# STRONG CONVERGENCE OF MODIFIED NOOR ITERATIONS

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In this paper, strong convergence theorem is obtained for the modified Noor iterations in the framework of uniformly smooth Banach spaces. Our results extend and improve the recent ones announced by Wittman, Kim, Xu, and some others.

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## 1. Introduction and preliminaries

Let $E$ be a real Banach space, $C$ a nonempty closed convex subset of $E$, and $T : C \to C$ a mapping. Recall that $T$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

A point $x \in C$ is a fixed point of $T$ provided $Tx = x$. Denote by $F(T)$ the set of fixed points of $T$; that is, $F(T) = \{x \in C : Tx = x\}$. It is assumed throughout the paper that $T$ is a nonexpansive mapping such that $F(T) \neq \emptyset$.

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping [1, 9]. More precisely, take $t \in (0,1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1-t)Tx, \quad x \in C, \tag{1.2}$$

where $u \in C$ is a fixed point. Banach's contraction mapping principle guarantees that $T_t$ has a unique fixed point $x_t$ in $C$. It is unclear, in general, what the behavior of $x_t$ is as $t \to 0$, even if $T$ has a fixed point. However, in the case of $T$ having a fixed point, Browder [1] proved that if $E$ is a Hilbert space, then $x_t$ does converge strongly to the fixed point of $T$ that is nearest to $u$. Reich [9] extended Browder's results to the setting of Banach spaces and proved that if $E$ is a uniformly smooth Banach space, then $x_t$ converges strongly to a fixed point of $T$ and the limit defines the (unique) sunny nonexpansive retraction from $C$ onto $F(T)$.

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Halpern [5] firstly introduced this iteration scheme:

$$x_0 = x \in C \quad \text{arbitrarily}, \qquad x_{n+1} = \alpha_n u + (1-\alpha_n) T x_n; \tag{1.3}$$

see also Browder [2]. He pointed out that the conditions $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$ are necessary in the sense that if the iteration scheme (1.3) converges to a fixed point of $T$, then these conditions must be satisfied. Ten years later, Lions [7] investigated the general case in Hilbert space under the conditions

$$\lim_{n\to\infty}\alpha_n = 0, \qquad \sum_{n=1}^{\infty}\alpha_n = \infty, \qquad \lim_{n\to\infty}\frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}^2} = 0 \tag{1.4}$$

on the parameters. However, Lions' conditions on the parameters were more restrictive and did not include the natural candidate $\{\alpha_n = 1/n\}$. Reich [9] gave the iteration scheme (1.3) in the case when $E$ is uniformly smooth and $\alpha_n = n^{-\delta}$ with $0 < \delta < 1$.

Wittmann [11] studied the iteration scheme (1.3) in the case when $E$ is a Hilbert space and $\{\alpha_n\}$ satisfies

$$\lim_{n\to\infty}\alpha_n = 0, \qquad \sum_{n=1}^{\infty}\alpha_n = \infty, \qquad \sum_{n=1}^{\infty}\|\alpha_{n+1} - \alpha_n\| < \infty. \tag{1.5}$$

Reich [10] obtained a strong convergence of the iterates (1.3) with two necessary and decreasing conditions on parameters for convergence in the case when $E$ is uniformly smooth with a weakly continuous duality mapping.

This paper introduces the composite iteration scheme:

$$\begin{aligned} w_n &= \delta_n x_n + (1-\delta_n) T x_n, & z_n &= \gamma_n x_n + (1-\gamma_n) T w_n, \\ y_n &= \beta_n x_n + (1-\beta_n) T z_n, & x_{n+1} &= \alpha_n u + (1-\alpha_n) y_n, \end{aligned} \tag{1.6}$$

where $u \in C$ is an arbitrary (but fixed) element in $C$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are sequences in $(0,1)$. We prove, under certain appropriate assumptions on the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ that $\{x_n\}$ defined by (1.6) converges to a fixed point of $T$.

