EXISTENCE OF SOLUTION FOR A SINGULAR ELLIPTIC EQUATION WITH CRITICAL SOBOLEV-HARDY EXPONENTS

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Received 9 June 2005 and in revised form 21 September 2005

Via the variational methods, we prove the existence of a nontrivial solution to a singular semilinear elliptic equation with critical Sobolev-Hardy exponent under certain conditions.

1. Introduction

In this paper, we consider the following elliptic problem:

$$-\Delta u-\mu\frac{u}{|x|^2}=\frac{|u|^{2^*(s)-2}}{|x|^s}u+a(x)|u|^{r-2}u+\lambda u,\quad x\in\mathbb{R}^N, \tag{1.1}$$

where $N\geq 3$, $0\leq\mu<\bar{\mu}\doteq((N-2)/2)^2$, $0\leq s<2$, $\lambda\geq 0$, and $2^*(s)\doteq 2(N-s)/(N-2)$ is the critical Sobolev-Hardy exponent; note that $2^*(0)=2^*\doteq 2N/(N-2)$ is the critical Sobolev exponent. The space $H\doteq H(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ in the norm

$$\|u\|\doteq\left(\int_{\mathbb{R}^N}\left(|\nabla u|^2-\mu\frac{u^2}{|x|^2}\right)dx\right)^{1/2}. \tag{1.2}$$

By the Hardy inequality [8, 9], this norm is equivalent to the usual norm $(\int_{\mathbb{R}^N}|\nabla u|^2dx)^{1/2}$. The scalar product in H is

$$(u,v)\doteq\int_{\mathbb{R}^N}\left(\nabla u\nabla v-\mu\frac{uv}{|x|^2}\right)dx\quad\forall u,v\in H. \tag{1.3}$$

We define $H_r\subset H$ with

$$H_r\doteq\{u\in H,\ u(x)=u(|x|)\}. \tag{1.4}$$

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International Journal of Mathematics and Mathematical Sciences 2005:20 (2005) 3213–3223
DOI: 10.1155/IJMMS.2005.3213

The hypothesis for $a(x)$ is as follows:

(A) $a(x)$ is nonnegative and locally bounded in $\mathbb{R}^N\setminus\{0\}$, $a(x) = O(|x|^{-s})$ in the bounded neighborhood G of the origin, $a(x) = O(|x|^{-t})$ as $|x| \to \infty$, $0 \le s < t < 2$, $2^*(t) < r < 2^*(s)$, where $2^*(t) \doteq 2(N-t)/(N-2)$ for $0 \le t < 2$.

The singular elliptic problems have received some attention in recent years. For example, Janneli [10] and Ferrero and Gazzolo [7] studied the semilinear elliptic equation

$$-\Delta u - \mu \frac{u}{|x|^2} = |u|^{2^*-2}u + \lambda u, \quad x \in \Omega, \qquad u(x) = 0, \quad x \in \partial\Omega, \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N (N \ge 3)$ is a smooth bounded domain containing the origin 0. They proved that (1.5) has a nontrivial solution under certain conditions for λ and μ. Moreover, Cao in [4, 5] and Chen in [6] also studied the semilinear elliptic equation (1.5). They show that (1.5) has nontrivial solutions and a sign-changing solution under some conditions for μ, λ. Ghoussoub and Yuan in [9] considered the quasilinear problem

$$-\Delta_p u = \mu \frac{|u|^{q-2}u}{|x|^s} + \lambda |u|^{r-2}u, \quad x \in \Omega, \qquad u(x) = 0, \quad x \in \partial\Omega. \tag{1.6}$$

They get that (1.6) has a positive solution and a sign-changing solution under some conditions for λ, μ, r, q.

