MIXED JACOBI-LIKE FORMS OF SEVERAL VARIABLES

MIN HO LEE

Received 4 November 2005; Accepted 26 March 2006

We study mixed Jacobi-like forms of several variables associated to equivariant maps of the Poincaré upper half-plane in connection with usual Jacobi-like forms, Hilbert modular forms, and mixed automorphic forms. We also construct a lifting of a mixed automorphic form to such a mixed Jacobi-like form.

1. Introduction

Jacobi-like forms of one variable are formal power series with holomorphic coefficients satisfying a certain transformation formula with respect to the action of a discrete subgroup Γ of $SL(2,\mathbb{R})$, and they are related to modular forms for Γ, which of course play a major role in number theory. Indeed, by using this transformation formula, it can be shown that that there is a one-to-one correspondence between Jacobi-like forms whose coefficients are holomorphic functions on the Poincaré upper half-plane and certain sequences of modular forms of various weights (cf. [1, 12]). More precisely, each coefficient of such a Jacobi-like form can be expressed in terms of derivatives of a finite number of modular forms in the corresponding sequence. Jacobi-like forms are also closely linked to pseudodifferential operators, which are formal Laurent series for the formal inverse ∂^{-1} of the differentiation operator ∂ with respect to the given variable (see, e.g., [1]). In addition to their natural connections with number theory and pseudodifferential operators, Jacobi-like forms have also been found to be related to conformal field theory in mathematical physics in recent years (see [2, 10]).

The generalization of Jacobi-like forms to the case of several variables was studied in [8] in connection with Hilbert modular forms, which are essentially modular forms of several variables. As it is expected, Jacobi-like forms of several variables correspond to sequences of Hilbert modular forms. Another type of generalization can be provided by considering mixed Jacobi-like forms of one variable for a discrete subgroup $\Gamma \subset SL(2,\mathbb{R})$, which are associated to a holomorphic map of the Poincaré upper half-plane that is equivariant with respect to a homomorphism of Γ into $SL(2,\mathbb{R})$ (cf. [7, 9]). Mixed Jacobi-like

Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2006, Article ID 31542, Pages 1–14
DOI 10.1155/IJMMS/2006/31542

forms are related to mixed automorphic forms, and examples of mixed automorphic forms include holomorphic forms of the highest degree on the fiber product of elliptic surfaces (see [6]).

In this paper, we study mixed Jacobi-like forms of several variables associated to equivariant maps of the Poincaré upper half-plane in connection with usual Jacobi-like forms, Hilbert modular forms, and mixed automorphic forms. We also construct a lifting of a mixed automorphic form to such a mixed Jacobi-like form.

2. Jacobi-like forms

In this section, we review Jacobi-like forms of several variables and describe some of their properties. We also describe Hilbert modular forms, which are closely linked to such Jacobi-like forms.

Throughout this paper, we fix a positive integer n. Let $(z_1,\dots,z_n)$ be the standard coordinate system for $\mathbb{C}^n$, and denote the associated partial differentiation operators by

$$\partial_1 = \frac{\partial}{\partial z_1},\dots,\partial_n = \frac{\partial}{\partial z_n}. \tag{2.1}$$

We will often use the multi-index notation. Thus, given $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}^n$ and $u = (u_1,\dots,u_n) \in \mathbb{C}^n$, we have

$$\partial^\alpha = \partial_1^{\alpha_1}\dots\partial_n^{\alpha_n}, \qquad u^\alpha = u_1^{\alpha_1}\dots u_n^{\alpha_n}, \tag{2.2}$$

and for $\beta = (\beta_1,\dots,\beta_n) \in \mathbb{Z}^n$, we write $\alpha \le \beta$ if $\alpha_i \le \beta_i$ for each $i = 1,\dots,n$. Furthermore, we also write $\boldsymbol{c} = (c,\dots,c) \in \mathbb{Z}^n$ if $c \in \mathbb{Z}$, and denote by $\mathbb{Z}_+$ the set of nonnegative integers. Given $\alpha \in \mathbb{Z}^n$ and $\beta \in \mathbb{Z}_+^n$, we write $\beta! = \beta_1!\dots\beta_n!$ and

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1}\cdots\binom{\alpha_n}{\beta_n}, \tag{2.3}$$

where for $1 \le i \le n$, we have $\binom{\alpha_i}{0} = 1$ and

$$\binom{\alpha_i}{\beta_i} = \frac{\alpha_i(\alpha_i-1)\cdots(\alpha_i-\beta_i+1)}{\beta_i!} \tag{2.4}$$

for $\beta_i > 0$.

Let $\mathcal{H} \subset \mathbb{C}$ be the Poincaré upper half-plane. Then the usual action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{H}$ by linear fractional transformations induces an action of $\mathrm{SL}(2,\mathbb{R})^n$ on the product $\mathcal{H}^n$ of n copies of $\mathcal{H}$. Thus, if $\gamma \in \mathrm{SL}(2,\mathbb{R})^n$ and $z = (z_1,\dots,z_n) \in \mathcal{H}^n$ with

$$\gamma = (\gamma_1,\dots,\gamma_n), \qquad \gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \quad (1 \le i \le n), \tag{2.5}$$

then we have

$$\gamma z = (\gamma_1 z_1,\dots,\gamma_n z_n) = \left(\frac{a_1z_1+b_1}{c_1z_1+d_1},\dots,\frac{a_nz_n+b_n}{c_nz_n+d_n}\right) \in \mathcal{H}^n. \tag{2.6}$$

For such γ and z, we set

$$J(\gamma,z) = (j(\gamma_1,z_1),\dots,j(\gamma_n,z_n)) \in \mathbb{C}^n, \qquad j(\gamma_i,z_i) = c_i z_i + d_i \tag{2.7}$$

for $1 \le i \le n$. We denote by $\widetilde{J}(\gamma,z)$ the diagonal matrix with diagonal entries $j(\gamma_i,z_i)$ with $1 \le i \le n$, that is,

$$\widetilde{J}(\gamma,z) = \operatorname{diag}(j(\gamma_1,z_1),\dots,j(\gamma_n,z_n)). \tag{2.8}$$

Then the map $(\gamma,z) \mapsto \widetilde{J}(\gamma,z)$ satisfies the cocycle condition

$$\widetilde{J}(\gamma\gamma',z) = \widetilde{J}(\gamma,\gamma'z)\widetilde{J}(\gamma',z) \tag{2.9}$$

for all $\gamma,\gamma' \in \mathrm{SL}(2,\mathbb{R})^n$ and $z \in \mathcal{H}^n$. Given an element $\eta = (\eta_1,\dots,\eta_n) \in \mathbb{Z}^n$ and a map $f : \mathcal{H}^n \to \mathbb{C}$, we set

$$(f|_\eta\gamma)(z) = J(\gamma,z)^{-\eta} f(\gamma z) \tag{2.10}$$

for all $z \in \mathcal{H}^n$ and $\gamma \in \mathrm{SL}(2,\mathbb{R})^n$. Let Γ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})^n$.