If $\delta_n = 1$ in (1.6), then iterative scheme (1.6) is a modified Ishikawa iteration:

$$z_n = \gamma_n x_n + (1-\gamma_n) T x_n, \qquad y_n = \beta_n x_n + (1-\beta_n) T z_n, \qquad x_{n+1} = \alpha_n u + (1-\alpha_n) y_n. \tag{1.7}$$

If $\gamma_n = 1$ in (1.6), then iterative scheme (1.6) is a modified Mann iteration:

$$y_n = \beta_n x_n + (1-\beta_n) T x_n, \qquad x_{n+1} = \alpha_n u + (1-\alpha_n) y_n. \tag{1.8}$$

If $\gamma_n = 1$ and $\beta_n = 0$ in (1.6), then iterative scheme (1.6) is a usual Halpern iteration defined by (1.3).

Kim and Xu [6] proved the iteration scheme (1.8) converges to fixed point of $T$ in uniformly smooth Banach spaces.

It is our purpose in this paper to introduce composite iteration scheme (1.6) for approximating a fixed point of nonexpansive mappings in the framework of uniformly smooth Banach spaces; we establish the strong convergence of the composite iteration scheme $\{x_n\}$ defined by (1.6). The results improve and extend results of Kim and Xu [6], Wittmann [11], and others.

Let $E$ be a real Banach space and let $J$ denote the normalized duality mapping from $E$ into $2^{E^*}$ given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad x \in E, \tag{1.9}$$

where $E^*$ denotes the dual space of $E$ and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing.

The norm of $E$ is said to be Gâteaux differentiable (and $E$ is said to be smooth) if

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t} \tag{1.10}$$

exists for each $x$, $y$ in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be uniformly Fréchet differentiable (and $E$ is said to be uniformly smooth) if the limit in (1.10) is attained uniformly for $(x, y) \in U \times U$.

We need the following definitions and lemmas for the proof of our main results.

Lemma 1.1. *A Banach space $E$ is uniformly smooth if and only if the duality map $J$ is single-valued and norm-to-norm uniformly continuous on bounded sets of $E$.*

Lemma 1.2. *In a Banach space $E$, there holds the inequality*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad x, y \in E, \tag{1.11}$$

*where $j(x+y) \in J(x+y)$.*

Recall that if $C$ and $D$ are nonempty subsets of a Banach space $E$ such that $C$ is nonempty closed convex and $D \subset C$, then a map $Q : C \to D$ is sunny [3, 8] provided $Q(x+t(x-Q(x))) = Q(x)$ for all $x \in C$ and $t \ge 0$ whenever $x+t(x-Q(x)) \in C$. A sunny nonexpansive retraction is a sunny retraction, which is also nonexpansive. Sunny nonexpansive retractions play an important role in our argument. They are characterized as follows [3, 4, 8]: if $E$ is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if there holds the inequality

$$\langle x - Qx, J(y-Qx)\rangle \le 0, \quad \forall x \in C,\ y \in D. \tag{1.12}$$

Reich [9] showed that if $E$ is uniformly smooth and if $D$ is the fixed point set of a nonexpansive mapping from $C$ into itself, then there is a sunny nonexpansive retraction from $C$ onto $D$ and it can be constructed as follows.

Lemma 1.3 (see Reich [9]). *Let $E$ be a uniformly smooth Banach space and let $T : C \to C$ be a nonexpansive mapping with a fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1-t)tx$ that converges strongly as $t \to 0$ to a fixed point of $T$. Define $Q : C \to F(T)$ by $Qu = s-\lim_{t\to 0} x_t$. Then $Q$ is the unique sunny nonexpansive retract from $C$ onto $F(T)$; that is, $Q$ satisfies the property*

$$\langle u - Qu, J(z-Qu)\rangle \le 0, \quad u \in C,\ z \in F(T). \tag{1.13}$$

Lemma 1.4 (see Xu [12, 13]). *Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers satisfying the property*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \gamma_n\sigma_n, \quad n \geq 0, \tag{1.14}$$

