IMPLICIT ITERATION PROCESS OF NONEXPANSIVE NON-SELF-MAPPINGS

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Suppose C is a nonempty closed convex subset of real Hilbert space H. Let $T : C \to H$ be a nonexpansive non-self-mapping and P is the nearest point projection of H onto C. In this paper, we study the convergence of the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ satisfying $x_n = (1-\alpha_n)u + \alpha_n T[(1-\beta_n)x_n + \beta_n T x_n]$, $y_n = (1-\alpha_n)u + \alpha_n PT[(1-\beta_n)y_n + \beta_n PT y_n]$, and $z_n = P[(1-\alpha_n)u + \alpha_n TP[(1-\beta_n)z_n + \beta_n T z_n]]$, where $\{\alpha_n\} \subseteq (0,1)$, $0 \le \beta_n \le \beta < 1$ and $\alpha_n \to 1$ as $n \to \infty$. Our results extend and improve the recent ones announced by Xu and Yin, and many others.

1. Introduction

Let C be a nonempty closed convex subset of a Banach space E. Then a non-self-mapping T from C into E is called *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. Given $u \in C$ and $\{\alpha_n\}$ is a sequence such that $0 < \alpha_n < 1$, we can define a contraction $T_n : C \to E$ by

$$T_n x = (1-\alpha_n)u + \alpha_n T x, \quad x \in C. \tag{1.1}$$

If T is a self-mapping (i.e., $T(C) \subset C$), then T_n maps C into itself, and hence, by Banach's contraction principle, T_n has a unique fixed point x_n in C, that is, we have

$$x_n = (1-\alpha_n)u + \alpha_n T x_n, \quad \forall n \ge 1 \tag{1.2}$$

(such a sequence $\{x_n\}$ is said to be an approximating fixed point of T since it possesses the property that if $\{x_n\}$ is bounded, then $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$) whenever $\lim_{n\to\infty} \alpha_n = 1$. The strong convergence of $\{x_n\}$ as $\alpha_n \to 1$ for a self-mapping T of a bounded C was proved in a Hilbert space independently by Browder [1] and Halpern [3] and in a uniformly smooth Banach space by Reich [7]. Thereafter, Singh and Watson [8] extended the result of Browder and Halpern to nonexpansive non-self-mapping T satisfying Rothe's boundary condition $T(\partial C) \subset C$ (here ∂C denotes the boundary of C). Recently, Xu and Yin [11] proved that if C is a nonempty closed convex (not necessarily bounded) subset of Hilbert space H, if $T : C \to H$ is a nonexpansive non-self-mapping, and if $\{x_n\}$ is the sequence defined by (1.2) which is bounded, then $\{x_n\}$ converges strongly as $\alpha_n \to 1$ to a

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fixed point of T. Marino and Trombetta [5] defined contractions S_n and U_n from C into itself by

$$S_n x = (1-\alpha_n)u + \alpha_n PTx, \quad \forall x \in C, \tag{1.3}$$

$$U_n x = P[(1-\alpha_n)u + \alpha_n Tx], \quad \forall x \in C, \tag{1.4}$$

where P is the nearest point projection of H onto C. Then by the Banach contraction principle, there exists a unique fixed point y_n(resp., z_n) of S_n(resp., U_n) in C, that is,

$$y_n = (1-\alpha_n)u + \alpha_n PTy_n, \tag{1.5}$$

$$z_n = P[(1-\alpha_n)u + \alpha_n Tz_n]. \tag{1.6}$$

Xu and Yin [11] also proved that if C is a nonempty closed convex subset of a Hilbert space H, if $T : C \to H$ is a nonexpansive non-self-mapping satisfying the weak inwardness condition, and $\{x_n\}$ is bounded, then $\{y_n\}$ (resp., $\{z_n\}$) defined by (1.5) (resp., (1.6)) converges strongly as $\alpha_n \to 1$ to a fixed point of T.

