ON THE EXISTENCE OF MULTIPLE POSITIVE ENTIRE SOLUTIONS FOR A CLASS OF QUASILINEAR ELLIPTIC EQUATIONS

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Our goal is to establish the theorems of existence and multiple of positive entire solutions for a class quasilinear elliptic equations in $\mathbb{R}^N$ with the Schauder-Tychonoff fixed point theorem as the principal tool. In many articles, the theorems of existence and multiple of positive entire solutions for a class semilinear elliptic equations are established. The results of the semilinear equations are extended to the quasilinear ones and the results of semilinear equations are developed.

1. Introduction

In this paper, we consider the existence of multiple positive entire solutions for a class of quasilinear elliptic equation

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f(x,u,\nabla u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $p > 1$.

Equations of the above form are mathematical models occurring in the studies of the p-Laplace equation, generalized reaction-diffusion theory, non-Newtonian fluid theory [7], and the turbulent flow of a gas in porous medium [2]. In the non-Newtonian fluid theory, the quantity p is characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids.

By a positive entire solution of (1.1) we mean a function $u \in W^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ which satisfies (1.1) at every point of $\mathbb{R}^N$ in a weak sense with $u > 0$ in $\mathbb{R}^N$ (see [4] and references therein), that is $u \in W^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ which satisfies

$$-\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla\psi\, dx = \int_{\mathbb{R}^N} f(x,u,\nabla u)\psi\, dx \quad \forall \psi \in C_0^\infty\left(\mathbb{R}^N\right) \tag{1.2}$$

and $u > 0$ in $\mathbb{R}^N$.

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The existence and nonexistence of entire solutions, existence of multiple positive entire solutions of (1.1) for $f(x,u,\nabla u) = q(x)f(u)$ or $f(x,u,\nabla u) = -f(x,u)$, have been studied in previous papers (see [22, 24, 25]). Some other problems have also been treated by many other authors. See, for example, [5, 6, 8, 13–15, 18, 23, 26, 27].

When $f : (0,\infty) \to (0,\infty)$ and $q : \mathbb{R}^N \to (0,\infty)$ are continuous functions, and

$$\int_1^\infty \left(\int_0^u f(s)ds \right)^{-1/p} du = \infty, \tag{1.3}$$

it has been shown in [22] that there exist entire radially symmetric solutions of the problem

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = q(x)f(u), \quad x \in \mathbb{R}^N. \tag{1.4}$$

On the other hand, it was shown in [24] that the problem

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(x,u) = 0, \quad x \in \mathbb{R}^N, \tag{1.5}$$

possesses infinitely many positive entire solutions. When $f(x,u)$ defined on $\mathbb{R}^N$ is locally Hölder continuous in x and is locally Lipschitz continuous in u; there exist a locally Hölder continuous function $\psi(r) \geq 0$ on $[0,\infty)$, $\int_0^\infty (\int_0^s \psi(t)dt)^{1/(p-1)}ds < \infty$, and a locally Lipschitz continuous function $F(u) > 0$ on $(0,\infty)$ such that

$$f(x,u) \leq \psi(|x|)F(u), \quad (x,u) \in \mathbb{R}^N \times (0,\infty), \tag{1.6}$$

and $\lim_{u\to 0}(F(u)/u^{p-1}) = 0$.

Moreover, it was also shown in [25] that the problem

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + q(x)u^{-\gamma} = 0, \quad x \in \mathbb{R}^N, \tag{1.7}$$

has a positive entire solution if $1 < p < N$, $0 \leq \gamma < p-1$, and $q(x) \in C(\mathbb{R}^+)$ satisfy

(A_1) wherever $q(x_0) = 0$, $\exists r > 0$ such that $q(x) > 0$ on $\partial B(x_0,r)$, where $B(x_0,r)$ is the ball of radius r centered at x_0;

(A_2) for any $0 < \varepsilon < (N-p)(p-1-|\gamma|)/(p-1)$,

$$\int_1^\infty r^{p+\varepsilon-1+[(N-p)|\gamma|/(p-1)]} m(r)dr < \infty; \tag{1.8}$$

(A_3) for $r \in (0,1)$,

$$q(r) = O(r^{-\delta}), \quad \delta < 1. \tag{1.9}$$

Motivated by the results of the above-cited papers, we further study the existence of multiple positive entire solutions for (1.1), the results of the semilinear equations are extended to the quasilinear ones. We can find the related results for $p = 2$ in [10, 11, 17, 20, 21]. The main differences between $p = 2$ and $p \neq 2$ are known in [5, 6]. When $p = 2$, it is well known that all positive solutions in $C^2(B_R)$ of the problem

$$\begin{aligned} \triangle u + f(u) &= 0 \quad \text{in } B_R, \\ u(x) &= 0 \quad \text{on } \partial B_R \end{aligned} \tag{1.10}$$

are radially symmetric solutions for very general f (see [3]). Unfortunately, this result does not apply to the case $p \neq 2$. Kichenassamy and Smoller showed that there exist many positive nonradial solutions of the above problem for some f (see [9]). The major stumbling block in the case of $p \neq 2$ is that certain nice features inherent to the case $p = 2$ seem to be lost or at least difficult to verify. In this paper, we obtain the existence of multiple positive entire solutions for a class of f, extended to the results in [11, 20, 21] and complement the results by [22, 24, 25].

2. Some preliminary lemmas

Before we prove the main results, we need the following definitions and lemmas.

Definition 2.1. $\overline{u} \in W^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ is called a supersolution to problem (1.1) if

$$-\int_{\mathbb{R}^N} |\nabla \overline{u}|^{p-2} \nabla \overline{u} \cdot \nabla \psi \, dx \geq \int_{\mathbb{R}^N} f(x, \overline{u}, \nabla \overline{u}) \psi \, dx \quad \forall \psi \in C_0^\infty(\mathbb{R}^N) \tag{2.1}$$

and $\overline{u} > 0$ in $\mathbb{R}^N$. Similarly, $\underline{u} \in W^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ is called a subsolution to problem (1.1) if

$$-\int_{\mathbb{R}^N} |\nabla \underline{u}|^{p-2} \nabla \underline{u} \cdot \nabla \psi \, dx \leq \int_{\mathbb{R}^N} f(x, \underline{u}, \nabla \underline{u}) \psi \, dx \quad \forall \psi \in C_0^\infty(\mathbb{R}^N) \tag{2.2}$$

and $\underline{u} > 0$ in $\mathbb{R}^N$.

