COMMUTANTS OF THE POMMIEZ OPERATOR

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The Pommiez operator $(\Delta f)(z) = (f(z) - f(0))/z$ is considered in the space $\mathcal{H}(G)$ of the holomorphic functions in an arbitrary finite Runge domain G. A new proof of a representation formula of Linchuk of the commutant of Δ in $\mathcal{H}(G)$ is given. The main result is a representation formula of the commutant of the Pommiez operator in an arbitrary invariant hyperplane of $\mathcal{H}(G)$. It uses an explicit convolution product for an arbitrary right inverse operator of Δ or of a perturbation $\Delta - \lambda I$ of it. A relation between these two types of commutants is found.

1. The Pommiez operator and its shift operators

Let G be a finite Runge domain in the complex plane $\mathbb{C}$, that is, a finite domain with connected complement with the characteristic property that every holomorphic function can be approximated by polynomials. As usual, by $\mathcal{H}(G)$, the space of the holomorphic functions on G is denoted. Additionally, assume that $0 \in G$.

Definition 1.1. If $f \in \mathcal{H}(G)$, then the *Pommiez operator* Δ is defined by

$$(\Delta f)(z) = \begin{cases} \dfrac{f(z) - f(0)}{z} & \text{if } z \neq 0, \\ f'(0) & \text{if } z = 0. \end{cases} \tag{1.1}$$

Remark 1.2. The notation of Pommiez in [8] for Δ is $f_{(1)}$, and $f_{(n)}$ for the nth power Δ^n assuming that the operator Δ acts on the holomorphic functions in a disc $D_R = \{z : |z| < R\}$. The operator Δ is known also as the *backward shift operator* (see Douglas et al. [5]).

Definition 1.3. Let ζ be an arbitrary point of G. Then the operator

$$(T_\zeta f)(z) = \begin{cases} \dfrac{zf(z) - \zeta f(\zeta)}{z - \zeta} & \text{if } z \neq \zeta, \\ f(\zeta) + \zeta f'(\zeta) & \text{if } z = \zeta, \end{cases} \tag{1.2}$$

determined by ζ, is called a *shift operator for the Pommiez operator* in $\mathcal{H}(G)$.

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Remark 1.4. Such an operator appears in Linchuk's representation formula of the commutant of Δ in $\mathcal{H}(G)$ (see [7, Theorem 1]). The name of the functional shift operator for T_ζ is given by Binderman [1, 2].

Theorem 1.5. *T_ζ is a continuous linear operator in $\mathcal{H}(G)$ with the compact-open topology, that is, with respect to the uniform convergence on the compact subsets of G.*

Proof. According to Köthe [6, pages 375–378], it is enough to consider a sequence $\{G_n\}_{n=1}^{\infty}$ of connected domains such that $G_n \subset \overline{G_n} \subset G_{n+1}$, for all n, and which exhausts G, that is, $G = \bigcup_{n=1}^{\infty} G_n$. Then the sequence of norms $p_n(f) = \sup_{z\in G_n} |f(z)| = \max_{z\in\overline{G_n}} |f(z)|$ generates the topology. Since the continuity of an operator is equivalent to its boundedness, here the latter will be established on G_n for all sufficiently large n.

Let $\zeta \in G$. Then for some n_0, one has $\zeta \in G_n$ for all $n \geq n_0$. Using the definition of T_ζ, the following estimate holds:

$$\left| T_\zeta f(z) \right| \leq \left| f(z) \right| + |\zeta| \left| \frac{f(z) - f(\zeta)}{z - \zeta} \right|. \tag{1.3}$$

If z is close to ζ, then the right-hand side of (1.3) could be estimated approximately as $|f(\zeta)| + |\zeta||f'(\zeta)|$, but for holomorphic functions, the derivative f' can be estimated by the function f itself, that is, $|f'(\zeta)| \leq B_n \max_{z\in\overline{G_n}} |f(z)|$. In general, everywhere in $\overline{G_n}$,

$$\left| T_\zeta f(z) \right| \leq A_n \max_{\eta\in\overline{G_n}} \left| f(\eta) \right|. \tag{1.4}$$

Then (1.4) can be written as the desired boundedness estimate for the operator T_ζ,

$$p_n(T_\zeta f) \leq A_n \max_{z\in\overline{G_n}} \left| f(z) \right| = A_n p_n(f), \quad \forall f \in \mathcal{H}(G). \tag{1.5}$$

□

Lemma 1.6. *If G is an arbitrary domain in the complex plane $\mathbb{C}$ containing the origin, then T_ζ commutes with the Pommiez operator Δ, that is,*

$$[(T_\zeta \Delta) f](z) = [(\Delta T_\zeta) f](z) \tag{1.6}$$

for every $f \in \mathcal{H}(G)$.

The proof of this lemma is a matter of an elementary check.

