FINDING LINKING SETS

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We present the most general definition of the linking of sets in a Banach space and discuss necessary and sufficient conditions for sets to link.

1. Introduction

Many nonlinear problems in the physical and social sciences can be reduced to finding critical points (minima, maxima, and minimax points) of functionals (real-valued functions on various spaces). The first critical points to be studied were maxima and minima, and much of the activity in the calculus of variations has been devoted to the finding of such points. A more difficult problem is to find critical points that are neither maxima nor minima.

It was discovered that there exist sets having the property that critical points can be found when the sets separate a functional. This has led to the search for such sets. Specifically, it is desired to find sets with the following properties.

Definition 1.1. Let A, B be subsets of a Banach space E. Say that A *links* B if the class of $G \in C_U^1(E; \mathbb{R})$ satisfying

$$a_0 := \sup_A G < b_0 := \inf_B G \tag{1.1}$$

is nonempty, and for every such G, there are a sequence $\{u_k\} \subset E$ and a constant c such that

$$b_0 \le c < \infty, \tag{1.2}$$

$$G(u_k) \longrightarrow c, \qquad G'(u_k) \longrightarrow 0. \tag{1.3}$$

(In this definition, as well as in the rest of the paper, we consider functionals of the class $C_U^m \subset C^m$, i.e., functionals whose Fréchet derivatives up to the order m are uniformly continuous on bounded sets.)

2 Finding linking sets

The importance of the concept of linking stems from the fact that in many applications, a sequence satisfying (1.3) (called a Palais-Smale sequence) leads to a critical point of G. In particular, this is true if any sequence satisfying (1.3) has a convergent subsequence. (If this is the case, then G is said to satisfy the Palais-Smale condition.) Thus, a useful method of finding critical points of G is to find two subsets A, B of E such that A links B and (1.3) holds.

There are several sufficient conditions given in the literature which imply that a set A links a set B in this sense (cf., e.g., [1, 3, 5–8, 11, 14, 15]). The most comprehensive is that of [11].

In [11], our goal was to prove a theorem of the following form.

THEOREM 1.2. *Let G be a C^1-functional on a Banach space E, and let A, B be subsets of E such that A has a certain relationship to B. Assume that*

$$a_0 := \sup_A G < b_0 := \inf_B G. \tag{1.4}$$

Then there are a sequence $\{u_k\} \subset E$ and an $a \geq b_0$ such that

$$G(u_k) \longrightarrow a, \qquad G'(u_k) \longrightarrow 0. \tag{1.5}$$

The reason for the requirement that A have a relationship to B is that the theorem is obviously false if A and B are two arbitrary sets. The problem facing us was to find a very general relationship that would make the theorem true. We defined a relationship which indeed made the theorem true and was the most general known hypothesis that did so. We termed this relationship "linking." (Previous authors used this term to describe more restrictive definitions.) We suspect that our definition of linking is the most general possible definition.

In the present paper, we define two relationships which are close to each other. The stronger one is sufficient for linking while the weaker one is necessary. We use the following maps.

Definition 1.3. Say that a map $\varphi : E \to E$ is of class Λ, if it is a homeomorphism onto E, and both φ, φ^{-1} are bounded on bounded sets. If, furthermore, $\varphi, \varphi^{-1} \in C_U^1(E; E)$, say that $\varphi \in \Lambda_U$.

Definition 1.4. Say that a bounded set A is *chained* to a set B if $A \cap B = \phi$ and

$$\inf_{x \in B} \|\varphi(x)\| \leq \sup_{x \in A} \|\varphi(x)\|, \qquad \varphi \in \Lambda_U. \tag{1.6}$$

Definition 1.5. Say that a bounded A is *strongly chained* to B if $A \cap B = \phi$ and (1.6) holds for every $\varphi \in \Lambda$.

Definition 1.6. For $A \subset E$, define

$$\mathcal{K}_U(A) = \{\varphi^{-1}(B_R) : \varphi \in \Lambda_U, \ \varphi(A) \subset B_R\}, \tag{1.7}$$

where

$$B_R = \{u \in E : \|u\| < R\}. \tag{1.8}$$

Our main results are the following.