Definition 2.1. Given $\eta = (\eta_1,\dots,\eta_n) \in \mathbb{Z}^n_+$, a *Hilbert modular form of weight* η *for* Γ is a holomorphic function $f : \mathcal{H}^n \to \mathbb{C}$ such that

$$f|_\eta\gamma = f \tag{2.11}$$

for all $\gamma \in \Gamma$, where $f|_\eta\gamma$ is as in (2.10). Denote by $\mathcal{M}_\eta(\Gamma)$ the space of all Hilbert modular forms of weight η for Γ.

Remark 2.2. The usual definition of Hilbert modular forms also includes the regularity condition at the cusps, which is satisfied automatically for $n > 1$ according to Koecher's principle (cf. [3, 4]).

We denote by R the ring of holomorphic functions $f(z_1,\dots,z_n)$ on $\mathcal{H}^n$ and by $R[[X]] = R[[X_1,\dots,X_n]]$ the set of all formal power series in $X_1,\dots,X_n$ with coefficients in R. Thus, using the multi-index notation, an element of $R[[X]]$ can be written in the form

$$\Phi(z,X) = \sum_{\alpha \ge \mathbf{0}} f_\alpha(z) X^\alpha \tag{2.12}$$

with $z = (z_1,\dots,z_n) \in \mathcal{H}^n$ and $X^\alpha = X_1^{\alpha_1}\dots X_n^{\alpha_n}$ for $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}^n_+$.

Let $\mathbb{C}^\times = \mathbb{C} - \{0\}$ be the set of nonzero complex numbers. Given $\lambda = (\lambda_1,\dots,\lambda_n) \in (\mathbb{C}^\times)^n$, we denote by $\widetilde{\lambda} = \operatorname{diag}(\lambda_1,\dots,\lambda_n)$ the associated $n \times n$ diagonal matrix, and set

$$\mathbb{C}^\times X = \{X\widetilde{\lambda} \mid \lambda \in (\mathbb{C}^\times)^n\} = \{(\lambda_1 X_1,\dots,\lambda_n X_n) \mid \lambda_1,\dots,\lambda_n \in \mathbb{C}^\times\}, \tag{2.13}$$

where $X = (X_1,\dots,X_n)$ is regarded as a row vector. Using (2.9), we see that $\mathrm{SL}(2,\mathbb{R})^n$ acts on $\mathcal{H}^n \times \mathbb{C}^\times X$ by

$$\gamma \cdot (z, X\widetilde{\lambda}) = (\gamma z, X\widetilde{J}(\gamma,z)^{-2}\widetilde{\lambda}) \tag{2.14}$$

for all $z \in \mathcal{H}^n$, $\lambda \in (\mathbb{C}^\times)^n$, and $\gamma \in \mathrm{SL}(2,\mathbb{R})^n$, where $\widetilde{J}(\gamma,z)$ is as in (2.8) so that

$$X\widetilde{J}(\gamma,z)^{-2}\widetilde{\lambda} = \left(j(\gamma_1,z_1)^{-2}\lambda_1 X_1,\ldots,j(\gamma_n,z_n)^{-2}\lambda_n X_n\right). \tag{2.15}$$

We now set

$$K_{\xi,\eta}(\gamma,(z,X\widetilde{\lambda})) = J(\gamma,z)^{\xi}\exp\left(\sum_{i=1}^{n} c_i\eta_i j(\gamma_i,z_i)^{-1}\lambda_i X_i\right) \tag{2.16}$$

for $z \in \mathcal{H}^n$, γ as in (2.5), and $\lambda \in (\mathbb{C}^\times)^n$. Then it can be shown that

$$K_{\xi,\eta}(\gamma\gamma',(z,X\widetilde{\lambda})) = K_{\xi,\eta}(\gamma,\gamma'\cdot(z,X\widetilde{\lambda}))K_{\xi,\eta}(\gamma',(z,X\widetilde{\lambda})) \tag{2.17}$$

for all $\gamma,\gamma' \in \mathrm{SL}(2,\mathbb{R})^n$, where $\gamma'\cdot(z,X\widetilde{\lambda})$ is as in (2.14).

Definition 2.3. Given $\xi,\eta \in \mathbb{Z}^n$, a *Jacobi-like form for* Γ *of n variables of weight* ξ, *and index* η is an element,

$$\Phi(z,X) = \Phi(z,X_1,\ldots,X_n) \tag{2.18}$$

of $R[[X]]$ satisfying

$$\Phi(\gamma z,X\widetilde{J}(\gamma,z)^{-2}) = K_{\xi,\eta}(\gamma,(z,X))\Phi(z,X) \tag{2.19}$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{H}^n$. Denote by $\mathcal{J}_{\xi,\eta}(\Gamma)$ the space of all Jacobi-like forms of n variables for Γ of weight ξ and index η.

Remark 2.4. Jacobi-like forms of several variables in $\mathcal{J}_{\xi,\eta}(\Gamma)$ with $\xi = \mathbf{0}$ and $\eta = \mathbf{1}$ were considered in [8], while Jacobi-like forms of one variable with index 0 were studied in [12].

Proposition 2.5. *Given* $\varepsilon \in \mathbb{Z}_+^n$, *consider a formal power series*

$$\Phi(z,X) = \sum_{\alpha\geq\varepsilon}\phi_\alpha(z)X^\alpha \in R[[X]]. \tag{2.20}$$

Then the following conditions are equivalent.

(i) *The power series* $\Phi(z,X)$ *is a Jacobi-like form belonging to* $\mathcal{J}_{\xi,\eta}(\Gamma)$.

(ii) *The coefficient functions* $\phi_\alpha : \mathcal{H} \to \mathbb{C}$ *satisfy*

$$(\phi_\alpha|_{2\alpha+\xi}\gamma)(z) = \sum_{\delta=\mathbf{0}}^{\alpha-\varepsilon}\frac{1}{\delta!}\frac{c^\delta\eta^\delta}{J(\gamma,z)^\delta}\phi_{\alpha-\delta}(z) \tag{2.21}$$

for all $z \in \mathcal{H}^n$ *and* $\alpha \geq \varepsilon$, *where* $\gamma \in \Gamma$ *is as in (2.5) with* $c = (c_1,\ldots,c_n)$.

(iii) *There exist modular forms* $f_\nu \in \mathcal{M}_{2\nu+\xi}(\Gamma)$ *for* $\nu \geq \varepsilon$ *such that*

$$\phi_\alpha(z) = \sum_{\beta=\mathbf{0}}^{\alpha-\varepsilon}\frac{\eta^\beta}{\beta!(2\alpha+\xi-\beta-\varepsilon)!}\partial^\beta f_{\alpha-\beta}(z) \tag{2.22}$$

for all $\alpha \geq \varepsilon$.