Lemma 1.7. *Let $p(z)$ be a polynomial of degree n. Then,*

$$(T_\zeta p)(z) = \sum_{k=0}^{n} (\Delta^k p)(z) \cdot \zeta^k. \tag{1.7}$$

Proof. It is sufficient to check (1.7) for an arbitrary power z^k. Obviously,

$$\Delta^s z^k = \begin{cases} z^{k-s} & \text{for } 0 \leq s \leq k, \\ 0 & \text{for } s > k. \end{cases} \tag{1.8}$$

If $z \neq \zeta$, then

$$\begin{aligned} T_\zeta(z^k) &= \frac{z \cdot z^k - \zeta \cdot \zeta^k}{z - \zeta} = z^k + z^{k-1}\zeta + \cdots + z\zeta^{k-1} + \zeta^k \\ &= (\Delta^0 z^k)\zeta^0 + (\Delta^1 z^k)\zeta^1 + \cdots + (\Delta^{k-1} z^k)\zeta^{k-1} + (\Delta^k z^k)\zeta^k \\ &= \sum_{s=0}^{k} (\Delta^s z^k)\zeta^s. \end{aligned} \tag{1.9}$$

Finally, in order to obtain (1.7) for arbitrary polynomial p, it remains to use the linearity of T_ζ.

The check of (1.7) for $z = \zeta$ is also easy. □

Theorem 1.8 (see Linchuk [7, Theorem 1]). *A continuous linear operator $M : \mathcal{H}(G) \to \mathcal{H}(G)$ commutes with the Pommiez operator Δ in $\mathcal{H}(G)$ if and only if it has a representation of the form*

$$(Mf)(z) = \Phi_\zeta\{(T_\zeta f)(z)\} \tag{1.10}$$

with a continuous linear functional $\Phi : \mathcal{H}(G) \to \mathbb{C}$.

Proof. The sufficiency can be proved by a direct check. Only the necessity needs to be proved. Lemma 1.7 implies that if $M\Delta = \Delta M$, then $MT_\zeta = T_\zeta M$ for all $\zeta \in G$. Indeed, if p is a polynomial of degree n, then by (1.7),

$$(MT_\zeta p)(z) = \sum_{k=0}^{n} M(\Delta^k p)(z) = \sum_{k=0}^{n} \Delta^k (Mp)(z) = (T_\zeta Mp)(z). \tag{1.11}$$

Then the identity $(MT_\zeta f)(z) = (T_\zeta Mf)(z)$ for any $f \in \mathcal{H}(G)$ follows by an approximation argument. Using it and the obvious property

$$(T_\zeta f)(z) = (T_z f)(\zeta), \tag{1.12}$$

one has

$$(MT_\zeta f)(z) = (T_z Mf)(\zeta). \tag{1.13}$$

Define the continuous linear functional $\Phi : \mathcal{H}(G) \to \mathbb{C}$ by

$$\Phi\{f\} = (Mf)(0). \tag{1.14}$$

Substituting $z = 0$ in (1.13), one has

$$\Phi\{T_\zeta f\} = (T_0 Mf)(\zeta). \tag{1.15}$$

But $T_0 = I$, the identity operator. Hence,

$$(Mf)(\zeta) = \Phi\{T_\zeta f\}. \tag{1.16}$$

It remains to write the variable z instead of ζ, denoting the "dumb" variable in the functional Φ by ζ, and to use (1.12). Thus,

$$(Mf)(z) = \Phi_\zeta\{(T_z f)(\zeta)\} = \Phi_\zeta\{(T_\zeta f)(z)\}. \tag{1.17}$$

□

2. Characterization of linear operators $M : \mathcal{H}(G) \to \mathcal{H}(G)$ with a fixed invariant hyperplane $\Phi\{f\} = 0$ which commute with the Pommiez operator Δ on it

Let $\Phi : \mathcal{H}(G) \to \mathbb{C}$ be a fixed nonzero linear functional, and consider the hyperplane

$$\mathcal{H}_\Phi = \{f \in \mathcal{H}(G) : \Phi\{f\} = 0\}. \tag{2.1}$$

Our aim is to characterize the linear operators $M : \mathcal{H}(G) \to \mathcal{H}(G)$ such that $\Phi\{f\} = 0$ implies that $\Phi\{Mf\} = 0$ and $M\Delta = \Delta M$ in the hyperplane $\mathcal{H}_\Phi$. In other words, we are looking for the continuous linear operators $M : \mathcal{H}(G) \to \mathcal{H}(G)$ such that $M(\mathcal{H}_\Phi) \subset \mathcal{H}_\Phi$ and which commute with the Pommiez operator Δ in $\mathcal{H}_\Phi$.

A similar problem for the differentiation operators is considered in [3].