*where $\{\gamma_n\}_{n=0}^{\infty} \subset (0,1)$ and $\{\sigma_n\}_{n=0}^{\infty}$ such that*
(i) $\lim_{n\to\infty}\gamma_n = 0$ *and* $\sum_{n=0}^{\infty}\gamma_n = \infty$,
(ii) *either* $\limsup_{n\to\infty}\sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty}|\gamma_n\sigma_n| < \infty$.
*Then $\{\alpha_n\}_{n=0}^{\infty}$ converges to zero.*

## 2. Main results

Theorem 2.1. *Let $C$ be a closed convex subset of a uniformly smooth Banach space $E$ and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \varnothing$. Given a point $u \in C$, the initial guess $x_0 \in C$ is chosen arbitrarily, and given sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ in $(0,1)$ and $\{\gamma_n\}_{n=0}^{\infty}$, $\{\delta_n\}_{n=0}^{\infty}$ in $[0,1]$, the following conditions are satisfied:*
(i) $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\alpha_n \to 0$;
(ii) $\beta_n + (1+\beta_n)(1-\gamma_n)(2-\delta_n) \in [0,a)$ *for some* $a \in (0,1)$;
(iii) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$, $\sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n| < \infty$, $\sum_{n=0}^{\infty}|\gamma_{n+1}-\gamma_n| < \infty$, *and* $\sum_{n=0}^{\infty}|\delta_{n+1}-\delta_n| < \infty$.

*Let $\{x_n\}_{n=1}^{\infty}$ be a composite process defined by*

$$\begin{aligned} w_n &= \delta_n x_n + (1-\delta_n)Tx_n, \qquad & z_n &= \gamma_n x_n + (1-\gamma_n)Tw_n, \\ y_n &= \beta_n x_n + (1-\beta_n)Tz_n, \qquad & x_{n+1} &= \alpha_n u + (1-\alpha_n)y_n. \end{aligned} \tag{2.1}$$

*Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of $T$.*

*Proof.* First we observe that $\{x_n\}_{n=0}^{\infty}$ is bounded. Indeed, if we take a fixed point $p$ of $T$, note that

$$||w_n - p|| \leq \delta_n||x_n - p|| + (1-\delta_n)||Tx_n - p|| \leq ||x_n - p||. \tag{2.2}$$

It follows from (1.6) and (2.2) that

$$\begin{aligned} ||z_n - p|| &\leq \gamma_n||x_n - p|| + (1-\gamma_n)||Tw_n - p|| \\ &\leq \gamma_n||x_n - p|| + (1-\gamma_n)||w_n - p|| \\ &\leq \gamma_n||x_n - p|| + (1-\gamma_n)||x_n - p|| \\ &\leq ||x_n - p||, \\ ||y_n - p|| &\leq \beta_n||x_n - p|| + (1-\beta_n)||Tz_n - p|| \\ &\leq \beta_n||x_n - p|| + (1-\beta_n)||z_n - p|| \\ &\leq \beta_n||x_n - p|| + (1-\beta_n)||x_n - p|| \\ &\leq ||x_n - p||. \end{aligned} \tag{2.3}$$

Therefore

$$\begin{aligned}||x_{n+1}-p|| &\le \alpha_n||u-p||+(1-\alpha_n)||y_n-p|| \\ &\le \alpha_n||u-p||+(1-\alpha_n)||x_n-p|| \\ &\le \max\{\|u-p\|,||x_n-p||\}.\end{aligned} \tag{2.4}$$

Now, an induction yields

$$||x_n-p|| \le \max\{\|u-p\|,||x_0-p||\}, \quad n\ge 0. \tag{2.5}$$

Hence, $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$. Next, we claim that

$$||x_{n+1}-x_n|| \longrightarrow 0. \tag{2.6}$$

In order to prove (2.6), it follows from (1.6) that

$$x_{n+1}=\alpha_n u+(1-\alpha_n)y_n, \qquad x_n=\alpha_{n-1}u+(1-\alpha_{n-1})y_{n-1}. \tag{2.7}$$

