CALDERÓN'S REPRODUCING FORMULA FOR HANKEL CONVOLUTION

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Received 27 March 2006; Accepted 27 March 2006

Calderón-type reproducing formula for Hankel convolution is established using the theory of Hankel transform.

1. Introduction

Calderón's formula [3] involving convolutions related to the Fourier transform is useful in obtaining reconstruction formula for wavelet transform besides many other applications in decomposition of certain function spaces. It is expressed as follows:

$$f(x) = \int_0^\infty (\phi_t * \phi_t * f)(x)\frac{dt}{t}, \tag{1.1}$$

where $\phi : \mathbb{R}^n \to \mathbb{C}$ and $\phi_t(x) = t^{-n}\phi(x/t)$, $t > 0$. For conditions of validity of identity (1.1), we may refer to [3].

Hankel convolution introduced by Hirschman Jr. [5] related to the Hankel transform was studied at length by Cholewinski [1] and Haimo [4]. Its distributional theory was developed by Marrero and Betancor [6]. Pathak and Pandey [8] used Hankel convolution in their study of pseudodifferential operators related to the Bessel operator. Pathak and Dixit [7] exploited Hankel convolution in their study of Bessel wavelet transforms. In what follows, we give definitions and results related to the Hankel convolution [5] to be used in the sequel.

Let γ be a positive real number. Set

$$\begin{aligned} \sigma(x) &= \frac{x^{2\gamma+1}}{2^{\gamma+1/2}\Gamma(\gamma+3/2)}, \\ j(x) &= C_\gamma x^{1/2-\gamma} J_{\gamma-1/2}(x), \\ C_\gamma &= 2^{\gamma-1/2}\Gamma\left(\gamma+\frac{1}{2}\right), \end{aligned} \tag{1.2}$$

where $J_{\gamma-1/2}$ denotes the Bessel function of order $\gamma - 1/2$.

Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2006, Article ID 24217, Pages 1–7
DOI 10.1155/IJMMS/2006/24217

We define $L_{p,\sigma}(0,\infty)$, $1 \le p \le \infty$, as the space of those real measurable functions ϕ on $(0,\infty)$ for which

$$\|\phi\|_{p,\sigma} = \left[\int_0^\infty |\phi(x)|^p d\sigma(x)\right]^{1/p} < \infty, \quad 1 \le p < \infty,$$
$$\|\phi\|_{\infty,\sigma} = \operatorname{ess}\sup_{0<x<\infty} |\phi(x)| < \infty. \tag{1.3}$$

For each $\phi \in L_{1,\sigma}(0,\infty)$, the Hankel transform of ϕ is defined by

$$\mathcal{H}(\phi) = \widehat{\phi}(x) = \int_0^\infty j(xt)\phi(t)d\sigma(t), \quad 0 \le x < \infty. \tag{1.4}$$

From [5, page 314], we know that $\widehat{\phi}$ is bounded and continuous on $[0,\infty)$ and

$$\|\widehat{\phi}\|_{\infty,\sigma} \le \|\phi\|_{1,\sigma}. \tag{1.5}$$

If $\phi(x) \in L_{1,\sigma}(0,\infty)$ and if $\widehat{\phi}(t) \in L_{1,\sigma}(0,\infty)$, then, by inversion, we have [5, page 316]

$$\phi(x) = \int_0^\infty j(xt)\widehat{\phi}(t)d\sigma(t), \quad 0 < x < \infty. \tag{1.6}$$

If $\phi(x)$ and $\psi(x)$ are in $L_{1,\sigma}(0,\infty)$, then the following Parseval formula also holds [10, page 127]:

$$\int_0^\infty \widehat{\phi}(t)\widehat{\psi}(t)d\sigma(t) = \int_0^\infty \phi(x)\psi(x)d\sigma(x). \tag{1.7}$$

To introduce Hankel convolution #, we define

$$D(x,y,z) = \int_0^\infty j(xt)j(yt)j(zt)d\sigma(t)$$
$$= 2^{3\gamma-5/2}\left[\Gamma\left(\gamma+\frac{1}{2}\right)\right]^2\left[\Gamma(\gamma)\pi^{1/2}\right]^{-1}(xyz)^{-2\gamma+1}\left[\Delta(xyz)\right]^{2\gamma-2}, \tag{1.8}$$

where $\Delta(xyz)$ denotes the area of triangle with sides x, y, z if such a triangle exists and zero otherwise. Clearly $D(x,y,z) \ge 0$ and is symmetric in x, y, z. Applying inversion formulae (1.6) to (1.8), we get

$$\int_0^\infty D(x,y,z)j(zt)d\sigma(z) = j(xt)j(yt) \quad 0 < x,y < \infty,\ 0 \le t < \infty. \tag{1.9}$$

