ON STRONG COMMUTATIVITY-PRESERVING MAPS

M. S. SAMMAN

Received 4 July 2004 and in revised form 4 December 2004

We identify some strong commutativity-preserving maps on semiprime rings. Among other results, we prove the following. (i) A centralizing homomorphism f of a semiprime ring R onto itself is strong commutativity preserving. (ii) A centralizing antihomomorphism f of a 2-torsion-free semiprime ring R onto itself is strong commutativity preserving.

1. Introduction and preliminaries

Let R be a ring with center $Z(R)$. We write the commutator $[x,y] = xy - yx$, $(x,y \in R)$. The following commutator identities hold: $[xy,z] = x[y,z] + [x,z]y$; $[x,yz] = y[x,z] + [x,y]z$ for all $x,y,z \in R$. We recall that R is *prime* if $aRb = (0)$ implies that $a = 0$ or $b = 0$; it is *semiprime* if $aRa = (0)$ implies that $a = 0$. A prime ring is clearly a semiprime ring. A mapping $f : R \to R$ is called *centralizing* if $[f(x),x] \in Z(R)$ for all $x \in R$; in particular if $[f(x),x] = 0$ for all $x \in R$, then it is called *commuting*. A commuting map is centralizing but the converse is not true, in general. It is easy to see that if $f : R \to R$ is an additive and commuting map, then $[f(x),y] = [x, f(y)]$ for all $x,y \in R$.

A mapping $f : R \to R$ is called *commutativity preserving* if $[f(x), f(y)] = 0$ whenever $[x,y] = 0$. Commutativity-preserving maps have been extensively studied on operator algebras (see [7, 9, 11, 12, 13] and the references therein). Many authors have also worked on commutativity-preserving maps on rings (see [1, 2, 6, 8], where further references are also given).

There has also been considerable interest in strong commutativity-preserving maps. A mapping $f : R \to R$ is called *strong commutativity preserving* if $[f(x), f(y)] = [x,y]$ for all $x,y \in R$. A strong commutativity-preserving map is commutativity preserving but the converse does not hold, in general.

We recall that an additive map f from a ring R into itself is called an *antihomomorphism* if $f(xy) = f(y)f(x)$ for all $x,y \in R$. We will follow Herstein [10] for other undefined notations and terminology used here.

In this paper, we mainly study commutativity-preserving and strong commutativity-preserving properties of homomorphisms and antihomomorphisms of certain rings. We

International Journal of Mathematics and Mathematical Sciences 2005:6 (2005) 917–923
DOI: 10.1155/IJMMS.2005.917

show (Proposition 2.1) that an epimorphism of a semiprime ring is strong commutativity preserving if and only if it is centralizing. Furthermore, we prove that if R is a 2-torsion-free semiprime ring and f is a centralizing antihomomorphism of R onto itself, then f is in fact strong commutativity preserving (Proposition 2.4). These and some other related results are proved in Section 2.

2. The results

Proposition 2.1. *Let R be a semiprime ring and f an epimorphism of R. Then f is centralizing if and only if it is strong commutativity preserving.*

Proof. Assume that f is centralizing. Then, by [3, Lemma 2], f is commuting and hence $[f(x),y]=[x,f(y)]$ for all $x,y\in R$. So,

$$[f(xy),x]=[xy,f(x)]=x[y,f(x)]+[x,f(x)]y=x[y,f(x)]=x[f(y),x]. \tag{2.1}$$

That is,

$$[f(xy),x]=x[f(y),x]\quad \forall x,y\in R. \tag{2.2}$$

Also, $[f(xy),x]=[f(x)f(y),x]=f(x)[f(y),x]+[f(x),x]f(y)=f(x)[f(y),x]$. That is,

$$[f(xy),x]=f(x)[f(y),x]\quad \forall x,y\in R. \tag{2.3}$$

By (2.2) and (2.3), we get $f(x)[f(y),x]=x[f(y),x]$. Since f is onto, therefore we have $f(x)[y,x]=x[y,x]$ for all $x,y\in R$. That is,

$$(f(x)-x)[y,x]=0\quad \forall x,y\in R. \tag{2.4}$$

Replacing y by uy in (2.4) and using (2.4) again, we get

$$0=(f(x)-x)[uy,x]=(f(x)-x)u[y,x]+(f(x)-x)[u,x]y=(f(x)-x)u[y,x]. \tag{2.5}$$

So,

$$(f(x)-x)u[y,x]=0\quad \forall x,y,u\in R. \tag{2.6}$$

Replacing x by $x+z$ in (2.4), we get

$$\begin{aligned}0&=(f(x)-x)[y,x]+(f(x)-x)[y,z]+(f(z)-z)[y,x]+(f(z)-z)[y,z]\\&=(f(x)-x)[y,z]+(f(z)-z)[y,x].\end{aligned} \tag{2.7}$$