We have

$$\begin{aligned}x_{n+1}-x_n = {} & (1-\alpha_n)(1-\beta_n)(Tz_n-Tz_{n-1})+(1-\alpha_n)\beta_n(x_n-x_{n-1}) \\ & +[(\beta_n-\beta_{n-1})(1-\alpha_n)-(\alpha_n-\alpha_{n-1})\beta_{n-1}](x_{n-1}-Tz_{n-1}) \\ & +(\alpha_n-\alpha_{n-1})(u-Tz_{n-1}).\end{aligned} \tag{2.8}$$

Therefore

$$\begin{aligned}||x_{n+1}-x_n|| \le {} & (1-\beta_n)(1-\alpha_n)||Tz_n-Tz_{n-1}||+(1-\alpha_n)\beta_n||x_n-x_{n-1}|| \\ & +|(\beta_n-\beta_{n-1})(1-\alpha_n)-(\alpha_n-\alpha_{n-1})\beta_{n-1}|||x_{n-1}-Tz_{n-1}|| \\ & +|\alpha_n-\alpha_{n-1}|||u-Tz_{n-1}||.\end{aligned} \tag{2.9}$$

That is,

$$\begin{aligned}||x_{n+1}-x_n|| \le {} & (1-\beta_n)(1-\alpha_n)||z_n-z_{n-1}||+(1-\alpha_n)\beta_n||x_n-x_{n-1}|| \\ & +|(\beta_n-\beta_{n-1})(1-\alpha_n)-(\alpha_n-\alpha_{n-1})\beta_{n-1}|||x_{n-1}-Tz_{n-1}|| \\ & +|\alpha_n-\alpha_{n-1}|||u-Tz_{n-1}||.\end{aligned} \tag{2.10}$$

Again, from (1.6) we obtain

$$w_n=\delta_n x_n+(1-\delta_n)Tx_n, \qquad w_{n-1}=\delta_{n-1}x_{n-1}+(1-\delta_{n-1})Tx_{n-1}; \tag{2.11}$$

therefore

$$w_n-w_{n-1}=(1-\delta_n)(Tx_n-Tx_{n-1})+\delta_n(x_n-x_{n-1})+(\delta_n-\delta_{n-1})(x_{n-1}-Tx_{n-1}), \tag{2.12}$$

that is,

$$\|w_n - w_{n-1}\| \leq \|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}| \|x_{n-1} - Tx_{n-1}\|. \tag{2.13}$$

Similarly we obtain

$$z_n = \gamma_n x_n + (1-\gamma_n) Tw_n, \qquad z_{n-1} = \gamma_{n-1} x_{n-1} + (1-\gamma_{n-1}) Tw_{n-1}. \tag{2.14}$$

Thus,

$$z_n - z_{n-1} = (1-\gamma_n)(Tw_n - Tw_{n-1}) + \gamma_n (x_n - x_{n-1}) + (\gamma_n - \gamma_{n-1})(x_{n-1} - Tw_{n-1}), \tag{2.15}$$

that is,

$$\begin{aligned} \|z_n - z_{n-1}\| &\leq (1-\gamma_n)\|Tw_n - Tw_{n-1}\| + \gamma_n \|x_n - x_{n-1}\| + |\gamma_n - \gamma_{n-1}| \|x_{n-1} - Tw_{n-1}\| \\ &\leq (1-\gamma_n)\|w_n - w_{n-1}\| + \gamma_n \|x_n - x_{n-1}\| + |\gamma_n - \gamma_{n-1}| \|x_{n-1} - Tw_{n-1}\|. \end{aligned} \tag{2.16}$$