Let C be a nonempty convex subset of Banach space E. Then for $x \in C$, we define the inward set $I_c(x)$ as follows:

$$I_c(x) = \{y \in E : y = x + a(z-x) \text{ for some } z \in C\ a \geq 0\}. \tag{1.7}$$

A mapping $T : C \to E$ is said to be *inward* if $Tx \in I_c(x)$ for all $x \in C$. T is also said to be *weakly inward* if for each $x \in C$, Tx belongs to the closure of $I_c(x)$.

In this paper, we extend Xu and Yin's results [11] to study the contraction mappings T_n, S_n, and U_n define by

$$T_n x = (1-\alpha_n)u + \alpha_n T[(1-\beta_n)x + \beta_n Tx], \tag{1.8}$$

$$S_n x = (1-\alpha_n)u + \alpha_n PT[(1-\beta_n)x + \beta_n PTx], \tag{1.9}$$

$$U_n x = P[(1-\alpha_n)u + \alpha_n TP[(1-\beta_n)x + \beta_n Tx]], \tag{1.10}$$

where $\{\alpha_n\} \subseteq (0,1)$, $0 \leq \beta_n \leq \beta < 1$, and P is the nearest point projection of H onto C. Moreover, we also prove the strong convergence of the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ satisfying

$$x_n = (1-\alpha_n)u + \alpha_n T[(1-\beta_n)x_n + \beta_n Tx_n], \tag{1.11}$$

$$y_n = (1-\alpha_n)u + \alpha_n PT[(1-\beta_n)y_n + \beta_n PTy_n], \tag{1.12}$$

$$z_n = P[(1-\alpha_n)u + \alpha_n TP[(1-\beta_n)z_n + \beta_n Tz_n]], \tag{1.13}$$

where $\alpha_n \to 1$ as $n \to \infty$. We note that if $\beta_n \equiv 0$, then (1.11), (1.12), (1.13) reduce to (1.2), (1.5), and (1.6), respectively. The results presented in this paper extend and improve the corresponding onces announced by Xu and Yin [11], and others.

2. Main results

In this section, we prove the strong convergence theorems for nonexpansive non-self-mappings. To prove our results, we use the following theorem.

THEOREM 2.1. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H, and let $T : C \to H$ be a nonexpansive non-self-mapping. Suppose that for some $u \in C$, $\{\alpha_n\} \subseteq (0,1)$, and $0 \leq \beta_n \leq \beta < 1$, the mapping T_n defined by (1.8) has a (unique) fixed point $x_n \in C$ for all $n \geq 1$. Then T has a fixed point if and only if $\{x_n\}$ remains bounded as $\alpha_n \to 1$. In this case, $\{x_n\}$ converges strongly as $\alpha_n \to 1$ to a fixed point of T.*

Proof. We denote by $F(T)$ the fixed point set of T. Suppose that $F(T)$ is nonempty. Let $w \in F(T)$. Then for each $n \geq 1$, we have

$$\begin{aligned}||w - x_n|| &= ||w - (1-\alpha_n)u - \alpha_n T[(1-\beta_n)x_n + \beta_n T x_n]|| \\ &\leq (1-\alpha_n)\|w-u\| + \alpha_n ||w - T[(1-\beta_n)x_n + \beta_n T x_n]|| \\ &\leq (1-\alpha_n)\|w-u\| + \alpha_n ||w - (1-\beta_n)x_n - \beta_n T x_n|| \\ &\leq (1-\alpha_n)\|w-u\| + \alpha_n(1-\beta_n)||w - x_n|| + \alpha_n\beta_n ||w - x_n|| \\ &= (1-\alpha_n)\|w-u\| + \alpha_n ||w - x_n||,\end{aligned} \tag{2.1}$$

and hence $(1-\alpha_n)\|w - x_n\| \leq (1-\alpha_n)\|w-u\|$, for all $n \geq 1$. This implies that $\|w - x_n\| \leq \|w-u\|$ for all $n \geq 1$. Then $\{x_n\}$ is a bounded sequence. Conversely, suppose that $\{x_n\}$ is bounded, z is a weak cluster point of $\{x_n\}$, and $\alpha_n \to 1$ as $n \to \infty$. Then we show that $F(T) \neq \emptyset$ and $\{x_n\}$ converges strongly to a fixed point of T. We choose a subsequence $\{x_{n_i}\}$ of the sequence $\{x_n\}$ with $\alpha_{n_i} \to 1$ such that $x_{n_i} \to z$ weakly, we can define a real-valued function g on H given by