In the case when Ω is an unbounded domain in $\mathbb{R}^N$, the corresponding problem becomes more complicated since the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)(p \ge 2)$ is not compact for all $q \in [p, p^*]$. However, by the Strauss lemma (see [13]), the embedding $H_r(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is compact for all $q \in [2, 2^*)$. Therefore, we can discuss the nontrivial solutions of (1.1) in H_r by variational methods. But there are also some difficulties for (1.1), because the embedding $H_r \hookrightarrow L^{2^*(s)}(\mathbb{R}^N, |x|^{-s})$ is still not compact. In [11], as $\lambda = 0$, the existence of a nontrivial solution is given for (1.1) with $s = 0$, so it will be meaningful to study the existence of nontrivial solutions for (1.1) as $s \in [0, 2)$ and $\lambda \ne 0$. In this paper, we obtain the following existence results.

Theorem 1.1. *Suppose (A) and* $0 \le s < 2$, $0 \le \mu < \bar{\mu}$, $\lambda \ge 0$. *Assume that one of the following conditions holds:*

(i) $\lambda = 0$ *and*

$$\max\left\{\frac{N-s}{\sqrt{\bar{\mu}} + \sqrt{\bar{\mu}-\mu}}, \frac{N-s-2\sqrt{\bar{\mu}-\mu}}{\sqrt{\bar{\mu}}}, 2^*(t)\right\} < r < 2^*(s), \tag{1.7}$$

(ii) $0 < \lambda < \lambda_1(\mu)$ *and* $0 \le \mu \le \bar{\mu} - 1$, *where* $\lambda_1(\mu) \doteq \inf_{u \in H\setminus\{0\}}(\|u\|^2/\int_{\mathbb{R}^N} u^2 dx)$.

Then problem (1.1) has at least a nontrivial solution in H_r.

Throughout this paper, we will use the letter C to denote the natural various constants independent of u, and $\int \cdot dx$ instead of $\int_{\mathbb{R}^N} \cdot dx$.

2. Proof of the main result

We first give some definitions and lemmas.

Definition 2.1. Let $\{u_m\}$ be a sequence in H_r, if there exists a constant $c \in \mathbb{R}^1$ such that

$$J(u_m) \longrightarrow c, \qquad J'(u_m) \longrightarrow 0 \quad \text{in } H_r^{-1} \tag{2.1}$$

as $m \to \infty$, then $\{u_m\}$ is called a $(\mathrm{PS})_c$ sequence in H_r.

Lemma 2.2 (Hardy inequality [8, 9]). *Assume that $1 < p < N$ and $u \in W^{1,p}(\mathbb{R}^N)$. Then*

$$\int \frac{|u|^p}{|x|^p}dx \le \left(\frac{p}{N-p}\right)^p \int |\nabla u|^p dx. \tag{2.2}$$

Lemma 2.3 (Sobolev-Hardy inequality [9]). *Assume that $1 < p < N$ and that $p^*(s) \doteq ((N-s)/(N-p))p$, $0 \le s \le p$. Then there exists a constant $C > 0$ such that for any $u \in W^{1,p}(\mathbb{R}^N)$,*

$$\left(\int \frac{|u|^{p^*(s)}}{|x|^s}dx\right)^{p/p^*(s)} \le C\int |\nabla u|^p dx. \tag{2.3}$$

Lemma 2.4 [11]. *Assume that hypothesis (A) holds. Then the embedding $H \hookrightarrow L^r(\mathbb{R}^N, a(x))$ is compact.*

Consider the energy functional

$$J(u) = \frac{1}{2}\|u\| - \frac{1}{2^*(s)}\int \frac{|u|^{2^*(s)}}{|x|^s}dx - \frac{1}{r}\int a(x)|u|^r dx - \frac{\lambda}{2}\int |u|^2 dx, \tag{2.4}$$

by Lemma 2.4, $J(u)$ is well defined and $J \in C^1(H,\mathbb{R})$; the critical points of the functional J correspond to weak solutions of problem (1.1).