Proof. The proposition can be proved by slightly modifying the proofs of [8, Lemma 4.2 and Theorem 4.4]. □

If $\Phi(z,X) = \sum_{\alpha\geq\varepsilon}\phi_\alpha(z)X^\alpha \in \mathcal{J}_{\xi,\eta}(\Gamma)$, then (2.21) implies that

$$\phi_\varepsilon|_{2\varepsilon+\xi}\gamma = \phi_\varepsilon \tag{2.23}$$

for all $\gamma \in \Gamma$; hence the initial coefficient $\phi_\varepsilon(z)$ of the formal power series $\Phi(z,X)$ is a Hilbert modular form of weight $2\varepsilon+\xi$ for Γ. We set

$$\mathcal{J}_{\xi,\eta}(\Gamma)_\varepsilon = X^\varepsilon\mathcal{J}_{\xi,\eta}(\Gamma), \tag{2.24}$$

which is a subspace of $\mathcal{J}_{\xi,\eta}(\Gamma)$ consisting of the elements of the form $\sum_{\alpha\geq\varepsilon}\phi_\alpha(z)X^\alpha$.

Then we see that there is a linear map

$$\mathfrak{F} : \mathcal{J}_{\xi,\eta}(\Gamma)_\varepsilon \longrightarrow \mathcal{M}_{2\varepsilon+\xi}(\Gamma) \tag{2.25}$$

sending an element of $\mathcal{J}_{\xi,\eta}(\Gamma)_\varepsilon$ to its coefficient of X^ε.

3. Mixed Jacobi-like forms

In this section, we discuss Jacobi-like forms of several variables associated to holomorphic maps of the Poincaré upper half-plane $\mathcal{H}$ that are equivariant with respect to a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$. Such Jacobi-like forms are related to mixed automorphic forms.

Let Γ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$, and for each $k \in \{1,\dots,n\}$, let $\omega_k : \mathcal{H} \to \mathcal{H}$ and $\chi_k : \Gamma \to \mathrm{SL}(2,\mathbb{R})$ be a holomorphic map and a group homomorphism, respectively, satisfying

$$\omega_k(\gamma\zeta) = \chi_k(\gamma)\omega_k(\zeta) \tag{3.1}$$

for all $\zeta \in \mathcal{H}$ and $\gamma \in \Gamma$. By setting

$$\omega = (\omega_1,\dots,\omega_n), \qquad \chi = (\chi_1,\dots,\chi_n), \tag{3.2}$$

we obtain a holomorphic map $\omega : \mathcal{H} \to \mathbb{C}^n$ and a homomorphism $\chi : \Gamma \to \mathrm{SL}(2,\mathbb{R})^n$. Given $\eta = (\eta_1,\dots,\eta_n) \in \mathbb{Z}^n$, we define the map $J_{\omega,\chi} : \mathrm{SL}(2,\mathbb{R}) \times \mathcal{H} \to \mathbb{C}^n$ by

$$J_{\omega,\chi}(\gamma,\zeta) = \big(j(\chi_1(\gamma),\omega_1(\zeta)),\dots,j(\chi_n(\gamma),\omega_n(\zeta))\big) \tag{3.3}$$

for all $\gamma \in \mathrm{SL}(2,\mathbb{R})$ and $\zeta \in \mathcal{H}$, where $j : \mathrm{SL}(2,\mathbb{R}) \times \mathcal{H} \to \mathbb{C}$ is as in (2.7).

Definition 3.1. Given $\xi = (\xi_1,\dots,\xi_n) \in \mathbb{Z}^n$, a *mixed automorphic form of type* ξ *associated to* Γ, ω, *and* χ is a holomorphic map $f : \mathcal{H} \to \mathbb{C}$ satisfying

$$f(\gamma\zeta) = J_{\omega,\chi}(\gamma,\zeta)^\xi f(\zeta) = j(\chi_1(\gamma),\omega_1(\zeta))^{\xi_1}\cdots j(\chi_n(\gamma),\omega_n(\zeta))^{\xi_n} f(\zeta) \tag{3.4}$$

for all $\zeta \in \mathcal{H}$ and $\gamma \in \Gamma$. Denote by $\mathcal{M}_\xi(\Gamma,\omega,\chi)$ the space of mixed automorphic forms of type ξ associated to Γ, ω, and χ.

Definition 3.2. Let $\mathcal{F}$ be the set of holomorphic functions on $\mathcal{H}$, and let $\mathcal{F}[[X]]$ be the space of formal power series in $X = (X_1,\dots,X_n)$. Given $\xi = (\xi_1,\dots,\xi_n), \eta = (\eta_1,\dots,\eta_n) \in \mathbb{Z}^n$, a formal power series $F(\zeta, X) \in \mathcal{F}[[X]]$ is a *mixed Jacobi-like form of weight* ξ *and index* η *associated to* Γ, ω, *and* χ if it satisfies

$$F(\gamma\zeta, X\widetilde{J}_{\omega,\chi}(\gamma,\zeta)^{-2}) = J_{\omega,\chi}(\gamma,\zeta)^{\xi} \exp\left(\sum_{k=1}^{n} \frac{c_{\chi,k}\eta_k X_k}{j(\chi_k(\gamma),\omega_k(\zeta))}\right) F(\zeta,X) \tag{3.5}$$

for all $\zeta \in \mathcal{H}$ and $\gamma \in \Gamma$, where $\widetilde{J}_{\omega,\chi}(\gamma,\zeta)$ denotes the diagonal matrix

$$\operatorname{diag}\left(j(\chi_1(\gamma),\omega_1(\zeta)),\dots,j(\chi_n(\gamma),\omega_n(\zeta))\right) \tag{3.6}$$

and $c_{\chi,k}$ is the $(2,1)$-entry of the matrix $\chi_k(\gamma) \in \mathrm{SL}(2,\mathbb{R})$. Denote by $\mathcal{J}_{\xi,\eta}(\Gamma,\omega,\chi)$ the space of mixed Jacobi-like forms of weight ξ and index η associated to Γ, ω, and χ.

Given $\mu \in \mathbb{Z}^n$ and a function $h : \mathcal{H} \to \mathbb{C}$, set

$$(h|_{\mu}^{\omega,\chi}\gamma)(\zeta) = h(\gamma\zeta)J_{\omega,\chi}(\gamma,\zeta)^{-\mu} \tag{3.7}$$

for all $\zeta \in \mathcal{H}$ and $\gamma \in \Gamma$.

Lemma 3.3. *A formal power series* $F(\zeta,X) = \sum_{\alpha\ge\varepsilon} f_\alpha(\zeta)X^\alpha \in \mathcal{F}[[X]]$ *with* $\varepsilon \in \mathbb{Z}_+^n$ *is an element of* $\mathcal{J}_{\xi,\eta}(\Gamma,\omega,\chi)$ *if and only if*

$$(f_\alpha|_{2\alpha+\xi}^{\omega,\chi}\gamma)(\zeta) = \sum_{\delta=\mathbf{0}}^{\alpha-\varepsilon} \frac{1}{\delta!} \frac{c_\chi^\delta \eta^\delta}{J_{\omega,\chi}(\gamma,\zeta)^\delta} f_{\alpha-\delta}(\zeta) \tag{3.8}$$

for all $\gamma \in \Gamma$ *with* $c_\chi = (c_{\chi,1},\dots,c_{\chi,n})$, $\zeta \in \mathcal{H}^n$, *and* $\alpha \ge \varepsilon$, *where* $c_{\chi,j}$ *denotes the* $(2,1)$*-entry of the matrix* $\chi_j(\gamma) \in \mathrm{SL}(2,\mathbb{R})$ *for* $1 \le j \le n$. *In particular, the initial coefficient* $f_\varepsilon(\zeta)$ *of* $F(\zeta,X)$ *is an element of* $\mathcal{M}_{2\varepsilon+\xi}(\Gamma,\omega,\chi)$ *if* $F(\zeta,X) \in \mathcal{J}_{\xi,\eta}(\Gamma,\omega,\chi)$.