In order to find the operators commuting with Δ in $\mathcal{H}(G)$, the one-parameter family $\{T_\zeta\}_{\zeta \in G}$ of operators commuting with Δ was used. Now it is possible to use another one-parameter family of linear operators.

Definition 2.1. Let $\lambda \in \mathbb{C}$ be such that the elementary boundary value problem

$$\begin{gathered}(\Delta y)(z) - \lambda y(z) = f(z),\\ \Phi\{y\} = 0\end{gathered} \tag{2.2}$$

has a solution $y = R_\lambda f$. The operator $R_\lambda : \mathcal{H}(G) \to \mathcal{H}(G)$ is called the *resolvent operator of the Pommiez operator with the boundary value condition* $\Phi\{f\} = 0$.

From the first equation of (2.2) it is easy to obtain the solution

$$y(z) = \frac{z}{1 - \lambda z} f(z) + \frac{y(0)}{1 - \lambda z} \tag{2.3}$$

with unknown constant $y(0)$. Formally, its value can be determined from the boundary condition $\Phi\{y\} = 0$. This is always possible, when $1/(1 - \lambda z) \in \mathcal{H}(G)$. Then, for the next considerations, it is convenient to denote

$$E(\lambda) = \Phi_\zeta\left\{\frac{1}{1 - \lambda\zeta}\right\}. \tag{2.4}$$

The function $E(\lambda)$ is defined and holomorphic at least in a neighborhood of the origin $\lambda = 0$. Let $\lambda \in \mathbb{C}$ be such that $E(\lambda) \neq 0$ and $1/(1 - \lambda z) \in \mathcal{H}(G)$. Such a choice of λ is always possible since the zeros of $E(\lambda)$ form a countable set and G is a finite domain. It is sufficient to choose λ so close to the origin that $1/\lambda \notin G$.

Now the condition $\Phi\{y\} = 0$ allows to find $y(0)$ and to obtain

$$(R_\lambda f)(z) = \frac{z}{1 - \lambda z} f(z) - \frac{1}{E(\lambda)(1 - \lambda z)} \Phi_\zeta\left\{\frac{\zeta f(\zeta)}{1 - \lambda\zeta}\right\}. \tag{2.5}$$

Substituting $(\Delta - \lambda I)f$ for f in (2.5) gives the following lemma.

LEMMA 2.2. *If $f \in \mathcal{H}(G)$, then*

$$[R_\lambda(\Delta - \lambda I)f](z) = f(z) - \frac{\Phi\{f\}}{E(\lambda)(1-\lambda z)}. \tag{2.6}$$

From (2.6), it follows that

$$[(\Delta R_\lambda)f](z) = [(R_\lambda \Delta)f](z) \quad \text{iff } \Phi\{f\} = 0, \tag{2.7}$$

that is, the resolvent operator R_λ commutes with the Pommiez operator if and only if f is in the hyperplane $\mathcal{H}_\Phi$. Hence, the resolvent operators form a one-parameter family of the class considered above.

An important role in the sequel will play the functions of the form

$$\varphi_\lambda(z) = \frac{1}{1-\lambda z}, \quad \lambda \in \mathbb{C}, \tag{2.8}$$

and also their modifications

$$\widetilde{\varphi}_\lambda(z) = \frac{\varphi_\lambda(z)}{E(\lambda)} = \frac{1}{E(\lambda)(1-\lambda z)} = \frac{1}{\Phi_\zeta\{1/(1-\lambda\zeta)\}(1-\lambda z)}. \tag{2.9}$$

THEOREM 2.3. *The operation*

$$(f * g)(z) = \Phi_\zeta\{(z-\zeta)T_\zeta f(z) T_\zeta g(z)\} = \Phi_\zeta\left\{\frac{[zf(z)-\zeta f(\zeta)][zg(z)-\zeta g(\zeta)]}{z-\zeta}\right\} \tag{2.10}$$

is a bilinear, commutative, and associative operation in $\mathcal{H}(G)$ such that

$$\Phi\{f * g\} = 0 \quad \textit{for arbitrary } f, g \in \mathcal{H}(G), \tag{2.11}$$

that is, $f * g$ is in the hyperplane defined by the functional Φ, and

$$(R_\lambda f)(z) = (\widetilde{\varphi}_\lambda * f)(z) = \frac{1}{E(\lambda)}(\varphi_\lambda * f)(z). \tag{2.12}$$

Proof. The bilinearity and the commutativity of the operation $*$ defined by (2.10) are obvious and only the associativity will be proved.