For (1.1), the following hypotheses on f are adopted.

(A) $f(x,u,v)$ is a continuous function in $\mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^N$ and locally Lipschitz continuous.

(B) For every bounded domain $\Omega \subset \mathbb{R}^N$, for any $M > 0$, $\exists \rho(\Omega, M) > 0$ such that

$$|f(x,u,v)| < \rho(\Omega, M)\left(1 + |v|^p\right), \quad x \in \Omega,\ 0 \leq u \leq M,\ v \in \mathbb{R}^N. \tag{2.3}$$

(C) There exist nonnegative continuous functions defined in $(\mathbb{R}^+)^3$, $F_1(r,u,v)$, and $F_2(r,u,v)$, which are local Lipschitz continuous and satisfy

$$F_1\left(|x|, u, |v|\right) \leq f(x,u,v) \leq F_2\left(|x|, u, |v|\right), \tag{2.4}$$

where $(x,u,v) \in \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^N$, $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$.

Lemma 2.2. *Let $\overline{u}, \underline{u} \in W^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$, respectively, be supersolution and subsolution of (1.1) on $\mathbb{R}^N$ with $\underline{u}(x) \leq \overline{u}(x)$ on $\mathbb{R}^N$, and let (A), (B) hold. Then, (1.1) possesses an entire solution $u(x)$ with $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ on $\mathbb{R}^N$.*

Proof. Let B_R be the ball with radius R in $\mathbb{R}^N$. Consider the boundary value problem

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f(x,u,\nabla u), \quad x \in B_R, \tag{2.5}$$

$$u|_{\partial B_R} = g, \tag{2.6}$$

where $g(x)$ is a function that satisfies $\underline{u}(x) \le g(x) \le \overline{u}(x)$. For (2.5), (2.6), $\overline{u}(x)$ is still a supersolution (for every R) and $\underline{u}(x)$ is still a subsolution (for every R), and $\overline{u}(x) \ge \underline{u}(x)$ in B_R. By $C^{1,\alpha}(\overline{B}_R)$ estimates in [13] and monotonic iteration [16] or [1, 8], one concludes that there exists $u \in C^1(\overline{B}_R)$, which is a weak solution of (1.1) with $\underline{u}(x) \le u_R(x) \le \overline{u}(x)$ in B_R.

Now, we want to apply elliptic interior estimates together with a diagonal process to conclude that $\{u_R : R \ge 1\}$ has a subsequence $\{u_{R_k} : R_k \uparrow \infty\}$ such that $\{u_{R_k}\}$ converges to a function u in $\mathbb{R}^N$ (pointwise) and this convergence is in C^1 on every compact set in $\mathbb{R}^N$. (Therefore, $u \in C^1$ and $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(x,u,\nabla u)$ with $\underline{u}(x) \le u(x) \le \overline{u}(x)$, and this concludes the proof.)

Step 1. On $B_2, \{u_R : R \ge 2\}$ is uniformly bounded by $\overline{u}(x)$ and $\underline{u}(x)$. Since both $\overline{u}(x)$ and $\underline{u}(x)$ are bounded functions on B_2, there exists $M > 0$ such that $\|u_R(x)\|_{L^\infty(B_2)} \le M$ for all $R \ge 2$.

From (1.1), u_R satisfies

$$\int_{B_2} |\nabla u_R|^p = -\int_{B_2} f u_R. \tag{2.7}$$

Therefore,

$$\int_{B_2} |\nabla u_R|^p \le M(\operatorname{meas} B_2)^{1/q} C_1 \|\nabla u_R\|_p. \tag{2.8}$$

Here $1/q + 1/p = 1$, and C_1 is the Sobolev embedding constant. So, $\|u_R\|_{1,p} \le C_2$. When $1 < p < N$, the embedding of $W_0^{1,p}(B_2)$ in $L^{Np/(N-p)}(B_2)$ implies that $u_R \in L^{Np/(N-p)}(B_2)$. Applying [12, Theorem 7.1, pages 286, 287], we obtain the estimate

$$\sup\{|u_R|;\, x \in B_2\} \le C_3, \tag{2.9}$$

here $C_3 = C_3(\|f\|_0)$. If $p \ge N$, we get (2.9) from the Sobolev embedding theorem. Using [12, Theorem 1.1, page 251], we see that u_R belongs to $C^\alpha(\overline{B_2})$ for some $0 < \alpha < 1$, and

$$\|u_R\|_{C^\alpha} \le C_4, \tag{2.10}$$

here C_4 is determined by C_3. By [19, Proposition 3.7, page 806] we also know that u_R belongs to $C^{1,\alpha}(\overline{B_2})$ and

$$\|u_R\|_{C^{1,\alpha}} \le C_5, \tag{2.11}$$

here C_5 is determined by C_4.

From the arguments above we see that there exists $C > 0$ such that

$$\|u_R\|_{C^{1+\alpha}(B_1)} \le C \quad \forall R \ge 2. \tag{2.12}$$

Since the embedding $C^{1+\alpha}(B_1) \to C^1(B_1)$ is compact, there exists a sequence denoted by $\{u_{R_{1j}}\}_{j=1,\dots}$ (where $R_{1j} \uparrow \infty$), which converges in $C^1(B_1)$. Let $u_1(x) = \lim_{j\to\infty} u_{R_{1j}}(x)$ for $x \in B_1$; then u_1 is a solution of (2.5) with $\underline{u}(x) \le u_1 \le \underline{u}(x)$.
Step 2. Repeat Step 1 up to the existence of the sequence $\{u_{R_{1j}}\}_{j=1,\dots}$ to get a subsequence $\{u_{R_{2m}}\}_{m=1,2,\dots}$ converging in $C^1(B_2)$ to a limit u_2. Then likewise u_2 is a solution of (2.5), (2.6) and $u_2|_{B_1} = u_1$. Repeat Step 1 again on $B_3,\dots$, and so forth. In this way, we obtain a sequence $\{u_{R_{kj}}\}_{j=1,2,\dots}$ which converges in $C^1(B_k)$ and is a subsequence of $\{u_{R_{(k-1)j}}\}$. Let $u_k = \lim_{j\to\infty} u_{R_{kj}}$, then u_k is a solution of (2.5), (2.6) in B_k and $u_k|_{B_{k-1}} = u_{k-1}$.
Step 3. By a diagonal process, $\{u_{R_{mm}}\}_{m=1,2,\dots}$ is a subsequence of $\{u_{R_{kj}}\}_{j=1,2,\dots}$ for every k. Thus, on B_k for each k we have

$$\lim_{m\to\infty} u_{R_{mm}} = u_k. \tag{2.13}$$

So, if we define $u(x) = \lim_{m\to\infty} u_{R_{mm}}(x)$, then $u(x)$ satisfies

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f(x,u,\nabla u), \quad x \in \mathbb{R}^N, \tag{2.14}$$

and $\underline{u}(x) \le u(x) \le \overline{u}(x)$, since $\underline{u}(x) \le u_k(x) \le \overline{u}(x)$ for every k. This completes the proof of Lemma 2.2. □