So,

$$(f(x)-x)[y,z]=-(f(z)-z)[y,x]\quad \forall x,y,z\in R. \tag{2.8}$$

Equation (2.8) implies that for all $x,y,z,v\in R$, we have

$$(f(x)-x)[y,z]v(f(x)-x)[y,z]=-(f(x)-x)[y,z]v(f(z)-z)[y,x]. \tag{2.9}$$

Putting $u = [y,z]v(f(z) - z)$ in (2.6) and using (2.9), we get

$$(f(x) - x)[y,z]v(f(x) - x)[y,z] = 0 \quad \forall v \in R. \tag{2.10}$$

R being semiprime implies that

$$(f(x) - x)[y,z] = 0 \quad \forall x,y,z \in R. \tag{2.11}$$

Replacing y by wy in (2.11), we get

$$0 = (f(x) - x)[wy,z] = (f(x) - x)w[y,z] + (f(x) - x)[w,z]y = (f(x) - x)w[y,z]. \tag{2.12}$$

Thus,

$$(f(x) - x)w[y,z] = 0 \quad \forall x,y,z,w \in R. \tag{2.13}$$

Multiplying (2.13) on the left by $[y,z]$ and on the right by $(f(x) - x)$, we get $[y,z](f(x) - x)w[y,z](f(x) - x) = 0$ for all $w \in R$. By the semiprimeness of R, we get $[y,z](f(x) - x) = 0$ and hence by (2.11), we have $(f(x) - x)[y,z] = [y,z](f(x) - x) = 0$ for all $x,y,z \in R$. So, by Herstein [10, Lemma 1.1.8], $(f(x) - x) \in Z(R)$. Therefore, $[f(x) - x, y] = 0$ for all $x,y \in R$. That is,

$$[f(x), y] = [x, y] \quad \forall x,y \in R. \tag{2.14}$$

Replacing y by $f(y)$ in (2.14), and using (2.14) again, we get $[f(x), f(y)] = [x, f(y)] = [x,y]$ for all $x,y \in R$. This proves that f is strong commutativity preserving.

Conversely, assume that f is strong commutativity preserving. Then,

$$[f(x), f(y)] - [x,y] = 0 \quad \forall x,y \in R. \tag{2.15}$$

Replacing y by xy in (2.15) and using the strong commutativity-preserving property of f, we get

$$\begin{aligned} 0 = [f(x), f(xy)] - [x,xy] &= [f(x), f(x)f(y)] - [x,xy] \\ &= f(x)[f(x), f(y)] + [f(x), f(x)]f(y) - x[x,y] - [x,x]y \\ &= f(x)[x,y] - x[x,y] = (f(x) - x)[x,y]. \end{aligned} \tag{2.16}$$

So,

$$(f(x) - x)[x,y] = 0 \quad \forall x,y \in R. \tag{2.17}$$

Replacing y by zy in (2.17) and using (2.17) again, we get

$$0 = (f(x) - x)[x,zy] = (f(x) - x)z[x,y] + (f(x) - x)[x,z]y = (f(x) - x)z[x,y]. \tag{2.18}$$

That is,

$$(f(x)-x)z[x,y]=0 \quad \forall x,y,z\in R. \tag{2.19}$$

Replacing y by $f(x)$ in (2.19), we get

$$(f(x)-x)z[f(x),x]=0 \quad \forall x\in R. \tag{2.20}$$

Replacing z by xz in (2.20), we get

$$(f(x)x-x^2)z[f(x),x]=0 \quad \forall x\in R. \tag{2.21}$$

Multiplying (2.20) on the left by x, we get

$$(xf(x)-x^2)z[f(x),x]=0 \quad \forall x\in R. \tag{2.22}$$

Subtracting (2.22) from (2.21), we get $[f(x),x]z[f(x),x]=0$ for all $x,z\in R$. Since R is semiprime, therefore, $[f(x),x]=0$ for all $x\in R$. So, f is commuting and hence centralizing. □

Remark 2.2. In Proposition 2.1, the implication that f is strong commutativity preserving implying that it is centralizing also follows from Brešar and Miers [7, Theorem 1]; however, the proof in the case of homomorphisms is simple and we have included it here for the sake of completeness. Furthermore, it may be of independent interest.