THEOREM 1.7. *If a bounded set A is strongly chained to B, then A links B.*

THEOREM 1.8. *Let G be a $(C^2 \cap C_U)$-functional on E, and let A, B be nonempty subsets of E such that A is bounded and chained to B and*

$$a_0 := \sup_A G \leq b_0 := \inf_B G. \tag{1.9}$$

Let

$$a := \inf_{K \in \mathcal{K}_U(A)} \sup_K G. \tag{1.10}$$

Then there is a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \longrightarrow a, \qquad G'(u_k) \longrightarrow 0. \tag{1.11}$$

If $a = b_0$, then it can be also required

$$d(u_k, B) \longrightarrow 0. \tag{1.12}$$

THEOREM 1.9. *If E is a Hilbert space, and A links B, then it is chained to B.*

Theorem 1.9 with additional assumptions is due to the authors elsewhere [12]. There it was proved that in a Hilbert space a compact set links another set if and only if it is chained to it.

2. The strong case

Definition 2.1. For $A \subset E$, define

$$\begin{aligned}
\Lambda(A) &= \{\varphi \in \Lambda : \varphi(u) = u, \ u \in A\}, \\
\Lambda_U(A) &= \{\varphi \in \Lambda_U : \varphi(u) = u, \ u \in A\}.
\end{aligned} \tag{2.1}$$

LEMMA 2.2. *One has*

$$\begin{aligned}
\varphi_1, \varphi_2 \in \Lambda &\Longrightarrow \varphi_1 \circ \varphi_2 \in \Lambda, \\
\varphi_1, \varphi_2 \in \Lambda_U &\Longrightarrow \varphi_1 \circ \varphi_2 \in \Lambda_U, \\
\varphi_1, \varphi_2 \in \Lambda(A) &\Longrightarrow \varphi_1 \circ \varphi_2 \in \Lambda(A), \\
\varphi_1, \varphi_2 \in \Lambda_U(A) &\Longrightarrow \varphi_1 \circ \varphi_2 \in \Lambda_U(A).
\end{aligned} \tag{2.2}$$

Lemma 2.3. *One has*

$$\varphi \in \Lambda \quad \textit{iff } \varphi^{-1} \in \Lambda,$$
$$\varphi \in \Lambda_U \quad \textit{iff } \varphi^{-1} \in \Lambda_U,$$
$$\varphi \in \Lambda(A) \quad \textit{iff } \varphi^{-1} \in \Lambda(A),$$
$$\varphi \in \Lambda_U(A) \quad \textit{iff } \varphi^{-1} \in \Lambda_U(A).$$

(2.3)

Definition 2.4. For $A \subset E$, define

$$\mathcal{H}(A) = \{\varphi^{-1}(B_R) : \varphi \in \Lambda, \ \varphi(A) \subset B_R\}. \tag{2.4}$$

Lemma 2.5. *If A is strongly chained to B and $K \in \mathcal{H}(A)$, then $K \cap B \neq \phi$.*

Proof. There is a $\varphi \in \Lambda$ such that $K = \varphi^{-1}(B_R)$ with $\varphi(A) \subset B_R$. Since A is strongly chained to B, we have

$$\inf_{x \in B} \|\varphi(x)\| \leq \sup_{x \in A} \|\varphi(x)\| < R. \tag{2.5}$$

Hence, there is a $v \in B$ such that $\varphi(v) \in B_R$. Hence, $v = \varphi^{-1}\varphi(v) \in \varphi^{-1}(B_R)$. $\square$

Lemma 2.6. *If $K \in \mathcal{H}(A)$ and $\sigma \in \Lambda(A)$, then $\sigma(K) \in \mathcal{H}(A)$.*

Proof. There is a $\varphi \in \Lambda$ such that $K = \varphi^{-1}(B_R)$ with $\varphi(A) \subset B_R$. Let

$$\tilde{\varphi} = \varphi \circ \sigma^{-1}. \tag{2.6}$$

Then $\tilde{\varphi} \in \Lambda$ (Lemma 2.2). Now,

$$\tilde{\varphi}^{-1} = \sigma \circ \varphi^{-1}. \tag{2.7}$$

Hence,

$$\sigma(K) = \sigma[\varphi^{-1}(B_R)] = \tilde{\varphi}^{-1}(B_R). \tag{2.8}$$

Moreover,

$$\tilde{\varphi}(u) = \varphi(u), \quad u \in A. \tag{2.9}$$

Hence,

$$\sup_{x \in A} \|\tilde{\varphi}(x)\| = \sup_{x \in A} \|\varphi(x)\| < R. \tag{2.10}$$

Therefore, $\sigma(K) \in \mathcal{H}(A)$. $\square$

   We can now give the proof of Theorem 1.7.