$$g(x) = \limsup_{i\to\infty} ||x_{n_i} - x||^2, \quad \text{for every } x \in H, \tag{2.2}$$

observing that $\|x_{n_i} - x\|^2 = \|x_{n_i} - z\|^2 + 2\langle x_{n_i} - z, z - x\rangle + \|z - x\|^2$. Since $x_{n_i} \to z$ weakly, we immediately get

$$g(x) = g(z) + \|x - z\|^2, \quad \forall x \in H, \tag{2.3}$$

in particular,

$$g(Tz) = g(z) + \|Tz - z\|^2. \tag{2.4}$$

On the other hand, we have

$$\begin{aligned}||x_{n_i} - Tx_{n_i}|| &\leq (1-\alpha_{n_i})||u - Tx_{n_i}|| + \alpha_{n_i}||T[(1-\beta_{n_i})x_{n_i} + \beta_{n_i}Tx_{n_i}] - Tx_{n_i}|| \\ &\leq (1-\alpha_{n_i})||u - Tx_{n_i}|| + \alpha_{n_i}||(1-\beta_{n_i})x_{n_i} + \beta_{n_i}Tx_{n_i} - x_{n_i}|| \\ &\leq (1-\alpha_{n_i})||u - Tx_{n_i}|| + \beta_{n_i}||Tx_{n_i} - x_{n_i}||,\end{aligned} \tag{2.5}$$

for all $i \geq 1$. This implies that $(1-\beta_{n_i})\|x_{n_i} - Tx_{n_i}\| \leq (1-\alpha_{n_i})\|u - Tx_{n_i}\|$, and hence

$$\begin{aligned}||x_{n_i} - Tx_{n_i}|| &= \frac{(1-\alpha_{n_i})}{(1-\beta_{n_i})}||u - Tx_{n_i}|| \\ &\leq \frac{(1-\alpha_{n_i})}{(1-\beta)}||u - Tx_{n_i}|| \longrightarrow 0 \quad \text{as } i \longrightarrow \infty.\end{aligned} \tag{2.6}$$

Note that

$$\begin{aligned}||x_{n_i} - Tz||^2 &= ||x_{n_i} - Tx_{n_i} + Tx_{n_i} - Tz||^2 \\ &\leq (||x_{n_i} - Tx_{n_i}|| + ||Tx_{n_i} - Tz||)^2 \\ &= ||x_{n_i} - Tx_{n_i}||^2 + 2||x_{n_i} - Tx_{n_i}|| ||Tx_{n_i} - Tz|| + ||Tx_{n_i} - Tz||^2\end{aligned} \tag{2.7}$$

for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned}g(Tz) &= \limsup_{i\to\infty} ||x_{n_i} - Tz||^2 \\ &\leq \limsup_{i\to\infty} ||Tx_{n_i} - Tz||^2 \\ &\leq \limsup_{i\to\infty} ||x_{n_i} - z||^2 = g(z).\end{aligned} \tag{2.8}$$

This, together with (2.4), implies that $Tz = z$ and z is a fixed point of T. Now since $F(T)$ is nonempty, closed, and convex, there exists a unique $v \in F(T)$ that is closest to u; namely, v is the nearest point projection of u onto $F(T)$. For any $y \in F(T)$, we have

$$\begin{aligned}||(x_n - u) + \alpha_n(u - y)||^2 &= ||\big((1-\alpha_n)u + \alpha_n T[(1-\beta_n)x_n + \beta_n Tx_n] - u\big) + \alpha_n(u - y)||^2 \\ &= \alpha_n^2 ||T[(1-\beta_n)x_n + \beta_n Tx_n] - y||^2 \\ &\leq \alpha_n^2 ||(1-\beta_n)x_n + \beta_n Tx_n - y||^2 \\ &= \alpha_n^2 ||(1-\beta_n)(x_n - y) + \beta_n(Tx_n - y)||^2 \\ &\leq \alpha_n^2 \big((1-\beta_n)||x_n - y|| + \beta_n ||x_n - y||\big)^2 \\ &= \alpha_n^2 ||x_n - y||^2 \\ &= \alpha_n^2 ||x_n - u + u - y||^2,\end{aligned} \tag{2.9}$$