Now, substituting (2.13) into (2.16) we have

$$\begin{aligned} \|z_n - z_{n-1}\| &\leq (1-\gamma_n)\|w_n - w_{n-1}\| + \gamma_n \|x_n - x_{n-1}\| + |\gamma_n - \gamma_{n-1}| \|x_{n-1} - Tw_{n-1}\| \\ &\leq (1-\gamma_n)(\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}| \|x_{n-1} - Tx_{n-1}\|) \\ &\quad + \gamma_n \|x_n - x_{n-1}\| + |\gamma_n - \gamma_{n-1}| \|x_{n-1} - Tw_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}| \|x_{n-1} - Tx_{n-1}\| + |\gamma_n - \gamma_{n-1}| \|x_{n-1} - Tw_{n-1}\|. \end{aligned} \tag{2.17}$$

It follows that

$$\|z_n - z_{n-1}\| \leq \|x_n - x_{n-1}\| + (|\delta_{n-1} - \delta_n| + |\gamma_{n-1} - \gamma_n|) M_1, \tag{2.18}$$

where $M_1$ is a constant such that

$$M_1 \geq \max\{\|x_{n-1} - Tx_{n-1}\|, \|Tw_{n-1} - x_{n-1}\|\}. \tag{2.19}$$

Substituting (2.18) into (2.10), we get

$$\begin{aligned} \|x_{n+1} - x_n\| \leq{}& (1-\alpha_n)\|x_n - x_{n-1}\| + (1-\alpha_n)(1-\beta_n)(|\delta_{n-1} - \delta_n| + |\gamma_{n-1} - \gamma_n|) M_1 \\ &+ |(\beta_n - \beta_{n-1})(1-\alpha_n) - (\alpha_n - \alpha_{n-1})\beta_{n-1}| \|x_{n-1} - Tz_{n-1}\| \\ &+ |\alpha_n - \alpha_{n-1}| \|u - Tz_{n-1}\|, \end{aligned} \tag{2.20}$$

that is,

$$\begin{aligned} \|x_{n+1} - x_n\| \leq{}& (1-\alpha_n)\|x_n - x_{n-1}\| \\ &+ M(|\gamma_n - \gamma_{n-1}| + |\beta_n - \beta_{n-1}| + 2|\alpha_n - \alpha_{n-1}| + |\delta_n - \delta_{n-1}|), \end{aligned} \tag{2.21}$$

where $M$ is a constant such that

$$M \geq \max\{\|u - Tz_{n-1}\|, \|x_{n-1} - Tz_{n-1}\|, M_1\} \tag{2.22}$$

for all $n$. By assumptions (i)–(iii), we have that

$$\begin{gathered}\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \\ \sum_{n=1}^{\infty} \left( |\gamma_n - \gamma_{n-1}| + |\beta_n - \beta_{n-1}| + 2|\alpha_n - \alpha_{n-1}| + |\delta_n - \delta_{n-1}| \right) < \infty.\end{gathered} \tag{2.23}$$

Hence, Lemma 1.4 is applicable to (2.21) and we obtain

$$\|x_{n+1} - x_n\| \longrightarrow 0. \tag{2.24}$$

On the other hand, from (1.6) and condition (i), we have

$$\|x_{n+1} - y_n\| = \alpha_n \|u - y_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{2.25}$$