Proof. Let G be a $(C^1 \cap C_U)$-functional on E, and let A, B be nonempty subsets of E such that A is strongly chained to B and (1.9) holds. Then

$$a := \inf_{K \in \mathcal{K}(A)} \sup_K G \qquad (2.11)$$

is finite. By Lemma 2.5, $K \cap B \neq \phi$ for all $K \in \mathcal{K}(A)$. Hence, $a \geq b_0$. Assume first that $b_0 < a$. If (1.11) were false, there would exist a positive constant δ such that $3\delta < a - b_0$ and

$$\|G'(u)\| \geq 3\delta \qquad (2.12)$$

whenever

$$u \in Q = \{u \in E : |G(u) - a| \leq 3\delta\}. \qquad (2.13)$$

Since $G \in C^1(E, \mathbb{R})$, there is a locally Lipschitz continuous mapping $Y(u)$ of $\hat{E} = \{u \in E : G'(u) \neq 0\}$ into E such that

$$\|Y(u)\| \leq 1, \quad u \in \hat{E}, \qquad (2.14)$$

$$(G'(u), Y(u)) \geq 2\delta \qquad (2.15)$$

whenever u satisfies (2.13). Let

$$Q_0 = \{u \in E : |G(u) - a| \leq 2\delta\},$$

$$Q_1 = \{u \in E : |G(u) - a| \leq \delta\},$$

$$Q_2 = E \setminus Q_0, \qquad (2.16)$$

$$\eta(u) = \frac{d(u, Q_2)}{[d(u, Q_1) + d(u, Q_2)]}.$$

It is easily checked that $\eta(u)$ is locally Lipschitz continuous on E and satisfies

$$\eta(u) = 1, \quad u \in Q_1; \qquad \eta(u) = 0, \quad u \in \bar{Q}_2; \qquad 0 < \eta(u) < 1, \quad \text{otherwise.} \qquad (2.17)$$

Consider the differential equation

$$\sigma'(t) = W(\sigma(t)), \quad t \in \mathbb{R}, \ \sigma(0) = u, \qquad (2.18)$$

where

$$W(u) = -\eta(u) Y(u). \qquad (2.19)$$

The mapping W is locally Lipschitz continuous on the whole of E and is bounded in norm by 1. Hence by a well-known existence theorem for ordinary differential equations in a Banach space, (2.18) has a unique solution for all $t \in \mathbb{R}$. Let us denote the solution

of (2.18) by $\sigma(t)u$. The mapping $\sigma(t)$ is in $C(E \times \mathbb{R}, E)$ and is called the flow generated by (2.19). Note that

$$
\begin{aligned}
\frac{dG(\sigma(t)u)}{dt} &= (G'(\sigma(t)u), \sigma'(t)u) \\
&= -\eta(\sigma(t)u)(G'(\sigma(t)u), Y(\sigma(t)u)) \\
&\le -2\delta\eta(\sigma(t)u).
\end{aligned}
\tag{2.20}
$$

Thus,

$$
\begin{aligned}
G(\sigma(t)u) &\le G(u), \quad t \ge 0, \\
G(\sigma(t)u) &\le a_0, \quad u \in A, \ t \ge 0, \\
\sigma(t)u &= u, \quad u \in A, \ t \ge 0.
\end{aligned}
\tag{2.21}
$$

This follows from the fact that

$$
G(\sigma(t)u) \le a_0 \le b_0 < a - 3\delta, \quad u \in A, \ t \ge 0.
\tag{2.22}
$$

Hence, $\eta(\sigma(t)u) = 0$ for $u \in A$, $t \ge 0$. This means that

$$
\begin{aligned}
\sigma'(t)u &= 0, \quad u \in A, \ t \ge 0, \\
\sigma(t)u &= \sigma(0)u = u, \quad u \in A, \ t \ge 0.
\end{aligned}
\tag{2.23}
$$