Proof. Given $\gamma \in \Gamma$ as described by (3.4) and (3.5), the formal power series $F(\zeta,X) = \sum_{\alpha\ge\varepsilon} f_\alpha(\zeta)X^\alpha$ is an element of $\mathcal{J}_{\xi,\eta}(\Gamma,\omega,\chi)$ if and only if

$$\begin{aligned}\sum_{\alpha\ge\varepsilon} f_\alpha(\gamma\zeta)J_{\omega,\chi}(\gamma,\zeta)^{-2\alpha-\xi}X^\alpha &= \prod_{i=1}^{n}\left(\sum_{\mu_i=0}^{\infty} \frac{1}{\mu_i!} \frac{c_{\chi,i}^{\mu_i}\eta_i^{\mu_i}X_i^{\mu_i}}{j(\chi_i(\gamma),\omega_i(z))^{\mu_i}}\right) \cdot \sum_{\nu\ge\varepsilon} f_\nu(\zeta)X^\nu \\ &= \sum_{\mu\ge\mathbf{0}}\sum_{\nu\ge\varepsilon} \frac{1}{\mu!}\frac{c^\mu\eta^\mu}{J_{\omega,\chi}(\gamma,\zeta)^\mu} f_\nu(\zeta)X^{\mu+\nu}\end{aligned} \tag{3.9}$$

for all $\zeta \in \mathcal{H}$. Thus by comparing the coefficients of X^α, we obtain

$$f_\alpha(\gamma\zeta)J_{\omega,\chi}(\gamma,\zeta)^{-2\alpha-\xi} = \sum_{\delta=\mathbf{0}}^{\alpha-\varepsilon} \frac{1}{\delta!}\frac{c^\delta\eta^\delta}{J_{\omega,\chi}(\gamma,\zeta)^\delta} f_{\alpha-\delta}(\zeta), \tag{3.10}$$

and therefore the lemma follows. □

For each $\varepsilon \in \mathbb{Z}^n$ with $\varepsilon \geq \mathbf{0}$, we set

$$\mathcal{J}_{\xi,\eta}(\Gamma,\omega,\chi)_\varepsilon = X^\varepsilon \mathcal{J}_{\xi,\eta}(\Gamma,\omega,\chi). \tag{3.11}$$

Then by Lemma 3.3, we see that there is a linear map

$$\mathcal{F}_{\omega,\chi} : \mathcal{J}_{\xi,\eta}(\Gamma,\omega,\chi)_\varepsilon \longrightarrow \mathcal{M}_{2\varepsilon+\xi}(\Gamma,\omega,\chi) \tag{3.12}$$

sending an element $\sum_{\alpha\geq\varepsilon} f_\alpha(\zeta)X^\alpha$ of $\mathcal{J}_{\xi,\eta}(\Gamma,\omega,\chi)$ to its initial coefficient $f_\varepsilon(\zeta)$.

If R is the set of holomorphic functions on $\mathcal{H}^n$ as in Section 2, we define the maps

$$\Delta^\omega : R \longrightarrow \mathcal{F}, \qquad \Delta^\omega_X : R[[X]] \longrightarrow \mathcal{F}[[X]] \tag{3.13}$$

associated to the map $\omega : \mathcal{H} \to \mathcal{H}^n$ as in (3.2) by

$$(\Delta^\omega h)(\zeta) = h(\omega(\zeta)), \qquad (\Delta^\omega_X F)(\zeta,X) = F(\omega(\zeta),X) \tag{3.14}$$

for all $\zeta \in \mathcal{H}$, $h \in R$, and $F \in R[[X]]$. Given a discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$, let $\widetilde{\Gamma}_\chi$ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})^n$ such that

$$\chi(\Gamma) = \chi_1(\Gamma) \times \cdots \times \chi_n(\Gamma) \subset \widetilde{\Gamma}_\chi, \tag{3.15}$$

where $\chi = (\chi_1,\ldots,\chi_n)$ is as in (3.2).

Theorem 3.4. (i) *If $\Delta^\omega : R \to \mathcal{F}$ and $\Delta^\omega_X : R[[X]] \to \mathcal{F}[[X]]$ are as in (3.14), then*

$$\Delta^\omega\big(\mathcal{M}_\xi(\widetilde{\Gamma}_\chi)\big) \subset \mathcal{M}_\xi(\Gamma,\omega,\chi), \qquad \Delta^\omega_X\big(\mathcal{J}_{\xi,\eta}(\widetilde{\Gamma}_\chi)_\varepsilon\big) \subset \mathcal{J}_{\xi,\eta}(\Gamma,\omega,\chi)_\varepsilon \tag{3.16}$$

for all $\xi,\eta \in \mathbb{Z}^n$.

(ii) *If $\mathcal{F}$ and $\mathcal{F}_{\omega,\chi}$ are the linear maps in (2.25) and (3.12), respectively, then the diagram*

$$\begin{array}{ccc} \mathcal{J}_{\xi,\eta}(\widetilde{\Gamma}_\chi)_\varepsilon & \xrightarrow{\ \mathcal{F}\ } & \mathcal{M}_{2\varepsilon+\xi}(\widetilde{\Gamma}_\chi) \\ {\scriptstyle \Delta^\omega_X}\downarrow & & \downarrow{\scriptstyle \Delta^\omega} \\ \mathcal{J}_{\xi,\eta}(\Gamma,\omega,\chi)_\varepsilon & \xrightarrow{\ \mathcal{F}_{\omega,\chi}\ } & \mathcal{M}_{2\varepsilon+\xi}(\Gamma,\omega,\chi) \end{array} \tag{3.17}$$

is commutative.