Lemma 2.3. (i) *Let all $a,b > 0$ and $p \ge 2$, then*

$$\left| a^{1/(p-1)} - b^{1/(p-1)} \right| \le 2\frac{|a-b|}{a^{(p-2)/(p-1)} + b^{(p-2)/(p-1)}}. \tag{2.15}$$

(ii) *Let all $a,b \ge 0$ and $1 < p < 2$, then*

$$\left| a^{1/(p-1)} - b^{1/(p-1)} \right| \le 2^{1/(p-1)}|a-b|\left(a^{(2-p)/(p-1)} + b^{(2-p)/(p-1)}\right). \tag{2.16}$$

3. Main results

In this section we give the following main results.

Theorem 3.1. *Assume that f satisfies (A)–(C) and the following conditions hold.*

(I) *$F_1(r,u,v)$ and $F_2(r,u,v)$ are nonincreasing functions in $u \in \mathbb{R}^+$, nondecreasing in $v \in \mathbb{R}^+$ for each $r \ge 0$.*

(II)(a) *For $1 < p < 2$ and fixed $r \in \mathbb{R}^+$, $F_2^{1/(p-1)}(r,\lambda,m\lambda/p)/\lambda$ is nonincreasing for $\lambda \in (0,\infty)$ and satisfies that*

$$\lim_{\lambda\to+\infty} \frac{F_2^{1/(p-1)}(r,\lambda,m\lambda/p)}{\lambda} = 0, \tag{3.1}$$

where $m = ((2-p)/(N-p+1))^{(2-p)/(p-1)}$.

(b) *For* $2 \le p \le N+1$ *and fixed* $r \in \mathbb{R}^+$, $F_2^{p-1}(r,\lambda,m_1\lambda/p)/\lambda$ *is nonincreasing for* $\lambda \in (0,\infty)$ *and satisfies that*

$$\lim_{\lambda\to+\infty} \frac{F_2^{p-1}(r,\lambda,m_1\lambda/p)}{\lambda} = 0, \tag{3.2}$$

where $m_1 = ((p-2)/(N(p-1)-1))^{(p-2)/(p-1)^2}$.

(III)(a) *For* $1 < p < 2$, *there exists a positive constant* $c > 0$ *such that*

$$\int_0^\infty K(s) F_2^{1/(p-1)}\left(s,c,\frac{mc}{p}\right) ds < \infty, \tag{3.3}$$

where

$$K(s) = \begin{cases} 1 & \text{if } 0 \le s < 1, \\ s^{1/(p-1)} & \text{if } s > 1. \end{cases} \tag{3.4}$$

(b) *For* $2 \le p \le N+1$, *there exists a positive constant* c *such that*

$$\int_0^\infty K(s) F_2^{p-1}\left(s,c,\frac{m_1 c}{p}\right) ds < \infty, \tag{3.5}$$

where

$$K(s) = \begin{cases} 1 & \text{if } 0 \le s < 1, \\ s^{p-1} & \text{if } s > 1, \end{cases} \tag{3.6}$$

then (1.1) has infinitely many positive entire solutions $u(x)$.

Proof. It is easy to see that under conditions (A)–(C) a positive solution $\overline{u}(x)$ of the equation

$$\operatorname{div}\left(|\nabla \overline{u}|^{p-2}\nabla \overline{u}\right) = F_1(|x|,\overline{u},|\nabla \overline{u}|), \quad x \in \mathbb{R}^N, \tag{3.7}$$

is a supersolution of (1.1) in $\mathbb{R}^N$; and a positive solution $\underline{u}$ of the equation

$$\operatorname{div}\left(|\nabla \underline{u}|^{p-2}\nabla \underline{u}\right) = F_2(|x|,\underline{u},|\nabla \underline{u}|), \quad x \in \mathbb{R}^N, \tag{3.8}$$

is a subsolution of (1.1) in $\mathbb{R}^N$. Therefore, we only prove that (3.7), (3.8) have solutions and satisfy $\underline{u}(x) \le \overline{u}(x)$.

We first consider (3.7), in view of the spherical symmetry of $F_1(|x|,v,|\nabla v|)$, it is natural to seek spherically symmetric solutions of (3.7), and thus we are led to the one-dimensional initial value problems

$$\left(\Phi_p(y')\right)' + \frac{N-1}{r}\Phi_p(y') = F_1(r,y,|y'|), \quad r > 0, \tag{3.9}$$

$$y(0) = \eta, \qquad y'(0) = 0, \tag{3.10}$$

where $\Phi_p(y) = |y|^{p-2}y$, and η is a real number which is determined below. If $y(r)$ is a solution of (3.9) on $(0,\infty)$, then $v(x) = y(|x|)$ is a supersolution of (1.1) in $\mathbb{R}^N$.

It can be proved that the problem (3.9), (3.10) is equivalent to the following integral equation:

$$y(r) = \eta + \int_0^r \left(\int_0^s \left(\frac{t}{s}\right)^{N-1} F_1(t,y,|y'(t)|)\,dt \right)^{1/(p-1)} ds. \tag{3.11}$$

Therefore we only consider integral equation (3.11). To prove that (3.11) has a solution, we consider two cases here: (i) $1 < p \le 2$ and (ii) $p > 2$.