Since G is a finite domain, then for sufficiently small λ and μ, the functions $\varphi_\lambda(z) = 1/(1-\lambda z)$ and $\varphi_\mu(z) = 1/(1-\mu z)$ are in $\mathcal{H}(G)$. It is a matter of a simple algebra to show that if $\lambda \neq \mu$, then

$$(\varphi_\lambda * \varphi_\mu)(z) = \frac{E(\mu)\varphi_\lambda(z) - E(\lambda)\varphi_\mu(z)}{\lambda - \mu}. \tag{2.13}$$

From this representation, it follows immediately that

$$[(\varphi_\lambda * \varphi_\mu) * \varphi_\nu](z) = \frac{E(\mu)E(\nu)}{(\lambda-\mu)(\lambda-\nu)}\varphi_\lambda(z) + \frac{E(\nu)E(\lambda)}{(\mu-\nu)(\mu-\lambda)}\varphi_\mu(z) + \frac{E(\lambda)E(\mu)}{(\nu-\lambda)(\nu-\mu)}\varphi_\nu(z). \tag{2.14}$$

Due to the circular symmetry with respect to λ, μ, and ν, one has the same expression for $[\varphi_\lambda * (\varphi_\mu * \varphi_\nu)](z)$, and hence

$$(\varphi_\lambda * \varphi_\mu) * \varphi_\nu = \varphi_\lambda * (\varphi_\mu * \varphi_\nu). \tag{2.15}$$

Since

$$\frac{\partial}{\partial\lambda}(\varphi_\lambda * \varphi_\mu) = \frac{\partial\varphi_\lambda}{\partial\lambda} * \varphi_\mu, \qquad \frac{\partial}{\partial\mu}(\varphi_\lambda * \varphi_\mu) = \varphi_\lambda * \frac{\partial\varphi_\mu}{\partial\mu}, \tag{2.16}$$

then partial differentiations with respect to λ, μ, and ν of (2.15), l, m, and n times, respectively, yield

$$\left(\frac{\partial^l\varphi_\lambda}{\partial\lambda^l} * \frac{\partial^m\varphi_\mu}{\partial\mu^m}\right) * \frac{\partial^n\varphi_\nu}{\partial\nu^n} = \frac{\partial^l\varphi_\lambda}{\partial\lambda^l} * \left(\frac{\partial^m\varphi_\mu}{\partial\mu^m} * \frac{\partial^n\varphi_\nu}{\partial\nu^n}\right), \tag{2.17}$$

which is in fact the identity

$$\left[\frac{l!z^l}{(1-\lambda z)^{l+1}} * \frac{m!z^m}{(1-\mu z)^{m+1}}\right] * \frac{n!z^n}{(1-\nu z)^{n+1}} = \frac{l!z^l}{(1-\lambda z)^{l+1}} * \left[\frac{m!z^m}{(1-\mu z)^{m+1}} * \frac{n!z^n}{(1-\nu z)^{n+1}}\right]. \tag{2.18}$$

Letting λ, μ, and ν tend separately to 0, and dividing by $l!m!n!$, it follows that

$$(z^l * z^m) * z^n = z^l * (z^m * z^n). \tag{2.19}$$

The bilinearity of the convolution now ensures that the associativity is valid for arbitrary polynomials p, q, and r as follows:

$$[p(z) * q(z)] * r(z) = p(z) * [q(z) * r(z)]. \tag{2.20}$$

The final step is to use Runge's theorem to approximate arbitrary holomorphic functions f, g, and h from $\mathcal{H}(G)$ by polynomials in order to complete the proof of the associativity,

$$(f * g) * h = f * (g * h). \tag{2.21}$$

The proof of the second assertion (2.11) of the theorem follows from the fact that the function

$$h(z,\zeta) = \frac{[zf(z) - \zeta f(\zeta)][zg(z) - \zeta g(\zeta)]}{z-\zeta} \tag{2.22}$$

is antisymmetric with respect to z and ζ, that is, $h(z,\zeta) = -h(\zeta,z)$, and hence

$$\begin{aligned}\Phi\{f * g\} &= \Phi_z\{(f * g)(z)\} = \Phi_z\Phi_\zeta\{h(z,\zeta)\} = \Phi_z\Phi_\zeta\{-h(\zeta,z)\} = -\Phi_z\Phi_\zeta\{h(\zeta,z)\} \\ &= -\Phi_\zeta\Phi_z\{h(\zeta,z)\} = -\Phi_z\Phi_\zeta\{h(z,\zeta)\} = -\Phi\{f * g\}.\end{aligned} \tag{2.23}$$

Here it is used that the functional Φ has the Fubini property, that is, the possibility of interchanging of Φ_z and Φ_ζ. At the end, z and ζ are also interchanged, since they are "dumb" variables in the expression. Thus (2.23) gives $2\Phi\{f * g\} = 0$, and hence (2.11) holds.

The last assertion in the theorem (2.12) can be proved directly. It is enough to use (2.10) when expressing the right-hand side of (2.12) and to compare with (2.5).