Proof. If $f : \mathcal{H}^n \to \mathbb{C}$ is an element of $\mathcal{M}_\xi(\widetilde{\Gamma}_\chi)$, then by (3.14) we have

$$\begin{aligned}(\Delta^\omega f)(\gamma\zeta) &= f(\omega(\gamma\zeta)) = f(\chi_1(\gamma)\omega_1(\zeta),\ldots,\chi_n(\gamma)\omega_n(\zeta)) \\ &= J(\chi(\gamma),\omega(\zeta))^\xi f(\omega(\zeta)) = J(\chi(\gamma),\omega(\zeta))^\xi (\Delta^\omega f)(\zeta)\end{aligned} \tag{3.18}$$

for all $\zeta \in \mathcal{H}$ and $\gamma \in \Gamma$; hence $\Delta^\omega f$ is an element of $\mathcal{M}_\xi(\widetilde{\Gamma}_\chi, \omega, \chi)$. On the other hand, if F is an element of $\mathcal{J}_{\xi,\eta}(\widetilde{\Gamma}_\chi)$ by (3.5) and (3.14), we see that

$$\begin{aligned}(\Delta_X^\omega(F))(\gamma\zeta, X\widetilde{J}_{\omega,\chi}(\gamma,\zeta)^{-2}) &= F(\chi_1(\gamma)\omega_1(\zeta),\dots,\chi_n(\gamma)\omega_n(\zeta), X\widetilde{J}_{\omega,\chi}(\gamma,\zeta)^{-2}) \\ &= J(\chi(\gamma),\omega(\zeta))^\xi \exp\left(\sum_{k=1}^n \frac{c_{\chi,k}\eta_k X_k}{j(\chi_k(\gamma),\omega_k(\zeta))}\right) F(\omega(\zeta), X)\end{aligned} \tag{3.19}$$

for all $\zeta \in \mathcal{H}$ and $\gamma \in \Gamma$. Thus $\Delta_X^\omega F$ is an element of $\mathcal{J}_{\xi,\eta}(\Gamma, \omega, \chi)$, and $\Delta_X^\omega F \in \mathcal{J}_{\xi,\eta}(\Gamma, \omega, \chi)_\varepsilon$ if $F \in \mathcal{J}_{\xi,\eta}(\widetilde{\Gamma}_\chi)_\varepsilon$, which proves (i). In order to verify (ii), consider an element $\Phi(\zeta, X) = \sum_{\alpha \ge \varepsilon} \phi_\alpha(\zeta) X^\alpha \in \mathcal{J}_{\xi,\eta}(\widetilde{\Gamma}_\chi)_\varepsilon$. Then we have

$$((\Delta^\omega \circ \mathcal{F})(\Phi))(\zeta) = (\Delta^\omega \phi_\varepsilon)(\zeta) = \phi_\varepsilon(\omega_1(\zeta),\dots,\omega_n(\zeta)) \tag{3.20}$$

for $\zeta \in \mathcal{H}$. On the other hand, we have

$$\begin{aligned}(\Delta_X^\omega \Phi)(\zeta, X) = \Phi(\omega(\zeta), X) &= \Phi(\omega_1(\zeta),\dots,\omega_n(\zeta), X) \\ &= \sum_{\alpha \ge \varepsilon} \phi_\alpha(\omega_1(\zeta),\dots,\omega_n(\zeta)) X^\alpha.\end{aligned} \tag{3.21}$$

Thus we see that

$$((\mathcal{F}_{\omega,\chi} \circ \Delta_X^\omega)(\Phi))(\zeta) = \phi_\varepsilon(\omega_1(\zeta),\dots,\omega_n(\zeta)) = ((\Delta^\omega \circ \mathcal{F})(\Phi))(\zeta), \tag{3.22}$$

which implies (ii); hence the proof of the theorem is complete. □

4. Examples

In this section, we discuss two examples related to mixed Jacobi-like forms. The first one involves a fiber bundle over a Riemann surface whose generic fiber is the product of elliptic curves, and the second one is linked to solutions of linear ordinary differential equations.

Example 4.1. Let E be an elliptic surface (cf. [5]). Thus E is a compact surface over $\mathbb{C}$ that is the total space of an elliptic fibration $\pi : E \to X$ over a Riemann surface X. Let E_0 be the union of the regular fibers of π, and let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be the fundamental group of $X_0 = \pi(E_0)$. Then the universal covering space of X_0 may be identified with the Poincaré upper half-plane $\mathcal{H}$, and we have $X_0 = \Gamma\backslash\mathcal{H}$, where Γ is regarded as a subgroup of $\mathrm{SL}(2,\mathbb{R})$ and the quotient is taken with respect to the action given by linear fractional transformations. Given $z \in \mathcal{H}_0$, let Φ be a holomorphic 1-form on $E_z = \pi^{-1}(z)$, and choose an ordered basis $\{\alpha_1(z), \alpha_2(z)\}$ for $H_1(E_z, \mathbb{Z})$ which depends on the parameter z in a continuous manner. If we set

$$\omega_1(z) = \int_{\alpha_1(z)} \Phi, \qquad \omega_2(z) = \int_{\alpha_2(z)} \Phi, \tag{4.1}$$

then ω_1/ω_2 is a many-valued function from X_0 to $\mathcal{H}$ which can be lifted to a single-valued function $\omega : \mathcal{H} \to \mathcal{H}$ on the universal cover $\mathcal{H}$ of X_0. Then it can be shown that there is a group homomorphism $\chi : \Gamma \to \mathrm{SL}(2,\mathbb{R})$, called the monodromy representation for the elliptic surface E, such that

$$\omega(\gamma z) = \chi(\gamma)\omega(z) \tag{4.2}$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$. Thus the maps χ and ω form an equivariant pair.

Let (χ_j, ω_j) be an equivariant pair associated to an elliptic surface E of the type described above for each $j \in \{1,\dots,p\}$, and set

$$\widetilde{\chi} = (1,\chi_1,\dots,\chi_p), \qquad \widetilde{\omega} = (1,\omega_1,\dots,\omega_p). \tag{4.3}$$

Then, given a positive integer p and an element $\mathbf{m} = (m_1,\dots,m_p) \in \mathbb{Z}^q$ with $m_1,\dots,m_p > 0$, the semidirect product $\Gamma \ltimes_{\widetilde{\chi}} (\mathbb{Z}^2)^{|\mathbf{m}|p}$ with $|\mathbf{m}| = m_1 + \cdots + m_p$ associated to $\widetilde{\chi}$ acts on $\mathcal{H} \times \mathbb{C}^{|\mathbf{m}|p}$ by

$$(\gamma, \boldsymbol{\ell}_1,\dots,\boldsymbol{\ell}_p) \cdot (z, \boldsymbol{\zeta}_1,\dots,\boldsymbol{\zeta}_p) = (\gamma z, \widehat{\boldsymbol{\zeta}}_1,\dots,\widehat{\boldsymbol{\zeta}}_p) \tag{4.4}$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$, where

$$\begin{aligned} \boldsymbol{\ell}_j &= ((\mu_{1,j},\nu_{1,j}),\dots,(\mu_{m_j,j},\nu_{m_j,j})) \in (\mathbb{Z}^2)^{m_j}, \\ \boldsymbol{\zeta}_j &= (\zeta_{1,j},\dots,\zeta_{m_j,j}), \widehat{\boldsymbol{\zeta}}_j = (\widehat{\zeta}_{1,j},\dots,\widehat{\zeta}_{m_j,j}) \in \mathbb{C}^{m_j} \end{aligned} \tag{4.5}$$

for $1 \le j \le p$ with

$$\widehat{\zeta}_{r,j} = \frac{\zeta_{r,j} + \omega_j(z)\mu_{r,j} + \nu_{r,j}}{c_{\chi_j}\omega_j(z) + d_{\chi_j}} \tag{4.6}$$

for each $r \in \{1,\dots,m_j\}$ if

$$\chi_j(\gamma) = \begin{pmatrix} a_{\chi_j} & b_{\chi_j} \\ c_{\chi_j} & d_{\chi_j} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}). \tag{4.7}$$