Now setting $t = 0$, we obtain

$$\int_0^\infty D(x,y,z)d\sigma(z) = 1. \tag{1.10}$$

Let $p,q,r \in [1,\infty)$ and $1/r = 1/p + 1/q - 1$. The Hankel convolution of $\phi \in L_{p,\sigma}(0,\infty)$ and $\psi \in L_{q,\sigma}(0,\infty)$ is defined by [5, page 311]

$$(\phi\#\psi)(x) = \int_0^\infty\int_0^\infty \phi(y)\psi(z)D(x,y,z)d\sigma(y)d\sigma(z). \tag{1.11}$$

By [9, page 179], the integral is convergent for almost all $x \in (0,\infty)$ and

$$\|\phi\#\psi\|_{r,\sigma} \leq \|\phi\|_{p,\sigma}\|\psi\|_{q,\sigma}. \tag{1.12}$$

Moreover, if $p = \infty$, then $(\phi\#\psi)(x)$ is defined for all $x \in (0,\infty)$ and is continuous. If $\phi, \psi \in L_{1,\sigma}(0,\infty)$, then from [5, page 314]

$$\widehat{(\phi\#\psi)}(t) = \hat{\phi}(t)\hat{\psi}(t), \quad 0 \leq t < \infty. \tag{1.13}$$

In this paper, Hankel dilation $\mathscr{D}_a$ is defined by

$$\mathscr{D}_a\phi(x) = \phi_a(x) = a^{-2\gamma-1}\phi\left(\frac{x}{a}\right), \quad a > 0. \tag{1.14}$$

2. Calderón's formula

In this section, we obtain Calderón's reproducing identity using the properties of Hankel transform and Hankel convolutions.

Theorem 2.1. *Let ϕ and $\psi \in L_{1,\sigma}(0,\infty)$ be such that following admissibility condition holds:*

$$\int_0^\infty \hat{\phi}(\xi)\hat{\psi}(\xi)\frac{d\sigma(\xi)}{\xi^{2\gamma+1}} = 1 \tag{2.1}$$

for all $\xi \in (0,\infty)$. Then the following Calderón's reproducing identity holds:

$$f(x) = \int_0^\infty (f\#\phi_a\#\psi_a)(x)\frac{d\sigma(a)}{a^{2\gamma+1}} \quad \forall f \in L^1(\mathbb{R}). \tag{2.2}$$

Proof. Taking Hankel transform of the right-hand side of (2.2), we get

$$\begin{aligned}
&\mathscr{H}\left[\int_0^\infty (f\#\phi_a\#\psi_a)(x)\frac{d\sigma(a)}{a^{2\gamma+1}}\right] \\
&\quad = \int_0^\infty \hat{f}(\xi)\hat{\phi}_a(\xi)\hat{\psi}_a(\xi)\frac{d\sigma(a)}{a^{2\gamma+1}} = \hat{f}(\xi)\int_0^\infty \hat{\phi}_a(\xi)\hat{\psi}_a(\xi)\frac{d\sigma(a)}{a^{2\gamma+1}} \\
&\quad = \hat{f}(\xi)\int_0^\infty \hat{\phi}(a\xi)\hat{\psi}(a\xi)\frac{d\sigma(a)}{a^{2\gamma+1}} = \hat{f}(\xi).
\end{aligned} \tag{2.3}$$

Now, by putting $a\xi = \omega$, we get

$$\begin{aligned}
\int_0^\infty \hat{\phi}(a\xi)\hat{\psi}(a\xi)\frac{d\sigma(a)}{a^{2\gamma+1}} &= \int_0^\infty \hat{\phi}(\omega)\hat{\psi}(\omega)\frac{d\sigma(\omega/\xi)}{(\omega/\xi)^{2\gamma+1}} \\
&= \int_0^\infty \hat{\phi}(\omega)\hat{\psi}(\omega)\frac{d\sigma(\omega)}{\omega^{2\gamma+1}} \\
&= 1.
\end{aligned} \tag{2.4}$$

Hence, the result follows.

The equality (2.2) can be interpreted in the following L^2- sense. □

Theorem 2.2. *Suppose $\phi \in L_{1,\sigma}(0,\infty)$ is real valued and satisfies*

$$\int_0^\infty [\hat{\phi}(a\xi)]^2 \frac{d\sigma(a)}{a^{2\gamma+1}} = 1. \tag{2.5}$$

For $f \in L_{1,\sigma}(0,\infty) \cap L_{2,\sigma}(0,\infty)$, suppose that

$$f_{\epsilon,\delta}(x) = \int_\epsilon^\delta (f \# \phi_a \# \phi_a)(x) \frac{d\sigma(a)}{a^{2\gamma+1}}. \tag{2.6}$$

Then $\|f - f_{\epsilon,\delta}\|_{2,\sigma} \to 0$ as $\epsilon \to 0$ and $\delta \to \infty$.