(i) For $1 < p < 2$, condition (II) implies that

$$K(s)\frac{F_2^{1/(p-1)}(s,\lambda,m\lambda/p)}{\lambda} \le K(s)F_2^{1/(p-1)}\left(s,c,\frac{mc}{p}\right), \quad \lambda \ge c, \tag{3.12}$$

where c is defined in (III), and for all $s \in (0,\infty)$, as $\lambda \to \infty$, we have

$$K(s)\frac{F_2^{1/(p-1)}(s,\lambda,m\lambda/p)}{\lambda} \longrightarrow 0. \tag{3.13}$$

According to (III), by Lebesgue dominated convergence theorem, we have

$$\frac{1}{\lambda}\int_0^\infty K(s)F_2^{1/(p-1)}\left(s,\lambda,\frac{m\lambda}{p}\right)ds \longrightarrow 0 \quad (\lambda \longrightarrow \infty) \tag{3.14}$$

from (a) of (II) implying that

$$\frac{1}{\lambda}\int_0^\infty K(s)F_1^{1/(p-1)}\left(s,\lambda,\frac{m\lambda}{p}\right)ds \longrightarrow 0 \quad (\lambda \longrightarrow \infty) \tag{3.15}$$

for all $1 < p < 2$. From (3.15), we can choose sufficiently large constants $\eta > 0$, such that

$$\int_0^\infty K(s)F_1^{1/(p-1)}\left(s,\eta,\frac{m\eta}{p}\right)ds < \frac{\eta}{p}. \tag{3.16}$$

Let Y be a set defined by

$$Y = \left\{ y \in C^1[0,\infty) \mid \eta \le y(r) \le 2\eta,\ 0 \le y'(r) \le \frac{m\eta}{p} \right\}. \tag{3.17}$$

Clearly, Y is a closed convex subset of $C^1[0,\infty)$. Furthermore, the mapping $F : Y \to C^1[0,\infty)$ is defined by

$$Fy(r) = \eta + \int_0^r \left(\int_0^s \left(\frac{t}{s}\right)^{N-1} F_1(t,y,|y'(t)|)\,dt \right)^{1/(p-1)} ds, \quad r \ge 0, \tag{3.18}$$

where the value at 0 as well as the following limit supplement define

$$Fy(0) = \lim_{r\to 0}\left[\eta + \int_0^r \left(\int_0^s \left(\frac{t}{s}\right)^{N-1} F_1(t,y,|y'(t)|)\,dt \right)^{1/(p-1)} ds \right] = \eta. \tag{3.19}$$

Next we will prove step by step that F is a continuous operator, mapping Y into a compact subset of Y. The following propositions are essential.

Proposition 3.2. *F is a mapping from Y to Y.*

In fact, if all $y \in Y$, it follows from (3.16) and (3.18) that

$$\begin{aligned}\eta \le Fy(r) &\le \eta + \int_0^r s^{1/(p-1)} F_1^{1/(p-1)}\left(s,\eta,\frac{m\eta}{p}\right) ds \\ &\le \eta + \int_0^r K(s) F_1^{1/(p-1)}\left(s,\eta,\frac{m\eta}{p}\right) ds \le 2\eta,\end{aligned} \tag{3.20}$$

and from (3.18), we have

$$\begin{aligned}0 < (Fy)'(r) &= \left(\int_0^r \left(\frac{t}{r}\right)^{N-1} F_1(t,y,|y'|)\,dt\right)^{1/(p-1)} \\ &\le \left[\left(\int_0^r \left(\frac{t}{r}\right)^{(N-1)/(2-p)} dt\right)^{2-p} \left(\int_0^r F_1^{1/(p-1)}(t,y,|y'|)\,dt\right)^{p-1}\right]^{1/(p-1)} \\ &= mr^{(2-p)/(p-1)} \int_0^r F_1^{1/(p-1)}(t,y,|y'|)\,dt \\ &\le m\int_0^\infty K(s) F_1^{1/(p-1)}\left(s,\eta,\frac{m\eta}{p}\right) ds \le \frac{m\eta}{p}\end{aligned} \tag{3.21}$$

for $r > 0$. It follows from (3.21) that

$$\lim_{r\to 0^+} [(Fy)'(r)]^{p-1} = \lim_{r\to 0^+} \frac{\int_0^r t^{N-1} F_1(t,y,|y'|)\,dt}{r^{N-1}} = 0, \tag{3.22}$$

then $\lim_{r\to 0^+}(Fy)'(r) = 0$. Therefore, we have

$$(Fy)'(0) = \lim_{r\to 0^+} \frac{[Fy(r) - Fy(0)]}{r} = \lim_{\xi\to 0^+} (Fy)'(\xi) = 0; \tag{3.23}$$

and then let

$$\left.\left(\int_0^r \left(\frac{t}{r}\right)^{N-1} F_1(t,y,|y'|)\,dt\right)^{1/(p-1)}\right|_{r=0} = 0. \tag{3.24}$$

So (3.18) holds for $r \ge 0$, and we have

$$0 \le (Fy)'(r) \le \frac{m\eta}{p}, \quad r \ge 0. \tag{3.25}$$

Proposition 3.3. *F is continuous.*

Let $\{y_k\}$ be a sequence in Y converging to $y \in Y$. By (3.18) and Lemma 2.3, we have

$$
\begin{aligned}
&|Fy_k(r) - Fy(r)| \\
&\quad \le \left| \int_0^r \left[\Phi_p^{-1}\left(\int_0^s \left(\frac{t}{s}\right)^{N-1} F_1(t, y_k, |y_k'|)\,dt \right) - \Phi_p^{-1}\left(\int_0^s \left(\frac{t}{s}\right)^{N-1} F_1(t, y, |y'|)\,dt \right) \right] ds \right| \\
&\quad \le 2^{1/(p-1)} \left| \int_0^\infty \left(\int_0^s |F_1(t, y_k, |y_k'|) - F_1(t, y, |y'|)|\,dt \right) \left(A^{(2-p)/(p-1)} + B^{(2-p)/(p-1)}\right) ds \right|,
\end{aligned}
\tag{3.26}
$$

where

$$
A = \int_0^s \left(\frac{t}{s}\right)^{N-1} F_1(t, y_k, |y_k'|)\,dt, \qquad B = \int_0^s \left(\frac{t}{s}\right)^{N-1} F_1(t, y, |y'|)\,dt. \tag{3.27}
$$

Note that $\Psi_k(s) = 2^{1/(p-1)} \int_0^s |F_1(t, y_k, |y_k'|) - F_1(t, y, |y'|)|\,dt (A^{(2-p)/(p-1)} + B^{(2-p)/(p-1)})$ satisfies

$$
\Psi_k(s) \le 2^{(p+1)/(p-1)} \int_0^s F_1^{1/(p-1)}\left(t, \eta, \frac{m\eta}{p}\right) dt, \tag{3.28}
$$