and so

$$\begin{aligned}&||x_n - u||^2 + \alpha_n^2 \|u - y\|^2 + 2\alpha_n \langle x_n - u, u - y\rangle \\ &\quad\leq \alpha_n^2 \big(||x_n - u||^2 + \|u - y\|^2 + 2\langle x_n - u, u - y\rangle\big) \\ &\quad\leq \alpha_n ||x_n - u||^2 + \alpha_n \|u - y\|^2 + 2\alpha_n \langle x_n - u, u - y\rangle\end{aligned} \tag{2.10}$$

for all $n \geq 1$. It follows that

$$||x_n - u||^2 \leq \alpha_n \|y - u\|^2 \leq \|y - u\|^2, \quad \forall y \in F(T),\ \{\alpha_n\} \subseteq (0,1)\ \forall n \in \mathbb{N}. \tag{2.11}$$

Since the norm of H is weakly lower semicontinuous (w-l.s.c.), we get

$$\|z - u\| \leq \liminf_{i\to\infty} ||x_{n_i} - u|| \leq \|y - u\|, \quad \forall y \in F(T). \tag{2.12}$$

Therefore, we must have $z = v$ for v is the unique element in $F(T)$ that is closest to u. This shows that v is the only weak cluster point of $\{x_n\}$ with $\alpha_n \to 1$. It remains to verify that the convergence is strong. In fact, it follows that

$$\begin{aligned} \|x_n - v\|^2 &= \|x_n - u\|^2 - \|u - v\|^2 - 2\langle x_n - v, v - u\rangle \\ &\le -2\langle x_n - v, v - u\rangle \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{2.13}$$

This completes the proof. □

Corollary 2.2. *Let H, C, T be as in Theorem 2.1. Suppose in addition that C is bounded and that the weak inwardness condition is satisfied. Then for each $u \in C$, the sequence $\{x_n\}$ satisfying (1.11) converges strongly as $\alpha_n \to 1$ to a fixed point of T.*

Theorem 2.3. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, let $T : C \to H$ be a nonexpansive non-self-mapping satisfying the weak inwardness condition, and let $P : H \to C$ be the nearest point projection. Suppose that for some $u \in C$, each $\{\alpha_n\} \subseteq (0,1)$ and $0 \le \beta_n \le \beta < 1$. Then, a mapping S_n defined by (1.9) has a unique fixed point $y_n \in C$. Further, T has a fixed point if and only if $\{y_n\}$ remains bounded as $\alpha_n \to 1$. In this case, $\{y_n\}$ converges strongly as $\alpha_n \to 1$ to a fixed point of T.*

Proof. It is straightforward that $S_n : C \to C$ is a contraction for every $n \ge 1$. Therefore by the Banach contraction principle, there exists a unique fixed point y_n of S_n in C satisfying (1.12). Let w be a fixed point of T. Then as in the proof of Theorem 2.1, $\{y_n\}$ is bounded. Conversely, suppose that $\{y_n\}$ is bounded. Applying Theorem 2.1, we obtain that $\{y_n\}$ converges strongly to a fixed point z of PT. Next, let us show that $z \in F(T)$. Since $z = PTz$ and P is the nearest point projection of H onto C, it follows by [9] that

$$\langle Tz - z, J(z - v)\rangle \ge 0, \quad \forall v \in C. \tag{2.14}$$

On the other hand, Tz belongs to the closure of $I_c(z)$ by the weak inwardness conditions. Hence for each integer $n \ge 1$, there exist $z_n \in C$ and $a_n \ge 0$ such that the sequence

$$r_n := z + a_n(z_n - z) \longrightarrow Tz. \tag{2.15}$$

Thus it follows that

$$\begin{aligned} 0 &\le a_n\langle Tz - z, z - z_n\rangle \\ &= \langle Tz - z, a_n(z - z_n)\rangle \\ &= \langle Tz - z, z - r_n\rangle \longrightarrow \langle Tz - z, z - Tz\rangle \\ &= -\|Tz - z\|^2. \end{aligned} \tag{2.16}$$