Proof. Taking Hankel transform of both sides of (2.6) and using Fubini's theorem, we get

$$\hat{f}_{\epsilon,\delta}(\xi) = \hat{f}(\xi) \int_\epsilon^\delta [\hat{\phi}(a\xi)]^2 \frac{d\sigma(a)}{a^{2\gamma+1}}. \tag{2.7}$$

By [5, page 311], we have

$$\begin{aligned} ||\phi_a \# \phi_a \# f||_{2,\sigma} &\leq ||\phi_a \# \phi_a||_{1,\sigma} \|f\|_{2,\sigma} \\ &\leq ||\phi_a||_{1,\sigma}^2 \|f\|_{2,\sigma}. \end{aligned} \tag{2.8}$$

Now using above inequality and Minkowski's inequality [2, page 41], we get

$$\begin{aligned} \|f\|_{2,\sigma}^2 &= \int_0^\infty d\sigma(x) \left| \int_\epsilon^\delta (\phi_a \# \phi_a \# f)(x) \frac{d\sigma(a)}{a^{2\gamma+1}} \right|^2 \\ &\leq \int_\epsilon^\delta \int_0^\infty |(\phi_a \# \phi_a \# f)(x)|^2 d\sigma(x) \frac{d\sigma(a)}{a^{2\gamma+1}} \\ &\leq \int_\epsilon^\delta ||(\phi_a \# \phi_a \# f)(x)||_{2,\sigma} \frac{d\sigma(a)}{a^{2\gamma+1}} \\ &\leq ||\phi_a||_{1,\sigma}^2 \|f\|_{2,\sigma} \int_\epsilon^\delta \frac{dt}{t} \\ &= ||\phi_a||_{1,\sigma}^2 \|f\|_{2,\sigma} \log\left(\frac{\delta}{\epsilon}\right). \end{aligned} \tag{2.9}$$

Hence, by Parseval formula (1.7), we get

$$\begin{aligned} \lim_{\substack{\epsilon\to 0\\ \delta\to\infty}} \|f - f_{\epsilon,\delta}\|_{2,\sigma}^2 &= \lim_{\substack{\epsilon\to 0\\ \delta\to\infty}} \|\hat{f} - \hat{f}_{\epsilon,\delta}\|_{2,\sigma}^2 \\ &= \lim_{\substack{\epsilon\to 0\\ \delta\to\infty}} \int_0^\infty \left| \hat{f}(\xi)\left(1 - \int_\epsilon^\delta [\hat{\phi}(a\xi)]^2 \frac{d\sigma(a)}{a^{2\gamma+1}}\right) \right|^2 d\sigma(x) = 0. \end{aligned} \tag{2.10}$$

Since $|\hat{f}(\xi)(1 - \int_\epsilon^\delta [\hat{\phi}(a\xi)]^2 (d\sigma(a)/a^{2\gamma+1}))| \leq |\hat{f}(\xi)|$, therefore, by the dominated convergence theorem, the result follows.

The reproducing identity (2.2) holds in the pointwise sense under different sets of nice conditions. □

Theorem 2.3. *Suppose* $f, \widehat{f} \in L_{1,\sigma}(0,\infty)$. *Let* $\phi \in L_{1,\sigma}(0,\infty)$ *be real valued and satisfies*

$$\int_0^\infty [\widehat{\phi}(a\xi)]^2 \frac{d\sigma(a)}{a^{2\gamma+1}} = 1, \quad \xi \in \mathbb{R} - \{0\}. \tag{2.11}$$

Then

$$\lim_{\substack{\epsilon\to 0\\ \delta\to\infty}} \int_\epsilon^\delta (f\#\phi_a\#\phi_a)(x)\frac{d\sigma(a)}{a^{2\gamma+1}} = f(x). \tag{2.12}$$

Proof. Let

$$f_{\epsilon,\delta}(x) = \int_\epsilon^\delta (f\#\phi_a\#\phi_a)(x)\frac{d\sigma(a)}{a^{2\gamma+1}}. \tag{2.13}$$