$\Psi_k(s) \to 0$ pointwise on $[0, \infty)$ as $k \to \infty$. From the Lebesgue dominated convergence theorem, it follows that $Fy_k(r)$ converges to $Fy(r)$ uniformly on $[0, \infty)$ as $k \to \infty$, and hence $Fy_k(r) \to Fy(r)$ in $C[0, \infty)$ as $k \to \infty$. On the other hand,

$$
\begin{aligned}
&|(Fy_k)'(r) - (Fy)'(r)| \\
&\quad = \left| \Phi_p^{-1}\left(\int_0^s \left(\frac{t}{s}\right)^{N-1} F_1(t, y_k, |y_k'|)\,dt \right) - \Phi_p^{-1}\left(\int_0^s \left(\frac{t}{s}\right)^{N-1} F_1(t, y, |y'|)\,dt \right) \right|.
\end{aligned}
\tag{3.29}
$$

Similarly, we have $(Fy_k)'(r)$ converging to $(Fy)'(r)$ uniformly on $[0, \infty)$ as $k \to \infty$. Thus, F is continuous.

Proposition 3.4. *FY is relatively compact.*

It suffices to show that $\{Fy(r) \mid y \in Y\}$ and $\{(Fy)'(r) \mid y \in Y\}$ are uniformly bounded and equicontinuous on every subset $[0, M] \subset [0, \infty)$. From (3.20), (3.21), the uniform boundedness is obvious, so we only prove that $\{(Fy)'(r) \mid y \in Y\}$ is equicontinuous in $[0, M]$.

From (3.21), we obtain

$$
\begin{aligned}
(Fy)'(r) &= \left(\int_0^r \left(\frac{s}{r}\right)^{N-1} F_1(s, y(s), |y'(s)|)\,ds \right)^{1/(p-1)}, \\
(\Phi_p((Fy)'(r)))' &= F_1(r, y(r), |y'(r)|) - (N-1) \int_0^r s^{N-1} r^{-N} F_1(s, y, |y'(s)|)\,ds,
\end{aligned}
\tag{3.30}
$$

then

$$\left|(\Phi_p((Fy)'(r)))'\right| \leq F_1\left(r,\eta,\frac{m\eta}{p}\right)+\frac{N-1}{r}\int_0^r F_1\left(s,\eta,\frac{m\eta}{p}\right)ds. \tag{3.31}$$

From (3.30), we have

$$\lim_{r\to 0^+}(\Phi_p((Fy)'(r)))' = \frac{1}{N}F_1(0,y(0),|y'(0)|), \tag{3.32}$$

then

$$(\Phi_p(Fy)')'(0) = \lim_{r\to 0^+}\frac{[\Phi_p((Fy)'(r))-\Phi_p((Fy)'(0))]}{r} = \frac{1}{N}F_1(0,y(0),|y'(0)|). \tag{3.33}$$

Therefore,

$$\left|(\Phi_p((Fy)'))'(0)\right| = \frac{1}{N}F_1(0,y(0),|y'(0)|) \leq F_1\left(0,\eta,\frac{m\eta}{p}\right). \tag{3.34}$$

For (3.25), we let

$$\left[\frac{N-1}{r}\int_0^r F_1\left(s,\eta,\frac{m\eta}{p}\right)ds\right]\bigg|_{r=0} = (N-1)F_1\left(0,\eta,\frac{m\eta}{p}\right), \tag{3.35}$$

and then (3.31) holds for all $r \geq 0$, therefore

$$\left|(\Phi_p((Fy)')'(r)\right| \leq F_1\left(r,\eta,\frac{m\eta}{p}\right)+\frac{N-1}{r}\int_0^r\frac{F_1(s,\dots)}{r}ds, \quad r\geq 0. \tag{3.36}$$

Consequently, for $((Fy)'(r))'$ we have estimates in $[0,M]$:

$$\begin{aligned}\max_{0\le r\le M}&\left|\Phi_p((Fy)'(r))'\right| \\ &\leq \max_{0\le r\le M}F_1\left(r,\eta,\frac{m\eta}{p}\right)+\max_{0\le r\le M}\left|\frac{N-1}{r}\int_0^r F_1\left(s,\eta,\frac{m\eta}{p}\right)ds\right| = L_M,\end{aligned} \tag{3.37}$$

then

$$\left|\Phi_p((Fy)'(r_1))-\Phi_p((Fy)'(r_2))\right| \leq L_M|r_1-r_2|, \quad r_1,r_2\in[0,M]. \tag{3.38}$$

On the other hand,

$$\left|(Fy)'(r_1)-(Fy)'(r_2)\right| \leq \sup\left|(\Phi_p^{-1})'(x)\right||r_1-r_2|, \quad r_1,r_2\in[0,M], \tag{3.39}$$

then $\{(Fy)'(r) \mid y\in Y\}$ is equicontinuous in $[0,M]$.

To prove that FY is relatively compact in Y, we will prove every sequence $\{Fy_k(r)\}$ which has convergent subsequence in Y. We only need to use Ascoli-Arzela theorem in turn for the sequence of interval $[0,M_1]\subset[0,M_2]\subset\cdots\subset[0,M_j]\subset\cdots$ (where $M_j\to\infty$ as $j\to\infty$), and use the diagonalization argument as in [25]. Thus we are able to apply the

Schauder-Tychonoff fixed point theorem and conclude that F has a fixed point y in Y. This fixed point $y = y(r)$ is a solution of (3.9), and so we obtain a supersolution $y(x)$ of (1.1) in $\mathbb{R}^N$ defined by $y(x) = y(|x|)$.