For $0 \le \mu < \bar{\mu}$, define the best Sobolev-Hardy constant:

$$A_s \doteq A_s(\mu) = \inf_{u \in H\{0\}} \frac{\int (|\nabla u|^2 - \mu u^2/|x|^2)dx}{\left(\int |u|^{2^*(s)}/|x|^s dx\right)^{2/2^*(s)}}. \tag{2.5}$$

In [12], *the author found that A_s is attained by the functions*

$$y_\varepsilon(x) = \frac{\left(2\varepsilon(\bar{\mu}-\mu)(N-s)/\sqrt{\bar{\mu}}\right)^{\sqrt{\bar{\mu}}/(2-s)}}{|x|^{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}}\left(\varepsilon + |x|^{(2-s)\sqrt{\bar{\mu}-\mu}/\sqrt{\bar{\mu}}}\right)^{(N-2)/(2-s)}} \tag{2.6}$$

for all $\varepsilon > 0$. Moreover, the functions $y_\varepsilon(x)$ solve the equation

$$-\Delta u - \mu\frac{u}{|x|^2} = \frac{|u|^{2^*(s)-2}}{|x|^s}u \quad \text{in } \mathbb{R}^N \setminus \{0\}, \tag{2.7}$$

and satisfy

$$\int\left(|\nabla y_\varepsilon|^2-\mu\frac{|y_\varepsilon|^2}{|x|^2}\right)dx=\int\frac{|y_\varepsilon|^{2^*(s)}}{|x|^s}dx=A_s^{(N-s)/(2-s)}. \tag{2.8}$$

In the following, we first give some estimates for the extremal functions.
Let

$$C_\varepsilon=\left(\frac{2\varepsilon(\bar{\mu}-\mu)(N-s)}{\sqrt{\bar{\mu}}}\right)^{\sqrt{\bar{\mu}}/(2-s)}, \qquad U_\varepsilon(x)=\frac{y_\varepsilon(x)}{C_\varepsilon}, \tag{2.9}$$

$B_{2l}=\{x\in\mathbb{R}^N,\ |x|<2l\}\subset G$ with $l>0$ and G is the domain in hypothesis (A), let $0\le\phi\le 1$ be a cutting-off function in $C_0^\infty(\mathbb{R}^N)\bigcap H_r$, such that $\phi(x)=1$ in B_l and $\phi(x)=0$ in $\mathbb{R}^N\setminus B_{2l}$. Set $u_\varepsilon(x)=\phi(x)y_\varepsilon(x)$ and $v_\varepsilon=u_\varepsilon(x)/(\int|u_\varepsilon|^{2^*(s)}/|x|^s)^{1/2^*(s)}$, so that $\int(|v_\varepsilon|^{2^*(s)}/|x|^s)=1$. In [12], the author proved that the following estimates are true:

$$\|v_\varepsilon\|^2=A_s+O\left(\varepsilon^{(N-2)/(2-s)}\right), \tag{2.10}$$

$$\int|v_\varepsilon|^q dx=\begin{cases} O\left(\varepsilon^{\sqrt{\bar{\mu}}q/(2-s)}\right), & 1\le q<\dfrac{N}{\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}},\\ O\left(\varepsilon^{\sqrt{\bar{\mu}}q/(2-s)}|\ln\varepsilon|\right), & q=\dfrac{N}{\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}},\\ O\left(\varepsilon^{\sqrt{\bar{\mu}}(N-q\sqrt{\bar{\mu}})/((2-s)\sqrt{\bar{\mu}-\mu})}\right), & \dfrac{N}{\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}}<q<2^*. \end{cases} \tag{2.11}$$

Moreover, we also need the following results.

Lemma 2.5. *Suppose that $\gamma=\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}$, $\acute{\gamma}=\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}$, $0\le\mu<\bar{\mu}$, and $0\le s<2$, then, $v_\varepsilon(x)$ satisfies the following estimates:*

$$\int\frac{|v_\varepsilon|^q}{|x|^s}dx\ge\begin{cases} c_1\varepsilon^{\sqrt{\bar{\mu}}q/(2-s)}, & 1\le q<\dfrac{N-s}{\gamma},\\ c_2\varepsilon^{\sqrt{\bar{\mu}}q/(2-s)}|\ln\varepsilon|, & q=\dfrac{N-s}{\gamma},\\ c_3\varepsilon^{(\sqrt{\bar{\mu}}(N-s)-\bar{\mu}q)/(2-s)\sqrt{\bar{\mu}-\mu}}, & \dfrac{N-s}{\gamma}<q<2^*(s), \end{cases} \tag{2.12}$$

where c_i $(i=1,2,3)$ are positive constants.