Let

$$
E_\alpha = \{u \in E : G(u) \le \alpha\}.
\tag{2.24}
$$

There is a $T > 0$ such that

$$
\sigma(T)E_{a+\delta} \subset E_{a-\delta}.
\tag{2.25}
$$

In fact, we can take $T = 1$. Let u be any element in $E_{a+\delta}$. If there is a $t_1 \in [0, T]$ such that $\sigma(t_1)u \notin Q_1$, then

$$
G(\sigma(T)u) \le G(\sigma(t_1)u) < a - \delta
\tag{2.26}
$$

by (2.20). Hence $\sigma(T)u \in E_{a-\delta}$. On the other hand, if $\sigma(t)u \in Q_1$ for all $t \in [0, T]$, then $\eta(\sigma(t)u) = 1$ for all t, and (2.20) yields

$$
G(\sigma(T)u) \le G(u) - 2\delta T \le a - \delta.
\tag{2.27}
$$

Hence (2.25) holds. Now by (2.11), there is a $K \in \mathcal{H}(A)$ such that

$$
K \subset E_{a+\delta}.
\tag{2.28}
$$

Note that $\sigma(T) \in \Lambda(A)$. Let $\widetilde{K} = \sigma(T)(K)$. Then $\widetilde{K} \in \mathcal{K}(A)$ by Lemma 2.6. But

$$\sup_{\widetilde{K}} G = \sup_{u \in K} G(\sigma(T)u) < a - \delta, \tag{2.29}$$

which contradicts (2.11), proving the theorem for the case $b_0 < a$.

Now assume $b_0 = a$. If there did not exist a sequence satisfying both (1.11)-(1.12), then there would be positive numbers ϵ, δ, T such that $\delta < \epsilon T$ and (2.12) holds whenever

$$u \in Q = \{u \in E : d(u, B) \le 4T, \ |G(u) - a| \le 3\delta\}. \tag{2.30}$$

Let

$$\begin{aligned} Q_0 &= \{u \in E : d(u, B) \le 3T, \ |G(u) - a| \le 2\delta\}, \\ Q_1 &= \{u \in E : d(u, B) \le 2T, \ |G(u) - a| \le \delta\}. \end{aligned} \tag{2.31}$$

Since $a = b_0$, we see that $Q_1 \ne \phi$. Define Q_2 and $\eta(u)$ as before and let $\sigma(t)$ be the flow generated by the mapping (2.19) with everything now with respect to the new sets Q_j. Let u be any element in $E_{a+\delta}$. If there is a $t_1 \le T$ such that $\sigma(t_1)u \notin Q_1$, then either

$$G(\sigma(t_1)u) < a - \delta \tag{2.32}$$

or

$$d(\sigma(t_1)u, B) > 2T. \tag{2.33}$$

Since

$$\|\sigma(t)u - \sigma(t')u\| \le |t - t'| \tag{2.34}$$

by (2.14), (2.33) implies that

$$d(\sigma(t)u, B) > T, \quad 0 \le t \le T. \tag{2.35}$$

On the other hand, if $\sigma(t)u \in Q_1$ for all $t \in [0, T]$, then

$$G(\sigma(T)u) \le G(u) - 2\epsilon T \le a + \delta - 2\delta = a - \delta. \tag{2.36}$$

Thus we have either

$$G(\sigma(T)u) < a - \delta \tag{2.37}$$

or (2.35) holds. Since $b_0 = a$, this shows that

$$\sigma(T)E_{a+\delta} \cap B = \phi. \tag{2.38}$$

We also note that

$$\sigma(t)A \cap B = \phi, \quad 0 \le t \le T. \tag{2.39}$$

We have by (2.20) that

$$G(\sigma(t)u) \le a_0 - 2\epsilon \int_0^t \eta(\sigma(\tau)u)d\tau, \quad u \in A. \tag{2.40}$$