We denote by $E_0^{|\mathbf{m}|p}$ the associated quotient space, that is,

$$E_0^{|\mathbf{m}|p} = \Gamma \times (\mathbb{Z}^2)^{|\mathbf{m}|p} \backslash \mathcal{H} \times \mathbb{C}^{|\mathbf{m}|p}. \tag{4.8}$$

Given $\varepsilon \in \mathbb{Z}^{p+1}$, we set $\xi = (2,m_1,\dots,m_p) - 2\varepsilon$, and let $F(z,X) \in \mathcal{J}_{\xi,\eta}(\Gamma,\boldsymbol{\omega},\boldsymbol{\chi})_\varepsilon$. Then by Lemma 3.3, we see that $\mathcal{F}_{\omega,\chi}(F(z,X))$ is an element of $\mathcal{M}_{(2,m_1,\dots,m_p)}(\Gamma,\omega,\chi)$, and it can be shown that the associated holomorphic form

$$\omega_F(\mathbf{z}) = \mathcal{F}_{\omega,\chi}(F(z,X))\,dz \wedge d\boldsymbol{\zeta}_1 \wedge \cdots \wedge d\boldsymbol{\zeta}_p \tag{4.9}$$

on $\mathcal{H} \times \mathbb{C}^{|\mathbf{m}|p}$ with $\mathbf{z} = (z,\boldsymbol{\zeta}_1,\dots,\boldsymbol{\zeta}_p) \in \mathcal{H} \times \mathbb{C}^{|\mathbf{m}|p}$ is invariant under the action of $\Gamma \times (\mathbb{Z}^2)^p$. Hence $\omega_F(\mathbf{z})$ can be regarded as a holomorphic $(|\mathbf{m}|p+1)$-form on $E_0^{|\mathbf{m}|p}$, and therefore we obtain a canonical map

$$\mathcal{J}_{\xi,\eta}(\Gamma,\boldsymbol{\omega},\boldsymbol{\chi})_\varepsilon \longrightarrow \Omega^{p+1}\left(E_0^{|\mathbf{m}|p}\right) \tag{4.10}$$

from $\mathcal{J}_{\xi,\eta}(\Gamma,\boldsymbol{\omega},\boldsymbol{\chi})_\varepsilon$ to the space $\Omega^{p+1}\left(E_0^{|\mathbf{m}|p}\right)$ of holomorphic $(|\mathbf{m}|p+1)$-forms on $E_0^{|\mathbf{m}|p}$.

Example 4.2. Let Γ be a Fuchsian group of the first kind, and let $K(X)$ be the function field of the smooth complex algebraic curve $X = \Gamma\backslash\mathcal{H} \cup \{\text{cusps}\}$. Consider a second-order linear differential equation

$$\left(\frac{d^2}{dx^2} + \widetilde{P}(x)\frac{d}{dx} + \widetilde{Q}(x)\right)\widetilde{f} = 0 \tag{4.11}$$

for $x \in X$ and $\widetilde{P}(x), \widetilde{Q}(x) \in K(X)$ with regular singular points, whose singular points are contained in $\Gamma\backslash\{\text{cusps}\} \subset X$. Let

$$\Lambda f = \left(\frac{d^2}{dz^2} + P(z)\frac{d}{dz} + Q(z)\right) f = 0, \tag{4.12}$$

for $z \in \mathcal{H}$, be the differential equation obtained by pulling back (4.11) via the natural projection $\mathcal{H} \to \Gamma\backslash\mathcal{H} \subset X$. Let σ_1 and σ_2 be linearly independent solutions of (4.12), and let $S^m(\Lambda)$ be the linear ordinary differential operator of order $m+1$ such that the $m+1$ functions

$$\sigma_1^m, \sigma_1^{m-1}\sigma_2, \ldots, \sigma_1\sigma_2^{m-1}, \sigma_2^m \tag{4.13}$$

are linearly independent solutions of the corresponding linear homogeneous equation $S^m(\Lambda)f = 0$. Let $\chi : \Gamma \to \mathrm{SL}(2,\mathbb{R})$ be the monodromy representation of Γ for the second-order equation $\Lambda f = 0$. Then the period map $\omega : \mathcal{H} \to \mathcal{H}$ defined by $\omega(z) = \sigma_1(z)/\sigma_2(z)$ for all $z \in \mathcal{H}$ is equivariant with respect to χ. Let $\psi : \mathcal{H} \to \mathbb{C}$ be a function corresponding to an element of $K(X)$ satisfying the *parabolic residue condition* in the sense of [11, Definition 3.20], and let f^ψ be a solution of the nonhomogeneous equation $S^m(\Lambda)f = \psi$. Then the function

$$\frac{d^{m+1}}{d\omega(z)^{m+1}}\left(\frac{f^\psi(z)}{\sigma_2(z)^m}\right) \tag{4.14}$$

is a mixed automorphic form of type $(0, m+2)$ associated to Γ, ω, and χ (cf. [11, page 32]).

Given a positive integer p and $\mathbf{m} = (m_1,\ldots,m_p) \in \mathbb{Z}^p$ with $m_1,\ldots,m_p > 0$, we consider a system of ordinary differential equations

$$S^{m_j}(\Lambda_j) f_j(z_j) = \psi_j(z_j), \quad 1 \le j \le p, \tag{4.15}$$

of the type described above and for each $j \in \{1,\ldots,p\}$, choose a solution $f_j^{\psi_j}(z_j)$ for the jth equation. For $1 \le j \le p$, let $\chi_j : \Gamma_j \to \mathrm{SL}(2,\mathbb{R})$ and $\omega_j : \mathcal{H} \to \mathcal{H}$ be the monodromy representation and the period map, respectively, associated to the operator $S^{m_j}(\Lambda_j)$, and set

$$\tilde{\chi} = (\chi_1,\dots,\chi_p), \qquad \tilde{\omega} = (\omega_1,\dots,\omega_p), \qquad \Gamma = \Gamma_1 \cap \cdots \cap \Gamma_p. \tag{4.16}$$

Then we see that the function $\hat{f} : \mathcal{H} \to \mathbb{C}$ defined by

$$\hat{f}(z) = f_1(z)\cdots f_p(z) \tag{4.17}$$

for all $z \in \mathcal{H}$ is a mixed automorphic form belonging to $\mathcal{M}_{\mathbf{m}}(\Gamma,\tilde{\omega},\tilde{\chi})$.