Further, (2.12) can be expressed in other words saying that the resolvent operator R_λ is in fact the convolution operator $\widetilde{\varphi}_\lambda *$ and one may write $R_\lambda = \widetilde{\varphi}_\lambda *$. □

Theorem 2.4. *The commutant of Δ with the invariant hyperplane $\mathcal{H}_\Phi$ coincides with the commutant of the resolvent operators R_λ in $\mathcal{H}(G)$.*

Proof. Let $M : \mathcal{H}(G) \to \mathcal{H}(G)$ be a linear operator commuting with R_λ for some $\lambda \in \mathbb{C}$, that is, $MR_\lambda = R_\lambda M$. First, it will be proved that $\mathcal{H}_\Phi$ is an invariant hyperplane for M. Indeed, let f and g be functions from $\mathcal{H}(G)$ such that $R_\lambda g = f$. By (2.2), this means that

$$\Delta f - \lambda f = g. \tag{2.24}$$

Next $MR_\lambda g = Mf$, or

$$R_\lambda Mg = MR_\lambda g = Mf \tag{2.25}$$

and hence, applying $\Delta - \lambda I$ and Definition 2.1,

$$Mg = (\Delta - \lambda I)Mf. \tag{2.26}$$

Using (2.24), this can be written as

$$M(\Delta - \lambda I)f = (\Delta - \lambda I)Mf, \tag{2.27}$$

which yields

$$(M\Delta)f = (\Delta M)f. \tag{2.28}$$

Hence, M commutes with Δ in $\mathcal{H}_\Phi$. It remains to show that $\Phi(Mf) = 0$. This follows using the representation (2.12) of the resolvent as a convolutional operator, and (2.11).

Conversely, let $M : \mathcal{H}(G) \to \mathcal{H}(G)$ have the hyperplane $\mathcal{H}_\Phi$ as an invariant subspace and let $M\Delta = \Delta M$ in $\mathcal{H}_\Phi$. One has to prove that $MR_\lambda = R_\lambda M$ for $\lambda \in \mathbb{C}$ with $E(\lambda) \neq 0$.

Let $f \in \mathcal{H}(G)$ be arbitrary and denote $h = (MR_\lambda - R_\lambda M)f$. Then

$$(\Delta - \lambda I)h = (\Delta - \lambda I)MR_\lambda f - Mf = M(\Delta - \lambda I)R_\lambda f - Mf = 0, \tag{2.29}$$

and also

$$\Phi\{h\} = \Phi\{MR_\lambda f\} - \Phi\{R_\lambda Mf\} = 0, \tag{2.30}$$

according to our assumptions. Since λ is not an eigenvalue, then $h = 0$, or

$$MR_\lambda f = R_\lambda Mf. \tag{2.31}$$

□

Definition 2.5. A linear operator $M : \mathcal{H}(G) \to \mathcal{H}(G)$ is said to be a *multiplier of the convolution algebra* $(\mathcal{H}(G), *)$ *when for arbitrary* $f, g \in \mathcal{H}(G)$, *it holds that*

$$M(f * g) = (Mf) * g. \tag{2.32}$$

Theorem 2.6. *A linear operator* $M : \mathcal{H}(G) \to \mathcal{H}(G)$ *is a multiplier of the convolution algebra* $(\mathcal{H}(G), *)$ *if and only if it has a representation of the form*

$$Mf(z) = \mu f(z) + (m * f)(z), \tag{2.33}$$

where $\mu =$ *const and* $m \in \mathcal{H}(G)$.

Proof. The sufficiency is obvious.

In order to prove the necessity, let $\lambda \in \mathbb{C}$ be such that $E(\lambda) \neq 0$ and $\varphi_\lambda(z) = 1/(1 - \lambda z) \in \mathcal{H}(G)$. To this end, it is enough to take λ with $|\lambda|$ so small that $1/\lambda \notin G$. This is possible since G is assumed to be finite.

Let $M : \mathcal{H}(G) \to \mathcal{H}(G)$ be an arbitrary multiplier of $(\mathcal{H}(G), *)$. Applying (2.12), one has

$$MR_\lambda f = M(\tilde{\varphi}_\lambda * f) = (M\tilde{\varphi}_\lambda) * f = \tilde{\varphi}_\lambda * Mf = R_\lambda Mf, \tag{2.34}$$

that is, $MR_\lambda f = R_\lambda Mf$. Also, denoting $r_\lambda = M\tilde{\varphi}_\lambda$, (2.34) gives

$$R_\lambda Mf = r_\lambda * f. \tag{2.35}$$

It remains to apply the operator $\Delta_\lambda = \Delta - \lambda I$ and the definition of the resolvent operator to obtain

$$Mf = \Delta_\lambda(r_\lambda * f). \tag{2.36}$$

The right-hand side can be transformed using the identity

$$\Delta_\lambda(u * v) = (\Delta_\lambda u) * v + \Phi(u)v \tag{2.37}$$

which can be checked directly. Then

$$(Mf)(z) = [(\Delta_\lambda r_\lambda) * f](z) + \Phi(r_\lambda)f(z), \tag{2.38}$$

which is the representation (2.33) with $\mu = \Phi(r_\lambda) = \Phi\{M\tilde{\varphi}_\lambda\}$ and $m(z) = (\Delta_\lambda r_\lambda)(z) = [\Delta_\lambda M\tilde{\varphi}_\lambda](z)$. Thus the necessity is proved. □

In order to prove the next theorem, which is the main result of this paper, the following auxiliary result is needed.