(ii) For $2 < p \le N+1$, since $\Phi_p^{-1} \notin C^1(R)$ (Φ_p^{-1} is the inverse function of $\Phi_p(y) = |y|^{p-2}y$), then we consider the perturbation equation

$$(r^{N-1}g_\epsilon(y'))' = r^{N-1}F_1(r,y,|y'|), \tag{3.40}$$

$$y(0) = \eta, \qquad y'(0) = 0, \tag{3.41}$$

where $g_\epsilon(y) = \epsilon y + \Phi_p(y)$. From $g_\epsilon^{-1} \in C^1(R)$, and then similarly with (i), we have for all $\epsilon > 0$, (3.40), (3.41) has a bounded positive solution $y_\epsilon \in C^1(R)$, and similarly with (i), we have

$$\max_{0\le r\le M} \left|\left(g_\epsilon\left((Fy_\epsilon)'(r)\right)\right)'(r)\right| \le L_M \quad \text{(be independent of } \epsilon). \tag{3.42}$$

From Ascoli-Arzela theorem, we have $g_\epsilon(y_\epsilon') \to v$ ($\epsilon \to 0$), where $v \in C(R)$. Since $\|y_\epsilon'\|_0$ is bounded, it follows that $\Phi_p(y_\epsilon) \to v$ ($\epsilon \to 0$). Since Φ_p is $R \to R$ strictly increasing which implies that $y_\epsilon' \to \Phi_p^{-1}(v)$ ($\epsilon \to 0$), then we have

$$y_\epsilon = \eta + \int_0^r y_\epsilon'(s)ds \longrightarrow \eta + \int_0^t \Phi_p^{-1}(v(s))ds = y \quad (\epsilon \longrightarrow 0). \tag{3.43}$$

From (3.40), (3.41), we have y_ϵ satisfying

$$\begin{aligned} \epsilon r^{N-1}y_\epsilon' + r^{N-1}\Phi_p(y_\epsilon') &= \eta + \int_0^s s^{N-1}F_1(s,y_\epsilon,y_\epsilon')ds, \\ y_\epsilon(0) = \eta, \qquad &y_\epsilon'(0) = 0, \end{aligned} \tag{3.44}$$

and then for all $r \in [0,M]$, let $\epsilon \to 0$, $y \in C^1[0,M]$ a bounded entire solution of (3.7).

Similarly, we consider the initial value problem for (3.8):

$$\begin{aligned} (\Phi_p(z'))' + \frac{N-1}{r}\Phi_p(z') &= F_2(r,z,|z'|), \quad r > 0, \\ z(0) = \xi, \qquad &z'(0) = 0, \end{aligned} \tag{3.45}$$

where ξ is a real number which is determined below. Further we consider equivalence integral equation equivalent to (3.45):

$$z(r) = \xi + \int_0^r \left(\int_0^s \left(\frac{t}{s}\right)^{N-1} F_2(t,z,|z'|)dt\right)^{1/(p-1)} ds. \tag{3.46}$$

From (3.14), we can choose sufficiently large constant $\xi > 0$, satisfying

$$\int_0^\infty K(s)F_2^{1/(p-1)}\left(s,\xi,\frac{m\xi}{p}\right)ds < \frac{\xi}{p}. \tag{3.47}$$

Let us set

$$Z = \left\{ z \in C^1[0,\infty) \mid \xi \le z(r) \le 2\xi,\ 0 \le z'(r) \le \frac{m\xi}{p},\ r \ge 0 \right\}. \tag{3.48}$$

Define mapping Ψ:

$$\Psi(r) = \xi + \int_0^r \left(\int_0^s \left(\frac{t}{s}\right)^{N-1} F_2(t, z, |z'|)\, dt \right)^{1/(p-1)} ds. \tag{3.49}$$

Similarly, $\Psi(r)$ has a fixed point in $z \in Z$.

In the above, constants η and ξ have to satisfy not only (3.14) and (3.40), but also they need to satisfy $2\xi \le \eta$ (this we can realize choosing first ξ which satisfies (3.40), fixing, and finally adjusting ξ, so that we choose η which satisfies $2\xi \le \eta$ and (3.18)). Therewith, we have

$$0 < \xi \le z(r) \le 2\xi \le \eta \le y(r) \le 2\eta. \tag{3.50}$$

On the other hand, if the problem (3.9) has a solution $y(r)$, then $v(x) = y(|x|) = y(r)$ is a solution of (3.7). Similarly, if the initial value problem (3.45) has solutions $z(r)$, then $w(x) = z(|x|) = z(r)$ is a solution of (3.8). From (3.50), it follows that

$$0 < \xi \le w(x) \le v(x) \le 2\eta, \quad x \in \mathbb{R}^N. \tag{3.51}$$

From Lemma 2.2, (1.1) has at least a solution $u(x)$ and satisfies

$$w(x) \le u(x) \le v(x), \quad x \in \mathbb{R}^N. \tag{3.52}$$

Here and now we have proved that (1.1) exists for positive entire solution $u(x)$. From (3.51), (3.52), we see that all super- and subbounds of positive solutions are dependent choice of sufficiently large positive number ξ, η. If we choose the number pair (ξ_j, η_j) $(j = 1, 2, \dots)$, using the closed interval of the closed interval set which is each other nonintersect $\{(\xi_j, \eta_j) \mid j = 1, 2, \dots\}$, then we obtain each other difference bounded positive solutions of (1.1) $u_j(x)$, $j = 1, 2, \dots$, therefore (1.1) possesses infinitely many positive entire solutions. □

Remark 3.5. When $p = 2$, in [11, 20, 21] relative results were obtained and our results can be seen as their extensions.

Remark 3.6. If condition (II) of Theorem 3.1 is replaced by

(II)$_2$(a) for $1 < p < 2$ and fixed $r \in \mathbb{R}^+$, $F_2^{1/(p-1)}(r, \lambda, m\lambda/p)/\lambda$ is nondecreasing for $\lambda \in (0, \infty)$ and satisfies

$$\lim_{\lambda \to 0^+} \frac{F_2^{1/(p-1)}(r, \lambda, m\lambda/p)}{\lambda} = 0, \tag{3.53}$$

where $m = ((2-p)/(N-p+1))^{(2-p)/(p-1)}$;

(b) for $2 \le p \le N+1$ and fixed $r \in \mathbb{R}^+$, $F_2^{p-1}(r,\lambda,m_1\lambda/p)/\lambda$ is nondecreasing for $\lambda \in (0,\infty)$ and satisfies

$$\lim_{\lambda\to 0^+} \frac{F_2^{p-1}(r,\lambda,m_1\lambda/p)}{\lambda} = 0, \tag{3.54}$$

where $m_1 = ((p-2)/(N(p-1)-1))^{(p-2)/(p-1)^2}$;

then the conclusion of Theorem 3.1 holds.