If $\sigma(t)u \in B$, we must have $G(\sigma(t)u) \ge b_0 \ge a_0$. The only way this can happen is when

$$\eta(\sigma(\tau)u) \equiv 0, \quad 0 \le \tau \le t. \tag{2.41}$$

But this implies $\sigma(\tau)u \in \bar{Q}_2$ for such τ, and this in turn implies either

$$G(\sigma(\tau)u) < a - \delta, \quad 0 \le \tau \le t, \tag{2.42}$$

or

$$d(\sigma(\tau)u, B) > 2T, \quad 0 \le \tau \le t. \tag{2.43}$$

In either case, we cannot have $\sigma(t)u \in B$. Thus (2.39) holds. Let $K \in \mathcal{K}(A)$ satisfy (2.28), and let $\widetilde{K} = \sigma(T)K$. Then $\widetilde{K} \in \mathcal{K}(A)$. But (2.38) and (2.39) imply that $\widetilde{K} \cap B = \phi$, contradicting the fact that A is strongly chained to B. This completes the proof of the theorem. □

3. The remaining proofs

LEMMA 3.1. *If A is chained to B and $K \in \mathcal{K}_U(A)$, then $K \cap B \ne \phi$.*

Proof. There is a $\varphi \in \Lambda_U$ such that $K = \varphi^{-1}(B_R)$ with $\varphi(A) \subset B_R$. Since A is chained to B, we have

$$\inf_{x \in B} \|\varphi(x)\| \le \sup_{x \in A} \|\varphi(x)\| < R. \tag{3.1}$$

Hence, there is a $v \in B$ such that $\varphi(v) \in B_R$. Hence, $v = \varphi^{-1}\varphi(v) \in \varphi^{-1}(B_R)$. □

LEMMA 3.2. *If $K \in \mathcal{K}_U(A)$ and $\sigma \in \Lambda_U(A)$, then $\sigma(K) \in \mathcal{K}_U(A)$.*

We can now give the proof of Theorem 1.8.

Proof. Let G be a $(C^2 \cap C_U)$-functional on E, and let A, B be nonempty subsets of E such that A is chained to B and (1.9) holds. Then a given by (1.10) is finite. By Lemma 3.1, $K \cap B \ne \phi$ for all $K \in \mathcal{K}_U(A)$. Hence, $a \ge b_0$. Assume first that $b_0 < a$. If (1.11) were false, there would exist a positive constant δ such that $3\delta < a - b_0$ and

$$\|G'(u)\| \ge 3\delta \tag{3.2}$$

whenever

$$u \in Q = \{u \in E : |G(u) - a| \le 3\delta\}. \tag{3.3}$$

Let

$$Q_0 = \{u \in E : |G(u) - a| \le 2\delta\},$$

$$Q_1 = \{u \in E : |G(u) - a| \le \delta\},$$

$$Q_2 = E \setminus Q_0, \tag{3.4}$$

$$\eta(u) = \frac{d(u, Q_2)}{[d(u, Q_1) + d(u, Q_2)]}.$$

It is easily checked that $\eta(u)$ is locally Lipschitz continuous on E and satisfies

$$\eta(u) = 1, \quad u \in Q_1; \qquad \eta(u) = 0, \quad u \in \bar{Q}_2; \qquad 0 < \eta(u) < 1, \quad \text{otherwise.} \tag{3.5}$$

Consider the differential equation

$$\sigma'(t) = W(\sigma(t)), \quad t \in \mathbb{R}, \ \sigma(0) = u, \tag{3.6}$$

where

$$W(u) = \frac{-\eta(u) G'(u)}{\|G'(u)\|}. \tag{3.7}$$

Since $G \in C^2(E, \mathbb{R})$, the mapping W is locally Lipschitz continuous on the whole of E and is bounded in norm by 1. Hence by a well-known existence theorem for ordinary differential equations in a Banach space, (3.6) has a unique solution for all $t \in \mathbb{R}$. Let us denote the solution of (3.6) by $\sigma(t)u$. The mapping $\sigma(t)$ is in $C_U^1(E \times \mathbb{R}, E)$ and is called the flow generated by (3.7). Note that

$$\begin{aligned} \frac{dG(\sigma(t)u)}{dt} &= (G'(\sigma(t)u), \sigma'(t)u) \\ &= -\eta(\sigma(t)u) \|G'(\sigma(t)u)\| \\ &\le -2\delta\eta(\sigma(t)u). \end{aligned} \tag{3.8}$$