Lemma 2.7. *Let $\lambda \in \mathbb{C}$ be such that $\varphi_\lambda(z) \in \mathcal{H}(G)$. Then, φ_λ is a cyclic element of the operator R_λ in $\mathcal{H}(G)$.*

Proof. Let $f \in \mathcal{H}(G)$ be arbitrarily chosen. It is needed to prove that there is a sequence of functions of the form

$$f_n(z) = \sum_{k=0}^{n} \alpha_{nk} R_\lambda^k \varphi_\lambda(z), \quad n = 1,2,\ldots \tag{2.39}$$

converging to $f(z)$ uniformly on the compact subsets of G.

First, it is easy to show by induction that

$$R_\lambda^k \varphi_\lambda(z) = \varphi_\lambda^{*(k+1)}(z) = p_{k+1}[\varphi_\lambda(z)] = a_{k,k+1}\varphi_\lambda^{k+1}(z) + a_{k,k}\varphi_\lambda^k(z) + \cdots + a_{k,1}\varphi_\lambda(z). \tag{2.40}$$

The calculation for $k = 1$ will be skipped and only the inductive step will be made. Suppose that $R_\lambda^{k-1}\varphi_\lambda$ is a polynomial p_k of $\varphi_\lambda(z)$ of degree $k \geq 2$ with $p_k(0) = 0$, that is,

$$R_\lambda^{k-1}\varphi_\lambda = \varphi_\lambda^{*k}(z) = p_k[\varphi_\lambda(z)] = a_{k-1,k}\varphi_\lambda^k(z) + a_{k-1,k-1}\varphi_\lambda^{k-1}(z) + \cdots + a_{k-1,1}\varphi_\lambda(z). \tag{2.41}$$

Then

$$\begin{aligned}
R_\lambda^k \varphi_\lambda(z) &= \varphi_\lambda^{*(k+1)}(z) = \varphi_\lambda^{*k}(z) * \varphi_\lambda(z) \\
&= \Phi_\zeta\left\{\frac{\{zp_k[\varphi_\lambda(z)] - \zeta p_k[\varphi_\lambda(\zeta)]\}[z\varphi_\lambda(z) - \zeta\varphi_\lambda(\zeta)]}{z-\zeta}\right\} \\
&= \Phi_\zeta\left\{\frac{\{zp_k[\varphi_\lambda(z)] - \zeta p_k[\varphi_\lambda(\zeta)]\}[z/(1-\lambda z) - \zeta/(1-\lambda\zeta)]}{z-\zeta}\right\} \\
&= \Phi_\zeta\left\{\frac{[1/\lambda + (z-1/\lambda)]p_k[\varphi_\lambda(z)] - \zeta p_k[\varphi_\lambda(\zeta)]}{(1-\lambda z)(1-\lambda\zeta)}\right\} \\
&= \frac{1}{\lambda}\Phi_\zeta\{\varphi_\lambda(\zeta)\}\{p_k[\varphi_\lambda(z)]\varphi_\lambda(z)\} - \frac{1}{\lambda}\Phi_\zeta\{\varphi_\lambda(\zeta)\}p_k[\varphi_\lambda(z)] \\
&\quad - \Phi_\zeta\{p_k[\varphi_\lambda(\zeta)]\varphi_\lambda(\zeta)\}\varphi_\lambda(z),
\end{aligned} \tag{2.42}$$

which is a polynomial p_{k+1} of $\varphi_\lambda(z)$ of degree $k+1$ with $p_{k+1}(0) = 0$, as in (2.40).

Now let $f \in \mathcal{H}(G)$ be arbitrarily chosen. Note that

$$w = \varphi_\lambda(z) = \frac{1}{1-\lambda z} \quad \text{iff } z = \varphi_\lambda^{-1}(w) = \frac{w-1}{\lambda w} \tag{2.43}$$

and consider the transformation

$$Tf(z) = f\left(\frac{w-1}{\lambda w}\right) = g(w). \tag{2.44}$$

Then,

$$T(R_\lambda^k \varphi_\lambda(z)) = a_{k,k+1}w^{k+1} + a_{k,k}w^k + a_{k,k-1}w^{k-1} + \cdots + a_{k,1}w. \tag{2.45}$$