$$\begin{aligned}
\|Tx_n - x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - Tz_n\| + \|Tz_n - Tx_n\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - Tz_n\| + \|z_n - x_n\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n \|x_n - Tz_n\| + \|z_n - x_n\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n \|x_n - Tx_n\| \\
&\quad + \beta_n \|Tx_n - Tz_n\| + \|z_n - x_n\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n \|x_n - Tx_n\| \\
&\quad + \beta_n \|x_n - z_n\| + \|z_n - x_n\| \\
&= \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n \|x_n - Tx_n\| \\
&\quad + (1 + \beta_n) \|x_n - z_n\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n \|x_n - Tx_n\| \\
&\quad + (1 + \beta_n)(1 - \gamma_n) \|x_n - Tw_n\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n \|x_n - Tx_n\| \\
&\quad + (1 + \beta_n)(1 - \gamma_n) \|x_n - Tx_n\| + (1 + \beta_n)(1 - \gamma_n) \|Tx_n - Tw_n\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n \|x_n - Tx_n\| \\
&\quad + (1 + \beta_n)(1 - \gamma_n) \|x_n - Tx_n\| + (1 + \beta_n)(1 - \gamma_n) \|x_n - w_n\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n \|x_n - Tx_n\| \\
&\quad + (1 + \beta_n)(1 - \gamma_n) \|x_n - Tx_n\| + (1 + \beta_n)(1 - \gamma_n)(1 - \delta_n) \|Tx_n - x_n\|.
\end{aligned} \tag{2.26}$$

It follows that

$$\begin{aligned}\{1-[\beta_n+(1+\beta_n)(1-\gamma_n)+(1+\beta_n)(1-\gamma_n)(1-\delta_n)]\}&\|Tx_n-x_n\|\\ &\le\|x_n-x_{n+1}\|+\|x_{n+1}-y_n\|.\end{aligned}\tag{2.27}$$

That is,

$$\begin{aligned}\{1-[\beta_n+(1+\beta_n)(1-\gamma_n)(2-\delta_n)]\}&\|Tx_n-x_n\|\\ &\le\|x_n-x_{n+1}\|+\|x_{n+1}-y_n\|.\end{aligned}\tag{2.28}$$

From condition (ii), (2.24), and (2.25), we know

$$\|Tx_n-x_n\|\longrightarrow 0.\tag{2.29}$$

Next, we claim that

$$\limsup_{n\to\infty}\langle u-q,J(x_n-q)\rangle\le 0,\tag{2.30}$$

where $q=Qu=s-\lim_{t\to 0}z_t$ with $z_t$ being the fixed point of the contraction $z\mapsto tu+(1-t)Tz$.

From $z_t$, the fixed point equation

$$z_t=tu+(1-t)Tz_t,\tag{2.31}$$

is solved. Since $T_t$ defined by (1.2) is a contraction, we obtain that $T_t$ has a unique fixed point. Thus we have

$$\|z_t-x_n\|=\|(1-t)(Tz_t-x_n)+t(u-x_n)\|.\tag{2.32}$$

It follows from Lemma 1.2 that

$$\begin{aligned}\|z_t-x_n\|^2&\le(1-t)^2\|Tz_t-x_n\|^2+2t\langle u-x_n,J(z_t-x_n)\rangle\\ &\le(1-2t+t^2)\|z_t-x_n\|^2+f_n(t)\\ &\quad+2t\langle u-z_t,J(z_t-x_n)\rangle+2t\|z_t-x_n\|^2,\end{aligned}\tag{2.33}$$

where

$$f_n(t)=(2\|z_t-x_n\|+\|x_n-Tx_n\|)\|x_n-Tx_n\|\longrightarrow 0,\quad\text{as } n\longrightarrow 0.\tag{2.34}$$

It follows that

$$\langle z_t-u,J(z_t-x_n)\rangle\le\frac{t}{2}\|z_t-x_n\|^2+\frac{1}{2t}f_n(t).\tag{2.35}$$

Letting $n\to\infty$ in (2.35) and noting (2.34) yield

$$\limsup_{n\to\infty}\langle z_t-u,J(z_t-x_n)\rangle\le\frac{t}{2}M,\tag{2.36}$$

where $M > 0$ is a constant such that $M \geq \|z_t - x_n\|^2$ for all $t \in (0,1)$ and $n \geq 1$. Letting $t \to 0$ from (2.36), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \leq 0. \tag{2.37}$$

So, for any $\epsilon > 0$, there exists a positive number $\delta_1$ such that, for $t \in (0,\delta_1)$, we get

$$\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \leq \frac{\epsilon}{2}. \tag{2.38}$$