Proof. Let ω_N denote the surface area of the $(N-1)$ sphere S^{N-1} in $\mathbb{R}^N$. For $1 \le q < 2^*(s)$, we have

$$\begin{aligned}\int \frac{|v_\varepsilon|^q}{|x|^s}dx &= \int \frac{|u_\varepsilon(x)|^q}{|x|^s}dx \cdot \left(\int \frac{|u_\varepsilon|^{2^*(s)}}{|x|^s}dx\right)^{-q/2^*(s)} = B\int \frac{|\phi(x)C_\varepsilon U_\varepsilon|^q}{|x|^s}dx \\ &= BC_\varepsilon^q\left(O(1)+\omega_N\int_0^l\left(\varepsilon+r^{(2-s)\sqrt{\bar\mu-\mu}/\sqrt{\bar\mu}}\right)^{-q(N-2)/(2-s)}r^{N-s-1-q\acute{\gamma}}dr\right) \\ &= BC_\varepsilon^q\Bigg(O(1)+\omega_N\varepsilon^{-q((N-2)/(2-s))+(\sqrt{\bar\mu}(N-s-\acute{\gamma}q)/(2-s)\sqrt{\bar\mu-\mu})} \\ &\qquad\times\int_0^{l\varepsilon^{\sqrt{\bar\mu}/((s-2)\sqrt{\bar\mu-\mu})}}\left(1+r^{(2-s)\sqrt{\bar\mu-\mu}/\sqrt{\bar\mu}}\right)^{-q(N-2)/(2-s)}r^{N-s-1-q\acute{\gamma}}dr\Bigg),\end{aligned} \tag{2.13}$$

where $B = (\int |u_\varepsilon|^{2^*(s)}/|x|^s dx)^{-q/2^*(s)}$.

If $-2q\sqrt{\bar\mu-\mu}+N-s-\acute{\gamma}q = 0$, that is, $q = (N-s)/\gamma$,

$$\int \frac{|v_\varepsilon|^q}{|x|^s}dx = BC_\varepsilon^q\left(O(1)+\omega_N\int_1^{l\varepsilon^{\sqrt{\bar\mu}/((s-2)\sqrt{\bar\mu-\mu})}}\frac{1}{r}dr\right) \ge B\acute{c}_1\varepsilon^{\sqrt{\bar\mu}q/(2-s)}|\ln\varepsilon|, \tag{2.14}$$

where $\acute{c}_1 > 0$ is a constant.

If $-2q\sqrt{\bar\mu-\mu}+N-s-\acute{\gamma}q < 0$, that is, $q > (N-s)/\gamma$,

$$\begin{aligned}\int \frac{|v_\varepsilon|^q}{|x|^s}dx &= BC_\varepsilon^q\left(O(1)+O\left(\varepsilon^{-q((N-2)/(2-s))+(\sqrt{\bar\mu}(N-s-\acute{\gamma}q)/(2-s)\sqrt{\bar\mu-\mu})}\right)\right) \\ &\ge B\acute{c}_2\varepsilon^{(\sqrt{\bar\mu}(N-s)-\bar\mu q)/(2-s)\sqrt{\bar\mu-\mu}},\end{aligned} \tag{2.15}$$

where $c_2' > 0$ is a constant.

If $-2q\sqrt{\bar\mu-\mu}+N-s-\acute{\gamma}q > 0$, that is, $q < (N-s)/\gamma$,

$$\begin{aligned}\int \frac{|v_\varepsilon|^q}{|x|^s}dx &= BC_\varepsilon^q\left(O(1)+\omega_N\int_0^l\left(\varepsilon+r^{(2-s)\sqrt{\bar\mu-\mu}/\sqrt{\bar\mu}}\right)^{-q(N-2)/(2-s)}r^{N-s-1-q\acute{\gamma}}dx\right) \\ &= BC_\varepsilon^q\cdot O(1) \ge B\acute{c}_3\varepsilon^{\sqrt{\bar\mu}q/(2-s)},\end{aligned} \tag{2.16}$$

where $\acute{c}_3 > 0$ is a constant.