By [5, page 311], we have

$$\begin{aligned} ||\phi_a\#\phi_a\#f||_{1,\sigma} &\le ||\phi_a\#\phi_a||_{1,\sigma}||f||_{1,\sigma} \\ &\le ||\phi_a||_{1,\sigma}^2||f||_{1,\sigma}. \end{aligned} \tag{2.14}$$

Now,

$$\begin{aligned} ||f_{\epsilon,\delta}||_{1,\sigma} &= \int_0^\infty d\sigma(x)\left| \int_\epsilon^\delta (\phi_a\#\phi_a\#f)(x)\frac{d\sigma(a)}{a^{2\gamma+1}}\right| \\ &\le \int_\epsilon^\delta \int_0^\infty |(\phi_a\#\phi_a\#f)(x)|\, d\sigma(x)\frac{d\sigma(a)}{a^{2\gamma+1}} \\ &\le \int_\epsilon^\delta ||(\phi_a\#\phi_a\#f)(x)||_{1,\sigma}\frac{d\sigma(a)}{a^{2\gamma+1}} \\ &\le ||\phi_a||_{1,\sigma}^2\|f\|_{1,\sigma}\int_\epsilon^\delta \frac{dt}{t} \\ &= ||\phi_a||_{1,\sigma}^2\|f\|_{1,\sigma}\log\left(\frac{\delta}{\epsilon}\right). \end{aligned} \tag{2.15}$$

Therefore, $f_{\epsilon,\delta} \in L^1(0,\infty)$. Also using Fubini's theorem and taking Hankel transform of (2.13), we get

$$\begin{aligned} \widehat{f}_{\epsilon,\delta}(\xi) &= \int_0^\infty j(x\xi)\left(\int_\epsilon^\delta (\phi_a\#\phi_a\#f)(x)\frac{d\sigma(a)}{a^{2\gamma+1}}\right)d\sigma(x) \\ &= \int_\epsilon^\delta\int_0^\infty j(x\xi)(\phi_a\#\phi_a\#f)(x)d\sigma(x)\frac{d\sigma(a)}{a^{2\gamma+1}} \\ &= \int_\epsilon^\delta \widehat{\phi}_a(\xi)\widehat{\phi}_a(\xi)\widehat{f}(\xi)\frac{d\sigma(a)}{a^{2\gamma+1}} \\ &= \widehat{f}(\xi)\int_\epsilon^\delta [\widehat{\phi}(a\xi)]^2\frac{d\sigma(a)}{a^{2\gamma+1}}. \end{aligned} \tag{2.16}$$

Therefore, by (2.11), $|\hat{f}_{\epsilon,\delta}(\xi)| \le |\hat{f}(\xi)|$. It follows that $\hat{f}_{\epsilon,\delta} \in L_{1,\sigma}(0,\infty)$. By inversion, we have

$$f(x) - f_{\epsilon,\delta}(x) = \int_0^\infty j(x\xi)\big[\hat{f}(\xi) - \hat{f}_{\epsilon,\delta}(\xi)\big]d\sigma(\xi), \quad x \in (0,\infty). \tag{2.17}$$

Putting

$$\begin{aligned} h_{\epsilon,\delta}(\xi;x) &= j(x\xi)\big[\hat{f}(\xi) - \hat{f}_{\epsilon,\delta}(\xi)\big] \\ &= j(x\xi)\hat{f}(\xi)\left[1 - \int_\epsilon^\delta [\hat{\phi}(a\xi)]^2 \frac{d\sigma(a)}{a^{2\gamma+1}}\right], \end{aligned} \tag{2.18}$$

we get

$$\begin{aligned} f(x) - f_{\epsilon,\delta}(x) &= \int_0^\infty j(x\xi)\big[\hat{f}(\xi) - \hat{f}_{\epsilon,\delta}(\xi)\big]d\sigma(\xi) \\ &= \int_0^\infty h_{\epsilon,\delta}(\xi;x)d\sigma(\xi). \end{aligned} \tag{2.19}$$

Now using (2.11) in (2.18), we get

$$\lim_{\substack{\epsilon\to 0\\ \delta\to\infty}} h_{\epsilon,\delta}(\xi;x) = 0, \quad \xi \in \mathbb{R} - \{0\}. \tag{2.20}$$

Since $|h_{\epsilon,\delta}(\xi;x)| \le |\hat{f}(\xi)|$, the Lebsegue dominated convergence theorem yields

$$\lim_{\substack{\epsilon\to 0\\ \delta\to\infty}} \big[f(x) - f_{\epsilon,\delta}(x)\big] = 0, \quad \forall x. \tag{2.21}$$

$\square$

Acknowledgment

This work is supported by CSIR (New Delhi), Grant no. 9/13(04)/2003/ EMR-I.

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