Hence we have $Tz = z$. □

Corollary 2.4 (see [11, Theorem 2]). *Let H, C, T, P, u, and $\{\alpha_n\}$ be as in Theorem 2.3. Then, a mapping S_n given by (1.3) has a unique fixed point $y_n \in C$ such that $y_n = (1 - \alpha_n)u + \alpha_n PTy_n$. Further, T has a fixed point if and only if $\{y_n\}$ remains bounded as $\alpha_n \to 1$. In this case, $\{y_n\}$ converges strongly as $\alpha_n \to 1$ to a fixed point of T.*

THEOREM 2.5. *Let H, C, T, P, u, $\{\alpha_n\}$, and $\{\beta_n\}$ be as in Theorem 2.3. Then a mapping U_n defined by (1.10) has a unique fixed point $z_n \in C$. Further, T has a fixed point if and only if $\{z_n\}$ remains bounded as $\alpha_n \to 1$ and $\beta_n \to 0$. In this case, $\{z_n\}$ converges strongly as $\alpha_n \to 1$ and $\beta_n \to 0$ to a fixed point of T.*

Proof. It follows by the Banach contraction principle that there exists a unique fixed point z_n of U_n such that

$$z_n = P[(1-\alpha_n)u + \alpha_n TP[(1-\beta_n)z_n + \beta_n Tz_n]]. \tag{2.17}$$

Let $w \in F(T)$. Then for each $n \geq 1$, we have

$$\begin{aligned}
||w - z_n|| &= \|Pw - P[(1-\alpha_n)u + \alpha_n TP((1-\beta_n)z_n + \beta_n Tz_n)]||\\
&\leq ||w - (1-\alpha_n)u - \alpha_n TP[(1-\beta_n)z_n + \beta_n Tz_n]||\\
&\leq (1-\alpha_n)\|w-u\| + \alpha_n||w - TP[(1-\beta_n)z_n + \beta_n Tz_n]||\\
&\leq (1-\alpha_n)\|w-u\| + \alpha_n(1-\beta_n)||w-z_n|| + \alpha_n\beta_n||w - Tz_n||\\
&\leq (1-\alpha_n)\|w-u\| + \alpha_n(1-\beta_n)||w-z_n|| + \alpha_n\beta_n||w - z_n||\\
&= (1-\alpha_n)\|w-u\| + \alpha_n||w-z_n||,
\end{aligned} \tag{2.18}$$

and hence $(1-\alpha_n)\|w-z_n\| \leq (1-\alpha_n)\|w-u\|$, for all $n > 1$. This implies that $\|w-z_n\| \leq \|w-u\|$, for all $n > 1$. Then $\{z_n\}$ is bounded. Conversely, suppose that $\{z_n\}$ is bounded, $\alpha_n \to 1$, and $\beta_n \to 0$. We show that $F(T) \neq \emptyset$. For any subsequence $\{z_{n_i}\}$ of the sequence $\{z_n\}$ converging weakly to $\bar{z}$ such that $\alpha_{n_i} \to 1$, we can define a real-valued function g on H given by

$$g(z) = \limsup_{i\to\infty} ||z_{n_i} - z||^2, \quad \text{for every } z \in H, \tag{2.19}$$

observing that $\|z_{n_i} - z\|^2 = \|z_{n_i} - \bar{z}\|^2 + 2\langle z_{n_i} - \bar{z}, \bar{z} - z\rangle + \|\bar{z} - z\|^2$. Since $z_{n_i} \to \bar{z}$ weakly, we get