Thus,

$$G(\sigma(t)u) \le G(u), \quad t \ge 0,$$

$$G(\sigma(t)u) \le a_0, \quad u \in A, \ t \ge 0, \tag{3.9}$$

$$\sigma(t)u = u, \quad u \in A, \ t \ge 0.$$

Again, this follows from the fact that

$$G(\sigma(t)u) \le a_0 \le b_0 < a - 3\delta, \quad u \in A, \ t \ge 0. \tag{3.10}$$

Hence, $\eta(\sigma(t)u) = 0$ for $u \in A, \ t \ge 0$. This means that

$$\sigma'(t)u = 0, \quad u \in A, \ t \ge 0,$$

$$\sigma(t)u = \sigma(0)u = u, \quad u \in A, \ t \ge 0. \tag{3.11}$$

Let

$$E_\alpha = \{u \in E : G(u) \le \alpha\}. \tag{3.12}$$

We note that there is a $T > 0$ such that

$$\sigma(T)E_{a+\delta} \subset E_{a-\delta}. \tag{3.13}$$

(In fact, we can take $T = 1$.) Let u be any element in $E_{a+\delta}$. If there is a $t_1 \in [0, T]$ such that $\sigma(t_1)u \notin Q_1$, then

$$G(\sigma(T)u) \le G(\sigma(t_1)u) < a - \delta \tag{3.14}$$

by (3.8). Hence $\sigma(T)u \in E_{a-\delta}$. On the other hand, if $\sigma(t)u \in Q_1$ for all $t \in [0, T]$, then $\eta(\sigma(t)u) = 1$ for all t, and (3.8) yields

$$G(\sigma(T)u) \le G(u) - 2\delta T \le a - \delta. \tag{3.15}$$

Hence (3.13) holds. Now by (1.10), there is a $K \in \mathcal{H}_U(A)$ such that

$$K \subset E_{a+\delta}. \tag{3.16}$$

Note that $\sigma(T) \in \Lambda_U(A)$. Let $\tilde{K} = \sigma(T)(K)$. Then $\tilde{K} \in \mathcal{H}_U(A)$ by **Lemma 3.2**. But

$$\sup_{\tilde{K}} G = \sup_{u \in K} G(\sigma(T)u) < a - \delta, \tag{3.17}$$

which contradicts (1.10), proving the theorem for the case $b_0 < a$.

Now assume that $b_0 = a$. If there did not exist a sequence satisfying both (1.11) and (1.12), then there would be positive numbers ϵ, δ, T such that $\delta < \epsilon T$ and (2.12) holds whenever

$$u \in Q = \{u \in E : d(u, B) \le 4T, \ |G(u) - a| \le 3\delta\}. \tag{3.18}$$

Let

$$\begin{aligned} Q_0 &= \{u \in E : d(u, B) \le 3T, \ |G(u) - a| \le 2\delta\}, \\ Q_1 &= \{u \in E : d(u, B) \le 2T, \ |G(u) - a| \le \delta\}. \end{aligned} \tag{3.19}$$

Since $a = b_0$, we see that $Q_1 \ne \phi$. Define Q_2 and $\eta(u)$ as before and let $\sigma(t)$ be the flow generated by the mapping (3.7) with everything now with respect to the new sets Q_j. Let u be any element in $E_{a+\delta}$. If there is a $t_1 \le T$ such that $\sigma(t_1)u \notin Q_1$, then either

$$G(\sigma(t_1)u) < a - \delta \tag{3.20}$$

or

$$d(\sigma(t_1)u, B) > 2T. \tag{3.21}$$

Since

$$\|\sigma(t)u - \sigma(t')u\| \le |t - t'|, \tag{3.22}$$

(3.21) implies that

$$d(\sigma(t)u, B) > T, \quad 0 \le t \le T. \tag{3.23}$$

On the other hand, if $\sigma(t)u \in Q_1$ for all $t \in [0, T]$, then

$$G(\sigma(T)u) \le G(u) - 2\epsilon T \le a + \delta - 2\delta = a - \delta. \tag{3.24}$$