Since $w = 0 \notin T(G)$, then by Runge's theorem, there exists a polynomial sequence $\{q_n(w) = \sum_{k=0}^n b_{n,k}w^k\}_{n=1}^\infty$ converging to $(1/w)g(w)$ in $\mathcal{H}(T(G))$. Then the sequence $\{wq_n(w)\}_{n=1}^\infty$ converges to $g(w)$. But

$$wq_n(w) = \sum_{k=0}^n c_{n,k}T(R_\lambda^k \varphi_\lambda(z)) \tag{2.46}$$

with constants $c_{n,0}, c_{n,1}, \ldots, c_{n,n}$. Hence, the sequence $\{r_n(z) = \sum_{k=0}^n c_{n,k}R_\lambda^k \varphi_\lambda(z)\}_{n=0}^\infty$ converges to $f(z)$ in $\mathcal{H}(G)$. Therefore, φ_λ is a cyclic element of R_λ in $\mathcal{H}(G)$. □

Theorem 2.8. *A linear operator $M : \mathcal{H}(G) \to \mathcal{H}(G)$ with an invariant hyperplane $\mathcal{H}_\Phi = \{f \in \mathcal{H}(G) : \Phi\{f\} = 0\}$ commutes with Δ in $\mathcal{H}_\Phi$ if and only if it has a representation of the form*

$$(Mf)(z) = \mu f(z) + (m * f)(z) \tag{2.47}$$

with a constant $\mu \in \mathbb{C}$ and $m \in \mathcal{H}(G)$.

Proof. Since $\Phi\{f * g\} = 0$ for $f, g \in \mathcal{H}(G)$ (see (2.11)), then each operator of the form (2.47) has $\mathcal{H}_\Phi$ as an invariant subspace. It commutes with Δ in $\mathcal{H}_\Phi$. Indeed, if $f \in \mathcal{H}_\Phi$, then (2.37) gives

$$\Delta(m * f) = m * [\Delta(f)] + \Phi\{f\}m, \tag{2.48}$$

and using (2.47),

$$(\Delta M)f = \mu\Delta(f) + m * [\Delta(f)] + \Phi\{f\}m = \mu\Delta(f) + m * [\Delta(f)] = (M\Delta)(f). \tag{2.49}$$

The sufficiency is proved.

In order to prove the necessity of (2.47), according to Theorem 2.4, $MR_\lambda = R_\lambda M$ for $\lambda \in \mathbb{C}$ with $E(\lambda) \neq 0$. As it is shown in the book [4, Theorem 1.3.11, page 33], the commutant of R_λ coincides with the ring of the multipliers of the convolution algebra $(\mathcal{H}(G), *)$ since R_λ has a cyclic element. By Lemma 2.7 such a cyclic element is the function $\varphi_\lambda(z) = 1/(1 - \lambda z)$ for which $R_\lambda f = \tilde{\varphi}_\lambda * f = (1/E(\lambda))[\varphi_\lambda * f]$. □

Remark 2.9. The constant μ and the function $m \in \mathcal{H}(G)$ in (2.47) are uniquely determined. Indeed, assume that $\mu f + m * f = \mu_1 f + m_1 * f$. Take f such that $\Phi(f) \neq 0$. Then, using (2.11), $\mu\Phi(f) = \mu_1\Phi(f)$, and hence $\mu = \mu_1$. From $m * f = m_1 * f$ for arbitrary $f \in \mathcal{H}(G)$, it follows that $(m - m_1) * f = 0$, and hence $m = m_1$.

3. Relation between the two types of commutants

It is natural to ask how the two types of commutants of Δ described above are connected to each other. A partial answer is given by the following theorem.

THEOREM 3.1. *Let M be an arbitrary operator commuting with Δ in $\mathcal{H}(G)$. Then* $\ker M$ *is an ideal in the convolution algebra* $(\mathcal{H}(G), *)$.

Proof. By Theorem 1.8,

$$(Mf)(z) = \Phi_\zeta\left\{\frac{zf(z) - \zeta f(\zeta)}{z - \zeta}\right\}, \tag{3.1}$$

with $\Phi : \mathcal{H}(G) \to \mathbb{C}$ being a linear functional. From the representation

$$\frac{zf(z) - \zeta f(\zeta)}{z - \zeta} = f(z) + \zeta \frac{f(z) - f(\zeta)}{z - \zeta}, \tag{3.2}$$

it follows that

$$\Phi_\zeta\left\{\frac{zf(z) - \zeta f(\zeta)}{z - \zeta}\right\} = 0 \Longleftrightarrow \begin{cases} \Phi_\zeta\left\{\dfrac{f(z) - f(\zeta)}{z - \zeta}\right\} = 0, \\ \Phi_\zeta\{f(\zeta)\} = 0. \end{cases} \tag{3.3}$$