By

$$\begin{aligned}B = \left(\int \frac{|u_\varepsilon|^{2^*(s)}}{|x|^s}dx\right)^{-q/2^*(s)} &= \left(\int \frac{|\phi(x)y_\varepsilon|^{2^*(s)}}{|x|^s}dx\right)^{-q/2^*(s)} \\ &\ge \left(\int \frac{|y_\varepsilon|^{2^*(s)}}{|x|^s}dx\right)^{-q/2^*(s)} = A_s^{(2-N)q/2(2-s)},\end{aligned} \tag{2.17}$$

we have finished the proof of Lemma 2.5. □

Lemma 2.6. *Suppose (A) and* $0 \leq s < 2$, $0 \leq \mu < \bar{\mu}$, $\lambda \geq 0$. *Assume that one of the following conditions holds:*

(i) $\lambda = 0$ *and*

$$\max\left\{\frac{N-s}{\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}}, \frac{N-s-2\sqrt{\bar{\mu}-\mu}}{\sqrt{\bar{\mu}}}, 2^*(t)\right\} < r < 2^*(s), \tag{2.18}$$

(ii) $0 < \lambda < \lambda_1(\mu)$ *and* $0 \leq \mu \leq \bar{\mu} - 1$.

Then, there exists $u_0 \in H_r$, $u_0 \neq 0$, *such that the following inequality holds:*

$$0 < \sup_{t\geq 0} J(tu_0) < \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)}. \tag{2.19}$$

Proof. For $t \geq 0$, we consider the functions

$$g(t) \doteq J(tv_\varepsilon) = \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*(s)}}{2^*(s)} - \frac{t^r}{r}\int a(x)|v_\varepsilon|^r dx - \frac{\lambda t^2}{2}\int |v_\varepsilon|^2 dx,$$
$$\bar{g}(t) = \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*(s)}}{2^*(s)}. \tag{2.20}$$

Note that $\lim_{t\to\infty} g(t) = -\infty$, $g(0) = 0$, and $g(t) > 0$ as $t \to 0^+$, therefore, $\sup_{t\geq 0} g(t) > 0$ must be attained by some $0 < t_\varepsilon < +\infty$ and $g'(t_\varepsilon) = 0$. So we have

$$g'(t_\varepsilon) = t_\varepsilon\|v_\varepsilon\|^2 - t_\varepsilon^{2^*(s)-1} - t_\varepsilon^{r-1}\int a(x)|v_\varepsilon|^r dx - \lambda t_\varepsilon\int |v_\varepsilon|^2 dx = 0. \tag{2.21}$$

Then

$$\|v_\varepsilon\|^2 = t_\varepsilon^{2^*(s)-2} + t_\varepsilon^{r-2}\int a(x)|v_\varepsilon|^r dx + \lambda\int |v_\varepsilon|^2 dx \geq t_\varepsilon^{2^*(s)-2}, \quad t_\varepsilon \leq \|v_\varepsilon\|^{2/(2^*(s)-2)}. \tag{2.22}$$

Moreover, by hypothesis (A), we have

$$\|v_\varepsilon\|^2 \leq t_\varepsilon^{2^*(s)-2} + C\|v_\varepsilon\|^{2(r-2)/(2^*(s)-2)}\int_{B_{2l}} \frac{|v_\varepsilon|^r}{|x|^s} + \lambda\int |v_\varepsilon|^2 dx. \tag{2.23}$$