Theorem 3.7. *Suppose that f satisfies (A)–(C) and the following conditions hold.*

(I) *$F_1(r,u,v)$ and $F_2(r,u,v)$ are nondecreasing for $u \in \mathbb{R}^+$, and nonincreasing for $v \in \mathbb{R}^+$.*

(II)(a) *For $1 < p < 2$ and fixed $r \in \mathbb{R}^+$, $F_2^{1/(p-1)}(r,\lambda,0)/\lambda$ is nonincreasing for $\lambda \in (0,\infty)$ and satisfies*

$$\lim_{\lambda\to \infty} \frac{F_2^{1/(p-1)}(r,\lambda,0)}{\lambda} = 0. \tag{3.55}$$

(b) *For $2 \le p \le N+1$ and fixed $r \in \mathbb{R}^+$, $F_2^{p-1}(r,\lambda,0)/\lambda$ is nonincreasing for $\lambda \in (0,\infty)$ and satisfies*

$$\lim_{\lambda\to \infty} \frac{F_2^{p-1}(r,\lambda,0)}{\lambda} = 0. \tag{3.56}$$

(III)(a) *For $1 < p < 2$, there exists a positive constant c such that*

$$\int_0^\infty K(s)F_2^{1/(p-1)}(s,c,0)\,ds < \infty, \tag{3.57}$$

where

$$K(s) = \begin{cases} 1 & \text{if } 0 \le s < 1, \\ s^{1/(p-1)} & \text{if } s > 1. \end{cases} \tag{3.58}$$

(b) *For $2 \le p \le N+1$, there exists a positive constant c such that*

$$\int_0^\infty K(s)F_2^{p-1}(s,c,0)\,ds < \infty, \tag{3.59}$$

where

$$K(s) = \begin{cases} 1 & \text{if } 0 \le s < 1, \\ s^{p-1} & \text{if } s > 1. \end{cases} \tag{3.60}$$

Then the conclusion of Theorem 3.1 holds.

Remark 3.8. If condition (II) of Theorem 3.7 is replaced by

$(\text{II})_2$(a) for $1 < p < 2$ and fixed $r \in \mathbb{R}^+$, $F_2^{1/(p-1)}(r,\lambda,0)/\lambda$ is nondecreasing for $\lambda \in (0,\infty)$ and satisfies

$$\lim_{\lambda \to 0^+} \frac{F_2^{1/(p-1)}(r,\lambda,0)}{\lambda} = 0; \tag{3.61}$$

(b) for $2 \le p \le N+1$ and fixed $r \in \mathbb{R}_+$, $F_2^{p-1}(r,\lambda,0)/\lambda$ is nondecreasing for $\lambda \in (0,\infty)$ and satisfies

$$\lim_{\lambda \to 0^+} \frac{F_2^{p-1}(r,\lambda,0)}{\lambda} = 0; \tag{3.62}$$

then the conclusion of Theorem 3.1 holds.

Theorem 3.9. *Assume that f satisfies (A)–(C) and the following conditions hold.*

(I) *$F_1(r,u,v)$ and $F_2(r,u,v)$ are nonincreasing in $u \in (0,\infty)$ and nonincreasing in $v \in (0,\infty)$ for all fixed $r \in \mathbb{R}^+$.*

(II)(a) *For $1 < p < 2$, there exists a positive constant c such that*

$$\int_0^\infty K(s) F_2^{1/(p-1)}(s,c,0)\,ds < \infty, \tag{3.63}$$

where

$$K(s) = \begin{cases} 1 & \text{if } 0 \le s < 1, \\ s^{1/(p-1)} & \text{if } s > 1. \end{cases} \tag{3.64}$$

(b) *For $2 \le p \le N+1$, there exists a positive constant c such that*

$$\int_0^\infty K(s) F_2^{p-1}(s,c,0)\,ds < \infty, \tag{3.65}$$

where

$$K(s) = \begin{cases} 1 & \text{if } 0 \le s < 1, \\ s^{p-1} & \text{if } s > 1. \end{cases} \tag{3.66}$$

Then the conclusion of Theorem 3.1 holds.

Theorem 3.10. *Assume that f satisfies (A)–(C) and the following conditions hold.*

(I) *$F_1(r,u,v)$ and $F_2(r,u,v)$ are nondecreasing in $u \in (0,\infty)$ and nondecreasing in $v \in (0,\infty)$ for all fixed $r \in \mathbb{R}^+$.*

(II)(a) *For $1 < p < 2$ and fixed $r \in \mathbb{R}^+$, $F_2^{1/(p-1)}(r, p\lambda, m\lambda/p)/\lambda$ is nonincreasing for $\lambda \in (0,\infty)$ and satisfies*

$$\lim_{\lambda\to+\infty} \frac{F_2^{1/(p-1)}(r, p\lambda, m\lambda/p)}{\lambda} = 0, \tag{3.67}$$

where $m = ((2-p)/(N-p+1))^{(2-p)/(p-1)}$.

(b) *For $2 \le p \le N+1$ and fixed $r \in \mathbb{R}^+$, $F_2^{p-1}(r, p\lambda, m_1\lambda/p)/\lambda$ is nonincreasing for $\lambda \in (0,\infty)$ and satisfies*

$$\lim_{\lambda\to+\infty} \frac{F_2^{p-1}(r, p\lambda, m_1\lambda/p)}{\lambda} = 0, \tag{3.68}$$

where $m_1 = ((p-2)/(N(p-1)-1))^{(p-2)/(p-1)^2}$.

(III)(a) *For $1 < p < 2$, there exists a positive constant $c > 0$ such that*

$$\int_0^\infty K(s) F_2^{1/(p-1)}\left(s, pc, \frac{mc}{p}\right) ds < \infty, \tag{3.69}$$

where

$$K(s) = \begin{cases} 1 & \text{if } 0 \le s < 1, \\ s^{1/(p-1)} & \text{if } s > 1. \end{cases} \tag{3.70}$$

(b) *For $2 \le p \le N+1$, there exists a positive constant c such that*

$$\int_0^\infty K(s) F_2^{p-1}\left(s, pc, \frac{m_1 c}{p}\right) ds < \infty, \tag{3.71}$$

where

$$K(s) = \begin{cases} 1 & \text{if } 0 \le s < 1, \\ s^{p-1} & \text{if } s > 1. \end{cases} \tag{3.72}$$

Then the conclusion of Theorem 3.1 holds.

4. Example

Example 4.1. Consider the equation

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \psi(x) e^{-|x|^2 u^\alpha (p-1)} |\nabla u|^{p-1}, \quad x \in \mathbb{R}^N,\ N \ge 3, \tag{4.1}$$

where $0 < \alpha < p-1$, $p > 1$, and $\psi(x) > 0$ is locally Lipschitz continuous in $\mathbb{R}^N$.