On the other hand, since $z_t \to q$ as $t \to 0$, from Lemma 1.1, there exists $\delta_2 > 0$ such that, for $t \in (0,\delta_2)$, we have

$$\begin{aligned}
&|\langle u - q, J(x_n - q)\rangle - \langle z_t - u, J(z_t - x_n)\rangle| \\
&\quad\leq |\langle u - q, J(x_n - q)\rangle - \langle u - q, J(x_n - z_t)\rangle| \\
&\qquad + |\langle u - q, J(x_n - z_t)\rangle - \langle z_t - u, J(z_t - x_n)\rangle| \\
&\quad\leq |\langle u - q, J(x_n - q) - J(x_n - z_t)\rangle| + |\langle z_t - q, J(x_n - z_t)\rangle| \\
&\quad\leq \|u - q\| \|J(x_n - q) - J(x_n - z_t)\| + \|z_t - q\| \|x_n - z_t\| < \frac{\epsilon}{2}.
\end{aligned} \tag{2.39}$$

Choosing $\delta = \min\{\delta_1, \delta_2\}$, for all $t \in (0,\delta)$, we have

$$\langle u - q, J(x_n - q)\rangle \leq \langle z_t - u, J(z_t - x_n)\rangle + \frac{\epsilon}{2}, \tag{2.40}$$

that is,

$$\limsup_{n\to\infty}\langle u - q, J(x_n - q)\rangle \leq \lim_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle + \frac{\epsilon}{2}. \tag{2.41}$$

It follows from (2.38) that

$$\limsup_{n\to\infty}\langle u - q, J(x_n - q)\rangle \leq \epsilon. \tag{2.42}$$

Since $\epsilon$ is chosen arbitrarily, we have

$$\limsup_{n\to\infty}\langle u - q, J(x_n - q)\rangle \leq 0. \tag{2.43}$$

Finally, we show that $x_n \to q$ strongly and this concludes the proof. Indeed, using Lemma 1.2 again, we obtain

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|(1-\alpha_n)(y_n - q) + \alpha_n(u - q)\|^2 \\
&\leq (1-\alpha_n)^2\|y_n - q\|^2 + 2\alpha_n\langle u - q, J(x_{n+1} - q)\rangle \\
&\leq (1-\alpha_n)\|x_n - q\|^2 + 2\alpha_n\langle u - q, J(x_{n+1} - q)\rangle.
\end{aligned} \tag{2.44}$$

Now we apply Lemma 1.4 and use (2.43) to see that $\|x_n - q\| \to 0$. □

As a corollary of Theorem 2.1, we have the following.

Corollary 2.2. *Let $C$ be a closed convex subset of a uniformly smooth Banach space $E$ and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point $u \in C$, the initial guess $x_0 \in C$ is chosen arbitrarily, and given sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ in $(0,1)$ and $\{\gamma_n\}_{n=0}^{\infty}$ in $[0,1]$, the following conditions are satisfied:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\alpha_n \to 0$;

(ii) $\beta_n + (1+\beta_n)(1-\gamma_n) \in [0,a)$ *for some* $a \in (0,1)$;

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, *and* $\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

*Let $\{x_n\}_{n=1}^{\infty}$ be the composite process defined by (1.7).*

*Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of $T$.*

*Proof.* By taking $\delta_n = 1$, we can obtain the desired conclusion. □

Corollary 2.3. *Let $C$ be a closed convex subset of a uniformly smooth Banach space $E$ and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point $u \in C$, the initial guess $x_0 \in C$ is chosen arbitrarily, and given sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ in (0,1), the following conditions are satisfied:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\alpha_n \to 0$;

(ii) $\beta_n \in [0,a)$ *for some* $a \in (0,1)$;

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

*Let $\{x_n\}_{n=1}^{\infty}$ be the composite process defined by (1.8).*

*Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of $T$.*

*Proof.* By taking $\gamma_n = 1$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of $T$. □

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