From (2.23) and (2.10)–(2.12), as ε small enough, we get

$$t_\varepsilon^{2^*(s)-2} \geq \frac{A_s}{2}. \tag{2.24}$$

By the simple computation, we know that the function $\bar{g}(t)$ attains its maximum at $t_0 = \|v_\varepsilon\|^{2/(2^*(s)-2)}$ and is increasing in the interval $[0, t_0]$. So, by (2.10), (2.22), and (2.24), we have

$$g(t_\varepsilon) \le \bar{g}(t_0) - \frac{1}{r}\left(\frac{A_s}{2}\right)^{r/(2^*(s)-2)} \int \frac{|v_\varepsilon|^r}{|x|^s}dx - \frac{\lambda}{2}\left(\frac{A_s}{2}\right)^{2/(2^*(s)-2)} \int |v_\varepsilon|^2 dx$$
$$\le \frac{2-s}{2(N-s)}\|v_\varepsilon\|^{2(N-s)/(2-s)} - C\int \frac{|v_\varepsilon|^r}{|x|^s} - C\int |v_\varepsilon|^2 dx \tag{2.25}$$
$$= \frac{2-s}{2(N-s)}A_s^{(N-s)/(2-s)} + O\left(\varepsilon^{(N-2)/(2-s)}\right) - C\int \frac{|v_\varepsilon|^r}{|x|^s} - C\int |v_\varepsilon|^2 dx.$$

In case (i), since

$$r > \max\left\{\frac{N-s}{\gamma}, \frac{N-s-2\sqrt{\bar{\mu}-\mu}}{\sqrt{\bar{\mu}}}, 2^*(t)\right\}, \tag{2.26}$$

by (2.12), we have

$$\int \frac{|v_\varepsilon|^r}{|x|^s} \ge c_3\varepsilon^{\sqrt{\bar{\mu}}(N-s-\sqrt{\bar{\mu}}r)/(2-s)\sqrt{\bar{\mu}-\mu}},$$
$$\frac{\sqrt{\bar{\mu}}(N-s-\sqrt{\bar{\mu}}r)}{(2-s)\sqrt{\bar{\mu}-\mu}} < \frac{N-2}{2-s}. \tag{2.27}$$

Let $u_0 = v_\varepsilon$, choosing ε small enough, from (2.25), we can deduce that

$$\sup_{t\ge 0} J(tu_0) = g(t_\varepsilon) < \frac{2-s}{2(N-s)}A_s^{(N-s)/(2-s)}. \tag{2.28}$$

In case (ii), $0 < \lambda < \lambda_1(\mu)$. By (2.11), as $\mu = \bar{\mu} - 1$,

$$\int |v_\varepsilon|^2 = O\left(\varepsilon^{(N-2)/(2-s)}|\ln\varepsilon|\right), \tag{2.29}$$

as $0 \le \mu < \bar{\mu} - 1$,

$$\int |v_\varepsilon|^2 = O\left(\varepsilon^{(N-2)/((2-s)\sqrt{\bar{\mu}-\mu})}\right). \tag{2.30}$$

Choosing ε small enough, we also get (2.28). The proof of Lemma 2.6 is completed. □

LEMMA 2.7. *Suppose that* $c \in (0, (2-s)/(2(N-s))A_s^{(N-s)/(2-s)})$. *Then* $J(u)$ *satisfies* $(\mathrm{PS})_c$ *condition.*

Proof. Let $\{u_m\} \in H_r$ be a $(\mathrm{PS})_c$ sequence. Then we have

$$J(u_m) = \frac{1}{2}\|u_m\|^2 - \frac{1}{2^*(s)}\int \frac{|u_m|^{2^*(s)}}{|x|^s}dx - \frac{1}{r}\int a(x)|u_m|^r dx - \frac{\lambda}{2}\int |u_m|^2 dx = c + o(1), \tag{2.31}$$

$$\langle J'(u_m), u_m\rangle = \|u_m\|^2 - \int \frac{|u_m|^{2^*(s)}}{|x|^s}dx - \int a(x)|u_m|^r dx - \lambda\int |u_m|^2 dx = o(1)\|u_m\|. \tag{2.32}$$

Let (2.31) $\times$ 2 $-$ (2.32), we have

$$2c+o(1)+o(1)\|u_m\| \geq \left(1-\frac{2}{2^*(s)}\right)\int \frac{|u_m|^{2^*(s)}}{|x|^s}dx+\left(1-\frac{2}{r}\right)\int a(x)|u_m|^r dx. \quad (2.33)$$