Remark 2.3. Let R be a ring and $f: R\to R$ an antihomomorphism. Then clearly, f is commutativity preserving.

The following proposition shows that under some additional assumptions, an antihomomorphism must be strong commutativity preserving.

Proposition 2.4. *Let R be a 2-torsion-free semiprime ring and f a centralizing antihomomorphism of R onto itself. Then f is strong commutativity preserving.*

Proof. By [5, Proposition 3.1], f is commuting and hence, $[f(x),y]=[x,f(y)]$ for all $x,y\in R$. So, $[f(xy),x]=[xy,f(x)]=x[y,f(x)]+[x,f(x)]y=x[y,f(x)]$. That is,

$$[f(xy),x]=x[y,f(x)] \quad \forall x,y\in R. \tag{2.23}$$

Also, $[f(xy),x]=[f(y)f(x),x]=f(y)[f(x),x]+[f(y),x]f(x)=[f(y),x]f(x)$. That is,

$$[f(xy),x]=[f(y),x]f(x) \quad \forall x,y\in R. \tag{2.24}$$

From (2.23) and (2.24), we get $[f(y),x]f(x)=x[y,f(x)]$; that is, $[f(y),x]f(x)=x[f(y),x]$ for all $x,y\in R$. Now f being onto implies that $[y,x]f(x)=x[y,x]$. So,

$$[y,x]f(x)=x[y,x] \quad \forall x,y\in R. \tag{2.25}$$

Replacing y by uy in (2.25), we get $[uy,x]f(x)=x[uy,x]$. That is,

$$u[y,x]f(x)+[u,x]yf(x)=xu[y,x]+x[u,x]y \quad \forall x,y\in R. \tag{2.26}$$

By (2.25) and (2.26), we get $ux[y,x]+[u,x]yf(x)=xu[y,x]+x[u,x]y$. That is, $ux[y,x]+[u,x]yf(x)=xu[y,x]+[u,x]f(x)y$. This implies that

$$ux[y,x]-xu[y,x]+[u,x]yf(x)-[u,x]f(x)y=0. \tag{2.27}$$

That is,

$$[u,x][y,x]+[u,x][y,f(x)]=0. \tag{2.28}$$

Using the fact that f is commuting, we get

$$0=[u,x][y,x]+[u,x][y,f(x)]=[u,x]([y,x]+[f(y),x])=[u,x][y+f(y),x]. \tag{2.29}$$

So,

$$[u,x][y+f(y),x]=0 \quad \forall x,y,u\in R. \tag{2.30}$$

Replacing u by uz in (2.30) and using (2.30) again, we get

$$\begin{aligned}0=[uz,x][y+f(y),x]&=[u,x]z[y+f(y),x]+u[z,x][y+f(y),x]\\&=[u,x]z[y+f(y),x].\end{aligned} \tag{2.31}$$

That is,

$$[u,x]z[y+f(y),x]=0 \quad \forall x,y,u,z\in R. \tag{2.32}$$

Replacing u by $y+f(y)$ in (2.32), we get $[y+f(y),x]z[y+f(y),x]=0$ for all $x,y,z\in R$. Since R is semiprime, we get

$$[y+f(y),x]=0 \quad \forall x,y\in R. \tag{2.33}$$

Rewriting (2.33), we get $0=[y,x]+[f(y),x]=[y,x]+[y,f(x)]=[y,x]-[f(x),y]$. So,

$$[f(x),y]=[y,x] \quad \forall x,y\in R. \tag{2.34}$$

That f is strong commutativity preserving follows from (2.34). Indeed, $[f(x),f(y)]=[f(y),x]=[x,y]$ for all $x,y\in R$. □

Remark 2.5. Brešar [4, Proposition 4.1] has proved the following result.

THEOREM 2.6. *Let R be a 2-torsion-free semiprime ring and let $f:R\to R$ be a centralizing antihomomorphism. Then,*
(a) $S=\{x\in R: f(x)=x\}\subseteq Z(R)$,
(b) *if R is prime and f does not map R into $Z(R)$, then $S=Z(R)$.*

We note that Theorem 2.6 can also be obtained as an application of Proposition 2.4 if f is onto. Thus our proof (below) can be regarded as an alternate argument for Theorem 2.6 which may also be of independent interest.