$$g(z) = g(\bar{z}) + \|\bar{z} - z\|^2, \quad \forall z \in H, \tag{2.20}$$

in particular,

$$g(PT\bar{z}) = g(\bar{z}) + \|PT\bar{z} - \bar{z}\|^2. \tag{2.21}$$

For instance, the straightforward verification gives

$$\begin{aligned}
||z_{n_i} - PTz_{n_i}|| &= ||P[(1-\alpha_{n_i})u + \alpha_{n_i}TP((1-\beta_{n_i})z_{n_i} + \beta_{n_i}Tz_{n_i})] - PTz_{n_i}||\\
&\leq (1-\alpha_{n_i})||u - Tz_{n_i}|| + \alpha_{n_i}\beta_{n_i}||Tz_{n_i} - z_{n_i}||, \quad \forall i \geq 1,
\end{aligned} \tag{2.22}$$

and this implies that $\|z_{n_i} - PTz_{n_i}\| \le (1-\alpha_{n_i})\|u - Tz_{n_i}\| + \alpha_{n_i}\beta_{n_i}\|Tz_{n_i} - z_{n_i}\| \to 0$ as $i \to \infty$. Moreover, we note that

$$\begin{aligned}||z_{n_i} - PT\bar{z}||^2 &= ||z_{n_i} - PTz_{n_i} + PTz_{n_i} - PT\bar{z}||^2 \\ &\le (||z_{n_i} - PTz_{n_i}|| + ||PTz_{n_i} - PT\bar{z}||)^2 \\ &= ||z_{n_i} - PTz_{n_i}||^2 + 2||z_{n_i} - PTz_{n_i}||\,||PTz_{n_i} - PT\bar{z}|| + ||PTz_{n_i} - PT\bar{z}||^2\end{aligned} \tag{2.23}$$

for all $i \in \mathbb{N}$. It follows that

$$\begin{aligned}g(PT\bar{z}) &= \limsup_{i\to\infty} ||z_{n_i} - PT\bar{z}||^2 \\ &\le \limsup_{i\to\infty} ||PTz_{n_i} - PT\bar{z}||^2 \\ &\le \limsup_{i\to\infty} ||z_{n_i} - \bar{z}||^2 = g(z)\end{aligned} \tag{2.24}$$

which in turn, together with (2.21), implies that $PT(\bar{z}) = \bar{z}$. Since T satisfies the weak inwardness condition, by the same argument as in the proof of Theorem 2.3, we see that $\bar{z}$ is a fixed point of T. For any $w \in F(T)$, we have

$$\begin{aligned}\alpha_n\Big[TP((1-\beta_n)w + \beta_n w) - u\Big] + u &= \alpha_n(w-u) + u \\ &= \alpha_n w + (1-\alpha_n)u \\ &= P(\alpha_n w + (1-\alpha_n)u)\end{aligned} \tag{2.25}$$

for all $n \in \mathbb{N}$. By following as in the proof of Theorem 2.1, we have

$$||z_n - u||^2 \le \alpha_n\|w-u\|^2 \le \|w-u\|^2, \quad \forall w \in F(T),\ \{\alpha_n\} \subseteq (0,1)\ \forall n \in \mathbb{N}. \tag{2.26}$$

From (2.26) and the w-l.s.c. of the norm of H, it follows that

$$\|\bar{z} - u\| \le \liminf_{n\to\infty} ||z_n - u|| \le \|w - u\| \tag{2.27}$$

for all $w \in F(T)$. Hence $\bar{z}$ is the nearest point projection z in $F(T)$ of u onto $F(T)$ which exists uniquely since $F(T)$ is nonempty, closed, and convex. Moreover,

$$\begin{aligned}||z_n - z||^2 &= ||z_n - u||^2 - \|u - z\|^2 - 2\langle z_n - z, z - u\rangle \\ &\le -2\langle z_n - z, z - u\rangle \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{2.28}$$

This completes the proof. □

Corollary 2.6 (see [11, Theorem 2]). *Let H, C, T, P, u, and $\{\alpha_n\}$ be as in Theorem 2.3. Then a mapping U_n defined by (1.4) has a unique fixed point $z_n \in C$. Further, T has a fixed point if and only if $\{z_n\}$ remains bounded as $\alpha_n \to 1$. In this case, $\{z_n\}$ converges strongly as $\alpha_n \to 1$ to a fixed point of T.*

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