Thus we have either

$$G(\sigma(T)u) < a - \delta \tag{3.25}$$

or (3.23) holds. Since $b_0 = a$, this shows that

$$\sigma(T)E_{a+\delta} \cap B = \phi. \tag{3.26}$$

We also note that

$$\sigma(t)A \cap B = \phi, \quad 0 \le t \le T. \tag{3.27}$$

We have by (3.8) that

$$G(\sigma(t)u) \le a_0 - 2\epsilon \int_0^t \eta(\sigma(\tau)u) d\tau, \quad u \in A. \tag{3.28}$$

If $\sigma(t)u \in B$, we must have $G(\sigma(t)u) \ge b_0 \ge a_0$. The only way this can happen is when

$$\eta(\sigma(\tau)u) \equiv 0, \quad 0 \le \tau \le t. \tag{3.29}$$

But this implies $\sigma(\tau)u \in \bar{Q}_2$ for such τ, and this in turn implies either

$$G(\sigma(\tau)u) < a - \delta, \quad 0 \le \tau \le t, \tag{3.30}$$

or

$$d(\sigma(\tau)u, B) > 2T, \quad 0 \le \tau \le t. \tag{3.31}$$

In either case, we cannot have $\sigma(t)u \in B$. Thus (3.27) holds. Let $K \in \mathcal{K}_U(A)$ satisfy (3.16), and let $\tilde{K} = \sigma(T)K$. Then $\tilde{K} \in \mathcal{K}_U(A)$. But (3.26) and (3.27) imply that $\tilde{K} \cap B = \phi$, contradicting the fact that A is chained to B. This completes the proof of the theorem. $\square$

Now we give the proof of Theorem 1.9.

Proof. Assume that $\varphi \in \Lambda_U$ does not satisfy (1.6), and let $G(u) = \|\varphi(u)\|^2$. Then by the definition of the class Λ_U, $G \in C^1_U(E, \mathbb{R})$, $\sup G(A) < \inf G(B)$, and G has no critical level

$c \geq \inf_{x \in B} G(x) > 0$. To show the latter, assume that there is a sequence u_k satisfying (1.3). Then we have for any bounded sequence $v_k \in E$,

$$(\varphi'(u_k)v_k, \varphi(u_k)) \longrightarrow 0. \tag{3.32}$$

Let $v_k := (\varphi'(u_k))^{-1}(\varphi(u_k))$. Then $(\varphi'(u_k)v_k, \varphi(u_k)) = G(u_k) \to c$. However, the sequence v_k is bounded: $G(u_k) \to c$ implies that $\varphi(u_k)$ is bounded, which implies that u_k, and consequently that $(\varphi'(u_k))^{-1}$, and v_k are also bounded. Hence, $G'(u_k) \to 0$ implies $G(u_k) \to 0$, showing that $c = 0$. Thus, A does not link B. $\qquad\square$

4. Equivalent linking conditions

In this section, we relate our present definition of linking to the one of [11].

Let $A, B \subset E$ be disjoint sets and assume that A is bounded. Let Φ be a set of all maps $\psi \in C_U([0,1] \times E; E)$ such that $\psi(0, \cdot) = \mathrm{id}_E$ and that there exists a point $\xi \in E$ such that $\psi(t, \cdot)$ converges to ξ as $t \to 1$ uniformly on bounded sets. Let $\Psi \subset \Phi$. We will say that A is Ψ-chained to B if for every $\psi \in \Psi$,

$$\psi(A, [0,1]) \cap B \neq \varnothing. \tag{4.1}$$

PROPOSITION 4.1. *If A is Φ-chained to B, then A is strongly chained to B.*

Proof. Assume that A is not strongly chained to B. Then there are a map $\varphi \in \Lambda$ and an $R > 0$ such that $\varphi(A) \subset B_R(0)$ and $\varphi(B) \cap B_R(0) = \varnothing$. Let $\psi(t, u) = (1 - t)u$. Then $\psi \in \Phi$ and $[\psi(t, \cdot) \circ \varphi](A) \subset B_R(0)$. Therefore $[\psi(t, \cdot) \circ \varphi](A) \cap \varphi(B) = \varnothing$. This implies that $[\varphi^{-1} \circ \psi(t, \cdot) \circ \varphi](A) \cap B = \varnothing$. Since $\varphi^{-1} \circ \psi(t, \cdot) \circ \varphi \in \Phi$, this means that A is not Φ-chained to B, a contradiction. $\qquad\square$