From

$$\|u_m\|^2 = 2J(u_m)+\frac{2}{2^*(s)}\int \frac{|u_m|^{2^*(s)}}{|x|^s}dx+\frac{2}{r}\int a(x)|u_m|^r dx+\lambda\int |u_m|^2 dx, \quad (2.34)$$

we get

$$\begin{aligned}\left(1-\frac{\lambda}{\lambda_1(\mu)}\right)\|u_m\|^2 &\leq 2J(u_m)+\frac{2}{2^*(s)}\int \frac{|u_m|^{2^*(s)}}{|x|^s}dx+\frac{2}{r}\int a(x)|u_m|^r dx \\ &\leq o(1)+o(1)\|u_m\|+C.\end{aligned} \quad (2.35)$$

So, we conclude that $\{u_m\}$ is bounded in H_r. Passing to a subsequence (still denoted by $\{u_m\}$), as $m \to \infty$, we get that

$$\begin{aligned} &u_m \rightharpoonup u \text{ weakly in } H_r, \\ &u_m \longrightarrow u \text{ strongly in } L^q(\mathbb{R}^N), \quad q \in [2,2^*), \\ &u_m \longrightarrow u \text{ a.e. in } \mathbb{R}^N, \\ &u_m \longrightarrow u \text{ strongly in } L^r(\mathbb{R}^N, a(x)). \end{aligned} \quad (2.36)$$

It follows from the Sobolev-Hardy inequality (see [9]) that $|u_m|^{2^*(s)-2}u_m$ is bounded in $L^{2^*(s)/(2^*(s)-1)}(\mathbb{R}^N,|x|^{-s})$, thus we have that

$$|u_m|^{2^*(s)-2}u_m \rightharpoonup |u|^{2^*(s)-2}u \text{ weakly in } L^{2^*(s)/(2^*(s)-1)}(\mathbb{R}^N,|x|^{-s}). \quad (2.37)$$

Since $J'(u_m) \to 0$, from (2.36) and (2.37), we obtain

$$\langle J'(u),u\rangle = \|u\|^2-\int \frac{|u|^{2^*(s)}}{|x|^s}dx-\int a(x)|u|^r dx-\lambda\int |u|^2 dx = \lim_{m\to\infty}\langle J'(u_m),u\rangle = 0. \quad (2.38)$$

Set $v_m \equiv u_m - u$, by Brezis-Lieb lemma [2], we have

$$\|u_m\|^2 = \|v_m\|^2+\|u\|^2+o(1), \quad (2.39)$$

$$\int \frac{|u_m|^{2^*(s)}}{|x|^s}dx = \int \frac{|u|^{2^*(s)}}{|x|^s}dx+\int \frac{|v_m|^{2^*(s)}}{|x|^s}dx+o(1). \quad (2.40)$$

It follows directly from (2.31)–(2.40) that

$$\begin{aligned} o(1)\|u_m\| &= \langle J'(u_m), u_m\rangle = \|u_m\|^2 - \int \frac{|u_m|^{2^*(s)}}{|x|^s}dx - \int a(x)|u_m|^r dx - \lambda\int |u_m|^2 dx \\ &= \langle J'(u), u\rangle + \|v_m\|^2 - \int \frac{|v_m|^{2^*(s)}}{|x|^s}dx + o(1) = \|v_m\|^2 - \int \frac{|v_m|^{2^*(s)}}{|x|^s}dx + o(1), \\ J(u) &= J(u_m) - \frac{1}{2}\|v_m\|^2 + \frac{1}{2^*(s)}\int \frac{|v_m|^{2^*(s)}}{|x|^s}dx + o(1) \\ &= c - \frac{1}{2}\|v_m\|^2 + \frac{1}{2^*(s)}\int \frac{|v_m|^{2^*(s)}}{|x|^s}dx + o(1). \end{aligned} \tag{2.41}$$