Proof. (a) By (2.33), $f(y)+y \in Z(R)$ for all $y \in R$. Therefore, for z in S, $f(z)+z = 2z \in Z(R)$. So, $[2z,x] = 2[z,x] = 0$ for all $x \in R$. As R is 2-torsion-free, so $[z,x] = 0$ for all $x \in R$. Therefore, $z \in Z(R)$ and hence $S \subseteq Z(R)$.

(b) Assume that R is prime and let $z \in Z(R)$. If $z = 0$, then $f(0) = 0$ implies that $0 \in S$. So, assume that $z \neq 0$. Then $f(z)+z \in Z(R)$, $z \in Z(R)$. So, $f(z) \in Z(R)$. Now replacing x by zx in (2.25), we get $[y,zx]f(zx) = (zx)[y,zx]$. That is,

$$z[y,x]f(x)f(z)+[y,z]xf(x)f(z) = zxz[y,x]+zx[y,z]x. \tag{2.35}$$

As $z \in Z(R)$, by (2.35), we get $z[y,x]f(x)f(z) = zxz[y,x]$. That is,

$$[y,x]f(x)f(z)z = x[y,x]z^2 \quad \forall x,y \in R,\ z \in Z(R). \tag{2.36}$$

By (2.25) and (2.36), we get $[y,x]f(x)f(z)z = [y,x]f(x)z^2$. That is,

$$[y,x]f(x)\big(f(z)z-z^2\big) = 0 \quad \forall x,y \in R,\ z \in Z(R). \tag{2.37}$$

Since R is prime, then any nonzero central element is not a zero divisor. Hence, if $f(z)z - z^2 \neq 0$, then $[y,x]f(x) = 0$ for all $x,y \in R$. Then by [10, corollary, page 8], either $f(x) = 0$ or $x \in Z(R)$. In any case, $f(x) \in Z(R)$ for all $x \in R$, a contradiction. So, $0 = f(z)z - z^2 = (f(z)-z)z$. As $z \neq 0$, therefore by the above argument, $f(z)-z = 0$ and hence $z \in S$. So, $Z(R) \subseteq S$ and by (a), we have $Z(R) = S$. □

Acknowledgments

The author is grateful to King Fahd University of Petroleum and Minerals for supporting this research. The author is also thankful to the referees for their valuable comments which improved the paper.

References

[1] R. Banning and M. Mathieu, *Commutativity preserving mappings on semiprime rings*, Comm. Algebra **25** (1997), no. 1, 247–265.

[2] H. E. Bell and M. N. Daif, *On commutativity and strong commutativity-preserving maps*, Canad. Math. Bull. **37** (1994), no. 4, 443–447.

[3] H. E. Bell and W. S. Martindale III, *Centralizing mappings of semiprime rings*, Canad. Math. Bull. **30** (1987), no. 1, 92–101.

[4] M. Brešar, *Centralizing mappings on von Neumann algebras*, Proc. Amer. Math. Soc. **111** (1991), no. 2, 501–510.

[5] ———, *Centralizing mappings and derivations in prime rings*, J. Algebra **156** (1993), no. 2, 385–394.

[6] ———, *Commuting traces of biadditive mappings, commutativity-preserving mappings and Lie mappings*, Trans. Amer. Math. Soc. **335** (1993), no. 2, 525–546.

[7] M. Brešar and C. R. Miers, *Commutativity preserving mappings of von Neumann algebras*, Canad. J. Math. **45** (1993), no. 4, 695–708.

[8] ———, *Strong commutativity preserving maps of semiprime rings*, Canad. Math. Bull. **37** (1994), no. 4, 457–460.

[9] M. D. Choi, A. A. Jafarian, and H. Radjavi, *Linear maps preserving commutativity*, Linear Algebra Appl. **87** (1987), 227–241.

[10] I. N. Herstein, *Rings with Involution*, The University of Chicago Press, Illinois, 1976.
[11] C. R. Miers, *Commutativity preserving maps of factors*, Canad. J. Math. **40** (1988), no. 1, 248–256.
[12] M. Omladič, *On operators preserving commutativity*, J. Funct. Anal. **66** (1986), no. 1, 105–122.
[13] W. Watkins, *Linear maps that preserve commuting pairs of matrices*, Linear Algebra and Appl. **14** (1976), no. 1, 29–35.

M. S. Samman: Department of Mathematical Sciences, College of Scinces, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
E-mail address: msamman@kfupm.edu.sa