
&\quad + \rho\xi\|u_n - \hat{u}\|^2 - \rho\sigma_2\delta\sigma_A\|u_n - \hat{u}\|\|u_{n+1} - u_n\| - \rho\nu\|u_n - \hat{u}\|^2 \\
&\quad - \rho\nu\|u_n - \hat{u}\|\|u_n - u_{n+1}\| - \rho\beta\|u_n - \hat{u}\|^2 + \rho\sigma\|u_n - \hat{u}\|^2 \\
&\quad - \rho\delta\sigma_3\sigma_B\|u_n - \hat{u}\|\|u_{n+1} - u_n\| - \rho\delta\sigma_4\sigma_C\|u_n - \hat{u}\|\|u_{n+1} - u_n\| \\
&\geq \rho\left[-\frac{\sigma_1^2\delta^2}{4\alpha}\right]\|u_{n+1} - u_n\|^2 \\
&\quad - \rho(\sigma_2\delta\sigma_A + \nu + \delta\sigma_3\sigma_B + \delta\sigma_4\sigma_C)\|u_n - \hat{u}\|\|u_{n+1} - u_n\| \\
&\quad + \rho(\xi + \sigma - \beta - \nu)\|u_n - \hat{u}\|^2.
\end{aligned}
\tag{4.12}
$$

Therefore, we have

$$
\begin{aligned}
\Lambda(u_n) &- \Lambda(u_{n+1}) \\
&\geq \frac{1}{2}\left(\mu - \frac{\rho\sigma_1^2\delta^2}{2\alpha}\right)\|u_{n+1} - u_n\|^2 \\
&\quad - \rho(\sigma_2\delta\sigma_A + \nu + \delta\sigma_3\sigma_B + \delta\sigma_4\sigma_C)\|u_n - \hat{u}\|\|u_{n+1} - u_n\| \\
&\quad + \rho(\xi + \sigma - \beta - \nu)\|u_n - \hat{u}\|^2 \\
&\geq \left[\rho(\xi + \sigma - \beta - \nu) - \frac{\rho^2(\sigma_2\delta\sigma_A + \nu + \delta\sigma_3\sigma_B + \delta\sigma_4\sigma_C)^2}{2(\mu - \rho\sigma_1^2\delta^2/2\alpha)}\right]\|u_n - \hat{u}\|^2.
\end{aligned}
\tag{4.13}
$$

The conditions (4.3), (4.7), and (4.13) show that the sequence $\{\Lambda(u_n)\}$ is non-negative and decreasing. Hence it converges to some number and the sequence $\{u_n\}$ converges strongly to $\hat{u}$ as $n \to \infty$. This completes the proof. $\qquad\square$

Remark 4.3. Theorem 4.2 improves and generalizes Ding [7, Theorem 4.1], Ding [8, Theorem 4.1 and Corollary 4.1], and Cohen [3, Theorem 2.2].

Acknowledgments

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Yali Zhao: Department of Applied Mathematics, Dalian University of Technology, Dalian, Liaoning 116024, China
E-mail address: yalizhao2000@yahoo.com.cn

Zunquan Xia: Department of Applied Mathematics, Dalian University of Technology, Dalian, Liaoning 116024, China
E-mail address: zqxiazhh@dlut.edu.cn

Zeqing Liu: Department of Mathematics, Liaoning Normal University, P.O. Box 200, Dalian, Liaoning 116029, China
E-mail address: zeqingliu@dl.cn

Shin Min Kang: Department of Mathematics and Research Institute of Natural Science, Gyeongsang National University, Jinju 660-701, Korea
E-mail address: smkang@nongae.gsnu.ac.kr