5. Liftings of mixed automorphic forms

Let $\omega = (\omega_1,\dots,\omega_n)$ and $\chi = (\chi_1,\dots,\chi_n)$ be as in Section 3. Thus $\omega_i : \mathcal{H} \to \mathcal{H}$ is a holomorphic map equivariant with respect to the homomorphism $\chi_i : \Gamma \to \mathrm{SL}(2,\mathbb{R})$ for each $i \in \{1,\dots,n\}$, where Γ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$. In this section, we construct liftings of mixed automorphic forms associated to Γ, ω, and χ of certain types to mixed Jacobi-like forms associated to Γ, ω, and χ.

We first consider discrete subgroups $\Gamma_1,\dots,\Gamma_n$ of $\mathrm{SL}(2,\mathbb{R})$ satisfying

$$\chi_i(\Gamma) \subset \Gamma_i \tag{5.1}$$

for all $i \in \{1,\dots,n\}$. Given $\xi = (\xi_1,\dots,\xi_n) \in \mathbb{Z}^n$ and $\mu = (\mu_1,\dots,\mu_n) \in \mathbb{Z}_+^n$, let $M_{2\mu_i+\xi_i}(\Gamma_i)$ denote the space of automorphic forms of one variable for Γ_i of weight $2\mu_i + \xi_i$. If Δ^{ω_i} is the map in (3.14) associated to $\omega_i : \mathcal{H} \to \mathcal{H}$ in the case of $n = 1$, then we see that

$$\Delta^{\omega_i}\big(M_{2\mu_i+\xi_i}(\Gamma_i)\big) = \big\{h \circ \omega_i \mid h \in M_{2\mu_i+\xi_i}(\Gamma_i)\big\} \tag{5.2}$$

for $1 \le i \le n$. We denote the tensor product of these spaces by

$$\mathcal{M}^0_{2\mu+\xi}(\Gamma,\omega,\chi) = \bigotimes_{i=1}^{n} \Delta^{\omega_i}\big(M_{2\mu_i+\xi_i}(\Gamma_i)\big), \tag{5.3}$$

and consider an element of the form

$$\mathfrak{h} = \sum_{k=1}^{p} C_k \bigotimes_{i=1}^{n} \big(h_{i,k} \circ \omega_i\big) \in \mathcal{M}^0_{2\mu+\xi}(\Gamma,\omega,\chi) \tag{5.4}$$

with $C_k \in \mathbb{C}$ and $h_{i,k} \in M_{2\mu_i+\xi_i}(\Gamma_i)$ for $1 \le i \le n$ and $1 \le k \le p$. Then we have

$$\begin{aligned}
\mathfrak{h}(\gamma z) &= \sum_{k=1}^{p} C_k \bigotimes_{i=1}^{n} \big(h_{i,k}(\omega_i(\gamma z))\big) = \sum_{k=1}^{p} C_k \bigotimes_{i=1}^{n} \big(h_{i,k}(\chi_i(\gamma)\omega_i(z))\big) \\
&= \sum_{k=1}^{p} C_k \bigotimes_{i=1}^{n} \big(j(\chi_i(\gamma),\omega_i(z))^{2\mu_i+\xi_i} h_{i,k}(\chi_i(\gamma)\omega_i(z))\big) \\
&= \left(\prod_{i=1}^{n} j(\chi_i(\gamma),\omega_i(z))^{2\mu_i+\xi_i}\right)\mathfrak{h}(z)
\end{aligned} \tag{5.5}$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$; hence $\mathfrak{h}$ is a mixed automorphic form belonging to $\mathcal{M}_{2\mu+\xi}(\Gamma,\omega,\chi)$. Thus we see that $\mathcal{M}^0_{2\mu+\xi}(\Gamma,\omega,\chi)$ is a subspace of $\mathcal{M}_{2\mu+\xi}(\Gamma,\omega,\chi)$.

We now discuss a lifting of an element of $\mathcal{M}^0_{2\mu+\xi}(\Gamma,\omega,\chi)$ to a Jacobi-like form belonging to $\mathcal{J}_{\xi,\eta}(\Gamma,\omega,\chi)_\varepsilon$ with $\varepsilon = (\varepsilon_1,\dots,\varepsilon_n) \in \mathbb{Z}^n_+$ and $\eta = (\eta_1,\dots,\eta_n) \in \mathbb{Z}^n$. Given $i \in \{1,\dots,n\}$ and $k \in \{1,\dots,p\}$, assuming that $\mu \geq \varepsilon$, we set

$$\widehat{h}_{i,k,\ell} = \frac{\eta_i^{\ell-\mu_i} h_{i,k}^{(\ell-\mu_i)}}{(\ell-\mu_i)!(\ell+\xi_i+\mu_i-\varepsilon_i)!} \tag{5.6}$$

for $\ell \geq \mu_i$ and

$$\widehat{h}^{\omega}_{k,\alpha}(z) = \left(\widehat{h}_{1,k,\alpha_1}(\omega_1(z)),\dots,\widehat{h}_{n,k,\alpha_n}(\omega_n(z))\right) \tag{5.7}$$

for all $z \in \mathcal{H}$ and $\alpha = (\alpha_1,\dots,\alpha_n) \geq \mu$. We define the formal power series $\Phi_{\mathfrak{h}}(z,X) \in R[[X]]$ associated to $\mathfrak{h}$ by

$$\Phi_{\mathfrak{h}}(z,X) = \sum_{k=1}^{p} C_k \sum_{\alpha \geq \mu} \left(\widehat{h}^{\omega}_{k,\alpha}(z)\right)^{\mathbf{1}} X^{\alpha}, \tag{5.8}$$

where $\mathbf{1} = (1,\dots,1) \in \mathbb{Z}^n$ so that

$$\left(\widehat{h}^{\omega}_{k,\alpha}(z)\right)^{\mathbf{1}} = \widehat{h}_{1,k,\alpha_1}(\omega_1(z)) \cdots \widehat{h}_{n,k,\alpha_n}(\omega_n(z)) \tag{5.9}$$

for $\alpha = (\alpha_1,\dots,\alpha_n)$.

Theorem 5.1. *The map $\mathfrak{h} \mapsto \Phi_{\mathfrak{h}}$ determines a lifting of an element of $\mathcal{M}^0_{2\mu+\xi}(\Gamma,\omega,\chi)$ to a Jacobi-like form belonging to $\mathcal{J}_{\xi,\eta}(\Gamma,\omega,\chi)_\mu \subset \mathcal{J}_{\xi,\eta}(\Gamma,\omega,\chi)_\varepsilon$ such that*

$$\mathcal{F}_{\omega,\chi}(\Phi_{\mathfrak{h}}) = \frac{\mathfrak{h}}{(2\mu+\xi-\varepsilon)!} \tag{5.10}$$

for all $\mathfrak{h} \in \mathcal{M}^0_{2\mu+\xi}(\Gamma,\omega,\chi)$, where $\mathcal{F}_{\omega,\chi}$ is the map sending $\Phi_{\mathfrak{h}}(z,X)$ to the coefficient of X^μ as in (3.12).