Since $\{\|v_m\|\}$ is bounded, without loss of generality, we may assume that

$$\lim_{m\to\infty} \|v_m\|^2 = k. \tag{2.42}$$

Then we get that

$$\lim_{m\to\infty} \int \frac{|v_m|^{2^*(s)}}{|x|^s}dx = k. \tag{2.43}$$

By the Sobolev-Hardy inequality,

$$\int \frac{|v_m|^{2^*(s)}}{|x|^s}dx \le A_s^{-2^*(s)/2}\|v_m\|^{2^*(s)} \tag{2.44}$$

for all $m \in N$. Then by taking $m \to +\infty$, we obtain

$$k \le A_s^{-2^*(s)/2} k^{2^*(s)/2}. \tag{2.45}$$

If $k > 0$, we have that $k \ge A_s^{2^*(s)/(2^*(s)-2)}$. By (2.41) we deduce that

$$J(u) = c - \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)k \le c - \frac{2^*(s)-2}{22^*(s)}A_s^{2^*(s)/(2^*(s)-2)} = c - \frac{2-s}{2(N-s)}A_s^{(N-s)(2-s)} < 0, \tag{2.46}$$

but from (2.38), we get

$$J(u) = J(u) - \frac{1}{2}\langle J'(u), u\rangle = \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\int \frac{|u|^{2^*(s)}}{|x|^s}dx + \left(\frac{1}{2} - \frac{1}{r}\right)\int a(x)|u|^r dx \ge 0, \tag{2.47}$$

this contradiction implies $k = 0$. By the definition of v_m, we conclude that $J(u)$ satisfies $(\mathrm{PS})_c$ condition. We have completed the proof of Lemma 2.7. □

Proof of Theorem 1.1. By the Sobolev-Hardy inequality and Lemma 2.4, for any $u \in H_r \backslash \{0\}$, we have

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2^*(s)} \int \frac{|u|^{2^*(s)}}{|x|^s} dx - \frac{1}{r} \int a(x)|u|^r dx - \frac{\lambda}{2} \int |u|^2 dx \\ &\geq \left(\frac{1}{2} - \frac{\lambda}{2\lambda_1(\mu)}\right)\|u\|^2 - \frac{C}{2^*(s)}\|u\|^{2^*(s)} - \frac{C}{r}\|u\|^r \\ &\geq \|u\|^2 \left(\frac{\lambda_1(\mu) - \lambda}{2\lambda_1(\mu)} - C\left(\|u\|^{2^*(s)-2} + \|u\|^{r-2}\right)\right). \end{aligned} \tag{2.48}$$

Clearly, for $\rho > 0$ small enough, there exists $\beta > 0$ such that $J(u) \geq \beta$ for all $u \in \partial B_\rho = \{u \in H_r,\ \|u\| = \rho\}$. For $u_0 \in H_r \backslash \{0\}$, $t \geq 0$, we have

$$J(tu_0) = \frac{t^2}{2}\|u_0\|^2 - \frac{t^{2^*(s)}}{2^*(s)} \int \frac{|u_0|^{2^*(s)}}{|x|^s} dx - \frac{t^r}{r} \int a(x)|u_0|^r dx - \frac{\lambda t^2}{2} \int |u_0|^2 dx. \tag{2.49}$$

Obviously, $\lim_{t\to+\infty} J(tu_0) = -\infty$, so we may choose t_0 large enough, such that $\|t_0u_0\| > \|u_0\| = \rho$ for some $u_0 \in \partial B_\rho$, and $J(t_0u_0) < 0$. By Lemmas 2.6 and 2.7 and the mountain pass theorem given in [1] (or [3]), we get a sequence $\{u_m\} \subset H_r$, $u_m \to u$ strongly for some $u \in H_r$, and $J(u) = c$, $J'(u) = 0$. Thus u is a nontrivial solution of problem (1.1). we have finished the proof of Theorem 1.1. □

Remark 2.8. If $\lambda = 0$, using similar ways, we can prove that problem (1.1) has at least a nontrivial solution in H when r, μ satisfy the condition (i) of Theorem 1.1.

Acknowledgment

This research was supported by National Natural Science Foundation of China under Grant 10271077, Science and Technology Committee of Shanghai under Grant 03JC14013, and National Natural Science Foundation of China (Mathematics Tianyuan Youth Foundation) under Grant A0324610.

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