The lower condition in (3.3) is easier to check:

$$\begin{aligned} \Phi_\zeta\{(f * g)(\zeta)\} &= \Phi_\zeta\left\{\Phi_\eta\left\{\frac{[\zeta f(\zeta) - \eta f(\eta)][\zeta g(\zeta) - \eta g(\eta)]}{\zeta - \eta}\right\}\right\} \\ &= \Phi_\eta\left\{\Phi_\zeta\left\{-\frac{[\eta f(\eta) - \zeta f(\zeta)][\eta g(\eta) - \zeta g(\zeta)]}{\eta - \zeta}\right\}\right\} \\ &= -\Phi_\eta\{(f * g)(\eta)\} = -\Phi_\zeta\{(f * g)(\zeta)\}. \end{aligned} \tag{3.4}$$

Here the Fubini property of the functional Φ is used. The function in the braces is antisymmetric with respect to ζ and η, which gives the minus sign in the braces. Thus, $2\Phi_\zeta\{(f * g)(\zeta)\} = 0$, and hence

$$\Phi_\zeta\{(f * g)(\zeta)\} = 0. \tag{3.5}$$

More algebra is needed to check the upper condition in (3.3). Let $f \in \ker M$ and consider

$$\begin{aligned} &\Phi_\zeta\left\{\frac{(f * g)(z) - (f * g)(\zeta)}{z - \zeta}\right\} \\ &\quad = \Phi_\zeta\Phi_\eta\left\{\frac{[zf(z) - \eta f(\eta)][zg(z) - \eta g(\eta)]}{(z - \zeta)(z - \eta)} - \frac{[\zeta f(\zeta) - \eta f(\eta)][\zeta g(\zeta) - \eta g(\eta)]}{(z - \zeta)(\zeta - \eta)}\right\} \\ &\quad = \Phi_\zeta\Phi_\eta\{\varphi_z(\zeta, \eta)\}. \end{aligned} \tag{3.6}$$

Here the function in the braces is denoted by $\varphi_z(\zeta, \eta)$. The proof of $\Phi_\zeta\Phi_\eta\{\varphi_z(\zeta, \eta)\} = 0$ goes easier by splitting $\varphi_z(\zeta, \eta)$ into symmetric and antisymmetric parts as follows:

$$\varphi_z(\zeta, \eta) = \frac{\varphi_z(\zeta, \eta) + \varphi_z(\eta, \zeta)}{2} + \frac{\varphi_z(\zeta, \eta) - \varphi_z(\eta, \zeta)}{2}. \tag{3.7}$$

The antisymmetric part can be treated as in the proof of (3.5) and in fact, one has

$$\Phi_\zeta\Phi_\eta\left\{\frac{\varphi_z(\zeta,\eta)-\varphi_z(\eta,\zeta)}{2}\right\}=0. \tag{3.8}$$

It remains to prove that the symmetric part also satisfies

$$\Phi_\zeta\Phi_\eta\left\{\frac{\varphi_z(\zeta,\eta)+\varphi_z(\eta,\zeta)}{2}\right\}=0. \tag{3.9}$$

After some usual algebraic calculations and suitable grouping, the expression $(\zeta-\eta)$ can be canceled from the numerator and the denominator of $\psi_z(\zeta,\eta)=\varphi_z(\zeta,\eta)+\varphi_z(\eta,\zeta)$ and it can be written as

$$\psi_z(\zeta,\eta)=\frac{[zf(z)-\zeta f(\zeta)][zg(z)-\eta g(\eta)]+[zf(z)-\eta f(\eta)][zg(z)-\zeta g(\zeta)]}{(z-\zeta)(z-\eta)}. \tag{3.10}$$

Now the left-hand side of (3.9) can be represented as

$$\begin{aligned}\Phi_\zeta\Phi_\eta\left\{\frac{\psi_z(\zeta,\eta)}{2}\right\}&=\frac{1}{2}\Phi_\zeta\left\{\frac{zf(z)-\zeta f(\zeta)}{z-\zeta}\right\}\Phi_\eta\left\{\frac{zg(z)-\eta g(\eta)}{z-\eta}\right\}\\&\quad-\frac{1}{2}\Phi_\eta\left\{\frac{zf(z)-\eta f(\eta)}{z-\eta}\right\}\Phi_\zeta\left\{\frac{zg(z)-\zeta g(\zeta)}{z-\zeta}\right\}=0.\end{aligned} \tag{3.11}$$

In (3.11), it was used that

$$\Phi_\zeta\left\{\frac{zf(z)-\zeta f(\zeta)}{z-\zeta}\right\}=\Phi_\eta\left\{\frac{zf(z)-\eta f(\eta)}{z-\eta}\right\}=0, \tag{3.12}$$

which expresses the fact that $f\in\ker M$. Thus (3.9) is also shown. □

Remark 3.2. Theorem 3.1 expresses a new property of $\ker M$. Other properties of $\ker M$ are studied in details by Linchuk [7].

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