Proof. For $1 \leq i \leq n$, applying Proposition 2.5 to the case of $n = 1$, we see that the formal power series

$$\Phi_i(z,X_i) = \sum_{\ell \geq \varepsilon_i} \phi_\ell(z) X_i^\ell \tag{5.11}$$

in X_i is a Jacobi-like form of one variable belonging to $\mathcal{J}_{\xi_i,\eta_i}(\Gamma_i)_{\varepsilon_i}$ if and only if there is a sequence of modular forms $\{f_r\}_{r\geq 0}$ with $f_r \in M_{2r+\xi_i}(\Gamma_i)$ satisfying

$$\phi_\ell = \sum_{j=0}^{\ell-\varepsilon_i} \frac{\eta_i^j f_{\ell-j}^{(j)}}{j!(2\ell+\xi_i-j-\varepsilon_i)!} \tag{5.12}$$

for all $\ell \geq \varepsilon_i$. We now consider an element $\mathfrak{h} \in \mathcal{M}_\varepsilon^0(\Gamma,\omega,\chi)$ given by (5.4). Given i and k, let $\{f_r\}_{r\geq 0}$ be the sequence of functions on $\mathcal{H}$ defined by

$$f_r = \begin{cases} h_{i,k} & \text{if } r = \mu_i, \\ 0 & \text{otherwise.} \end{cases} \tag{5.13}$$

Then clearly $f_r \in M_{2r+\xi_i}(\Gamma_i)$ for each $r \geq \varepsilon_i$. If $\Phi_{i,k}(z,X_i) = \sum_{\ell\geq\varepsilon_i}\phi_{i,k,\ell}(z)X_i^\ell$ is the corresponding Jacobi-like form belonging to $\mathcal{J}_{\xi_i,\eta_i}(\Gamma_i)$, then by (5.12) the coefficient function $\phi_{i,k,\ell}$ coincides with $\widehat{h}_{i,k,\ell}$ in (5.6). Thus for each k, we see that the product

$$\begin{aligned} \Phi_k^\omega(z,X) &= \Phi_{1,k}(\omega_1(z),X_1)\cdots\Phi_{n,k}(\omega_n(z),X_n) \\ &= \sum_{\alpha_1\geq\mu_1}\cdots\sum_{\alpha_n\geq\mu_n}\widehat{h}_{1,k,\alpha_1}(\omega_1(z))\cdots\widehat{h}_{n,k,\alpha_n}(\omega_n(z))X_1^{\alpha_1}\cdots X_n^{\alpha_n} \\ &= \sum_{\alpha\geq\mu}(\widehat{h}_{k,\alpha}^\omega(z))^{\mathbf{1}}X^\alpha \end{aligned} \tag{5.14}$$

is a Jacobi-like form belonging to $\mathcal{J}_{\xi,\eta}(\Gamma,\omega,\chi)_\mu \subset \mathcal{J}_{\xi,\eta}(\Gamma,\omega,\chi)_\varepsilon$. From this and the fact that the formal power series in (5.8) can be written in the form

$$\Phi_{\mathfrak{h}}(z,X) = \sum_{k=1}^{p} C_k\Phi_k^\omega(z,X), \tag{5.15}$$

we see that $\Phi_{\mathfrak{h}}(z,X)$ is a Jacobi-like form belonging to $\mathcal{J}_{\xi,\eta}(\Gamma,\omega,\chi)_\mu$. On the other hand, from (5.6) we have

$$\widehat{h}_{i,k,\mu_i} = \frac{h_{i,k}}{(2\mu_i+\xi_i-\varepsilon_i)!} \tag{5.16}$$

for $1 \leq i \leq n$ and $1 \leq k \leq p$, which implies that

$$\begin{aligned} \mathcal{F}_{\omega,\chi}(\Phi_k^\omega(z,X)) &= \frac{h_{1,k}(\omega_1(z))\cdots h_{n,k}(\omega_n(z))}{(2\mu_1+\xi_1-\varepsilon_1)!\cdots(2\mu_n+\xi_n-\varepsilon_n)!} \\ &= \frac{h_{1,k}(\omega_1(z))\cdots h_{n,k}(\omega_n(z))}{(2\mu+\xi-\varepsilon)!}. \end{aligned} \tag{5.17}$$

Combining this with (5.15), we obtain

$$\mathcal{F}_{\omega,\chi}(\Phi_{\mathfrak{h}}(z,X)) = \sum_{k=1}^{p} C_k\left(\frac{h_{1,k}(\omega_1(z))\cdots h_{n,k}(\omega_n(z))}{(2\mu+\xi-\varepsilon)!}\right) = \frac{\mathfrak{h}(z)}{(2\mu+\xi-\varepsilon)!}, \tag{5.18}$$

where we identified the tensor product with the usual product in $\mathbb{C}$; hence the proof of the theorem is complete. □

References

[1] P. B. Cohen, Y. Manin, and D. Zagier, *Automorphic pseudodifferential operators*, Algebraic Aspects of Integrable Systems, Progress in Nonlinear Differential Equations and Their Applications, vol. 26, Birkhäuser Boston, Massachusetts, 1997, pp. 17–47.

[2] C. Dong and G. Mason, *Transformation laws for theta functions*, Proceedings on Moonshine and Related Topics (Montréal, QC, 1999), CRM Proceedings & Lecture Notes, vol. 30, American Mathematical Society, Rhode Island, 2001, pp. 15–26.

[3] E. Freitag, *Hilbert Modular Forms*, Springer, Berlin, 1990.

[4] P. B. Garrett, *Holomorphic Hilbert Modular Forms*, The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth, California, 1990.

[5] K. Kodaira, *On compact analytic surfaces. II*, Annals of Mathematics. Second Series **77** (1963), 563–626.

[6] M. H. Lee, *Mixed cusp forms and holomorphic forms on elliptic varieties*, Pacific Journal of Mathematics **132** (1988), no. 2, 363–370.

[7] ______, *Mixed Jacobi-like forms*, Complex Variables. Theory and Application **42** (2000), no. 4, 387–396.

[8] ______, *Hilbert modular pseudodifferential operators*, Proceedings of the American Mathematical Society **129** (2001), no. 11, 3151–3160.

[9] ______, *Mixed Automorphic Forms, Torus Bundles, and Jacobi Forms*, Lecture Notes in Mathematics, vol. 1845, Springer, Berlin, 2004.

[10] M. Miyamoto, *A modular invariance on the theta functions defined on vertex operator algebras*, Duke Mathematical Journal **101** (2000), no. 2, 221–236.

[11] P. Stiller, *Special values of Dirichlet series, monodromy, and the periods of automorphic forms*, Memoirs of the American Mathematical Society **49** (1984), no. 299, iv+116.

[12] D. Zagier, *Modular forms and differential operators*, Proceedings of the Indian Academy of Sciences. Mathematical Sciences **104** (1994), no. 1, 57–75.

Min Ho Lee: Department of Mathematics, University of Northern Iowa, Cedar Falls, IA 50614-0506, USA

E-mail address: lee@math.uni.edu