COROLLARY 4.2. *If A is Φ-chained to B, then A links B.*

Let

$$\Phi_0 := \{\psi \in \Phi : \psi(t, \cdot) \in C_U^1 \text{ and } \forall t \in (0,1) \ \exists \varepsilon_t > 0 : \|\psi'(t, \cdot)\| \geq \varepsilon_t\}. \tag{4.2}$$

PROPOSITION 4.3. *If E is a Hilbert space and A links B, then A is Φ_0-chained to B.*

Proof. Assume that A is not Φ_0-chained to B. Then there are a map $\psi \in \Phi_0$ and a point $\xi \in E$ such that $\psi(t, \cdot) \to_{t \to 1} \xi$ and $\psi([0,1], A) \cap B = \varnothing$. Since A is bounded, $\psi(t, \cdot)$ converges to ξ uniformly on A. Then there are a $\tau \in (0,1)$ and an $r > 0$ such that $\psi(\tau, A) \subset B_r(\xi)$ and $B_{2r}(\xi) \cap B = \varnothing$. Let $G(u) = \|\psi(\tau, u) - \xi\|$. The functional G is in C_U^1 and it does not have a critical level $c > r$ since $\|\psi'(\tau, \cdot)\|$ is bounded away from zero. Thus, A does not link B. $\qquad\square$

COROLLARY 4.4. *If E is a Hilbert space, A is strongly chained to B, $d(A, B) > 0$, then A is Φ_0-chained to B.*

Acknowledgments

References

[1] P. Bartolo, V. Benci, and D. Fortunato, *Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity*, Nonlinear Analysis. Theory, Methods & Applications **7** (1983), no. 9, 981–1012.

[2] V. Benci and P. H. Rabinowitz, *Critical point theorems for indefinite functionals*, Inventiones Mathematicae **52** (1979), no. 3, 241–273.

[3] H. Brezis and L. Nirenberg, *Remarks on finding critical points*, Communications on Pure and Applied Mathematics **44** (1991), no. 8-9, 939–963.

[4] K.-C. Chang, *Infinite-Dimensional Morse Theory and Multiple Solution Problems*, Progress in Nonlinear Differential Equations and Their Applications, 6, Birkhäuser Boston, Massachusetts, 1993.

[5] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Applied Mathematical Sciences, vol. 74, Springer, New York, 1989.

[6] L. Nirenberg, *Variational and topological methods in nonlinear problems*, Bulletin of the American Mathematical Society. New Series **4** (1981), no. 3, 267–302.

[7] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conference Series in Mathematics, vol. 65, American Mathematical Society, Rhode Island, 1986.

[8] M. Schechter, *The intrinsic mountain pass*, Pacific Journal of Mathematics **171** (1995), no. 2, 529–544.

[9] _____, *New linking theorems*, Rendiconti del Seminario Matematico della Università di Padova **99** (1998), 255–269.

[10] _____, *Linking Methods in Critical Point Theory*, Birkhäuser Boston, Massachusetts, 1999.

[11] M. Schechter and K. Tintarev, *Pairs of critical points produced by linking subsets with applications to semilinear elliptic problems*, Bulletin de la Société Mathématique de Belgique. Série B **44** (1992), no. 3, 249–261.

[12] _____, *Linking in Hilbert space*, Proceedings of the American Mathematical Society **134** (2006), no. 2, 403–410.

[13] S. Z. Shi, *Ekeland's variational principle and the mountain pass lemma*, Acta Mathematica Sinica. New Series **1** (1985), no. 4, 348–355.

[14] E. A. de B. e. Silva, *Linking theorems and applications to semilinear elliptic problems at resonance*, Nonlinear Analysis. Theory, Methods & Applications **16** (1991), no. 5, 455–477.

[15] K. Tintarev, *Isotopic linking and critical points of functionals*, Nonlinear Analysis. Theory, Methods & Applications **30** (1997), no. 7, 4145–4149, Proceedings of the Second World Congress of Nonlinear Analysts, Part 7 (Athens, 1996).

[16] M. Willem, *Minimax Theorems*, Progress in Nonlinear Differential Equations and Their Applications, 24, Birkhäuser Boston, Massachusetts, 1996.

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