Let $f(x,u,\nabla u) = \psi(x)e^{-|x|^2u^\alpha(p-1)}|\nabla u|^{p-1}$, $\psi^*(r) = \max_{|x|=r}\psi(x)$, and $\psi_*(r) = \min_{|x|=r}\psi(x)$. Choose

$$F_1(|x|,u,|v|) = \begin{cases} \psi_*(|x|)e^{-|x|^2u^\alpha(p-1)}|v|^{p-1} & \text{if } 1<p<2, \\ \psi_*(|x|)e^{-|x|^2u^\alpha/(p-1)}|v|^{1/(p-1)} & \text{if } p\geq 2, \end{cases}$$
$$F_2(|x|,u,|v|) = \begin{cases} \psi^*(|x|)e^{-|x|^2u^\alpha(p-1)}|v|^{p-1} & \text{if } 1<p<2, \\ \psi^*(|x|)e^{-|x|^2u^\alpha/(p-1)}|v|^{1/(p-1)} & \text{if } p\geq 2. \end{cases} \tag{4.2}$$

It is easy to check that (i) $F_1(r,u,v)$, $F_2(r,u,v)$ satisfy condition (I) of Theorem 3.1; (ii) $F_2(r,u,v)$ satisfies condition (II) of Theorem 3.1, $F_1(r,u,t)$, $F_2(r,u,t)$ are nonincreasing functions for $u\in\mathbb{R}^+$, which is a nondecreasing function for $t\in\mathbb{R}^+$, and for $1<p<2$ and fixed $r\in\mathbb{R}^+$,

$$\frac{F_2^{1/(p-1)}(r,\lambda,m\lambda/p)}{\lambda} = (\psi^*(r))^{1/(p-1)}e^{-r^2\lambda^\alpha}\frac{p}{m} \tag{4.3}$$

is nonincreasing for $\lambda\in(0,\infty)$ and satisfies

$$\lim_{\lambda\to+\infty}\frac{F_2^{p-1}(r,\lambda,m\lambda/p)}{\lambda} = 0, \tag{4.4}$$

where $m = ((2-p)/(N-p+1))^{(2-p)/(p-1)}$. For $p\geq 2$, being similar to $1<p<2$, suppose that

$$\int_0^\infty K(s)F_2^{1/(p-1)}\left(s,c,\frac{mc}{p}\right)ds = \left(\frac{mc}{p}\right)^{p-1}\int_0^\infty K(s)(\psi^*(s))^{1/(p-1)}e^{-s^2c^\alpha}ds < \infty \tag{4.5}$$

(where c is a certain positive constant) holds. Then from Theorem 3.1 it follows that (4.1) possesses infinitely many positive entire solutions $u(x)$.

Example 4.2. Consider the equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \psi(x)u^{\alpha(p-1)}e^{-|x||\nabla u|^\beta(p-1)}, \quad x\in\mathbb{R}^N, \tag{4.6}$$

where $\alpha,\beta\in(0,p-1)$, $p>1$.

Let

$$f(x,u,\nabla u) = \psi(x)u^{\alpha(p-1)}e^{-|x||\nabla u|^\beta(p-1)},$$

$$F_1(|x|,u,|v|) = \begin{cases} \psi_*(|x|)u^{\alpha(p-1)}e^{-|x|^2|v|^\beta(p-1)} & \text{if } 1<p<2, \\ \psi_*(|x|)u^{\alpha/(p-1)}e^{-|x|^2|v|^\beta/(p-1)} & \text{if } p\geq 2, \end{cases} \tag{4.7}$$
$$F_1(|x|,u,|v|) = \begin{cases} \psi^*(|x|)u^{\alpha(p-1)}e^{-|x|^2|v|^\beta(p-1)} & \text{if } 1<p<2, \\ \psi^*(|x|)u^{\alpha/(p-1)}e^{-|x|^2|v|^\beta/(p-1)} & \text{if } p\geq 2, \end{cases}$$

where $\psi(x)$, $\psi_*(r)$, $\psi^*(r)$ are defined in Example 4.1. It is easy to prove that the conditions of Theorem 3.1 are satisfied. Suppose that

$$\int_0^\infty K(s)\left(\psi^*(s)\right)^{1/(p-1)}ds < \infty \tag{4.8}$$

holds. Then (4.6) possesses infinitely many positive entire solutions $u(x)$.

Example 4.3. Consider the equation

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \psi(x)\left(u^\alpha + u^\beta + |\nabla u|^\gamma\right), \quad x \in \mathbb{R}^N,\ N \geq 3, \tag{4.9}$$

where $\psi(x)$, $\psi_*(r)$, $\psi^*(r)$ are defined in Example 4.1, $\alpha,\beta,\gamma \in (0,1)$, and $\beta+\gamma < p-1$, $p > 1$. It is easy to prove that the conditions of Theorem 3.7 are satisfied. Moreover, suppose that

$$\int_0^\infty K(s)\left(\psi^*(s)\right)^{1/(p-1)}ds < \infty \tag{4.10}$$

holds. Then (4.9) possesses infinitely many positive entire solutions $u(x)$.

Example 4.4. Consider the equation

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \psi(x)\frac{|\nabla u|^\beta}{(1+u)^\alpha}, \tag{4.11}$$

where $\alpha,\beta > 0$ and $\beta - \alpha > p-1$, $p > 1$, with $\psi(x)$, $\psi_*(r)$, $\psi^*(r)$ are defined in Example 4.1. It is easy to prove that the conditions of Theorem 3.9 are satisfied. Moreover, suppose that

$$\int_0^\infty K(s)\left(\psi^*(s)\right)^{1/(p-1)}ds < \infty \tag{4.12}$$

holds. Then (4.11) possesses infinitely many positive entire solutions $u(x)$.

Example 4.5. Consider the equation

$$\triangle u = e^{-\gamma|x|}u^\alpha|\nabla u|^\beta, \quad x \in \mathbb{R}^2, \tag{4.13}$$

where α, β, γ are all positive constants. Let $f(r,u,v) = e^{\gamma r}u^\alpha v^\beta$. Clearly if $\gamma > 0$, $\beta \geq 0$, either $\alpha+\beta < p-1$ or $\alpha+\beta > p-1$, then it is easy to verify that the conditions of Theorem 3.10 are satisfied. Thus, Theorem 3.1 holds for (4.13).

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