EXPONENTIATING DERIVATIONS OF QUASI *-ALGEBRAS: POSSIBLE APPROACHES AND APPLICATIONS

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The problem of exponentiating derivations of quasi *-algebras is considered in view of applying it to the determination of the time evolution of a physical system. The particular case where observables constitute a proper CQ^*-algebra is analyzed.

1. Introduction

The unbounded nature of the operators describing observables of a quantum mechanical system with a finite or an infinite number of degrees of freedom is mathematically *a fact* which follows directly from the noncommutative nature of the quantum world in the sense that, as a consequence of the Wiener-von Neumann theorem, the commutation relation $[\hat{q},\hat{p}] = i\mathbf{1}$ for the position $\hat{q}$ and the momentum $\hat{p}$ is not compatible with the boundedness of both $\hat{q}$ and $\hat{p}$. Thus any operator representation of this commutation relation necessarily involves unbounded operators. Also the bosonic creation and annihilation operators $a^\dagger$ and a, $[a,a^\dagger] = \mathbf{1}$, or the hamiltonian of the simple harmonic oscillator $H = (1/2)(\hat{p}^2 + \hat{q}^2) = a^\dagger a + (1/2)\mathbf{1}$, just to mention few examples, are all unbounded operators.

However, when an experiment is carried out, what is measured is an eigenvalue of an observable, which is surely a *finite* real number: for instance, if the physical system $\mathcal{S}$ on which measurements are performed is in a laboratory, then if we measure the position of a particle of $\mathcal{S}$, we must get a finite number as a result. Also, if we measure the energy of a quantum particle in a, say, harmonic potential, we can only get a finite measure since the probability that the particle has infinite energy is zero. Moreover, in a true relativistic world, since the velocity of a particle cannot exceed the velocity of light c, any measurement of its momentum can only give, again, a finite result. From the mathematical point of view, this may correspond to restricting the operator to some *spectral subspaces* where the unboundedness is in fact removed. This procedure supports the practical point of view where it seems enough to deal, from the very beginning, with bounded operators only.

It is then reasonable to look for a compromise within these opposite approaches and the compromise could be the following. Given a system $\mathcal{S}$, we consider a slightly modified

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version of it in which all the operators related to $\mathcal{S}$ are replaced by their *regularized* version (e.g., their *finite-volume* version, natural choice if $\mathcal{S}$ *lives* in a laboratory!), obtained by means of some given cutoff, we compute all those quantities which are relevant for our purposes, and then in order to check whether this procedure has modified the original physical nature of $\mathcal{S}$, we try to see whether these results are *stable* under the removal of the cutoff. As an example, if $\mathcal{S}$ is contained inside a box of volume V, we do expect that all the results become independent of the volume cutoff as soon as the value of this cutoff W becomes larger than V, since what happens outside the box has almost no role in the behavior of $\mathcal{S}$.

As it is extensively discussed in [12], the full description of a physical system $\mathcal{S}$ implies the knowledge of three basic ingredients: the set of the observables, the set of the states and, finally, the dynamics that describes the time evolution of the system by means of the time dependence of the expectation value of a given observable on a given state. Originally, the set of the observables was considered to be a C^*-algebra, see [10]. In many applications, however, this was shown not to be the most convenient choice and the C^*-algebra was replaced by a von Neumann algebra, because the role of the representations turns out to be crucial mainly when long-range interactions are involved, see [4] and references therein. Here we use a different algebraic structure, similar to the one considered in [9], which is suggested by the considerations above: because of the relevance of the unbounded operators in the description of $\mathcal{S}$, we will assume in Sections 2 and 3 that the observables of the system belong to a quasi *-algebra $(\mathcal{A},\mathcal{A}_0)$, see [15] and references therein, while, in order to have a richer mathematical structure, in Section 4 we will use a slightly different algebraic structure: $(\mathcal{A},\mathcal{A}_0)$ will be assumed to be a proper CQ^*-algebra, which has nicer topological properties. In particular, for instance, $\mathcal{A}_0$ is a C^*-algebra. The set of states over $(\mathcal{A},\mathcal{A}_0)$, Σ, is described again in [15], while the dynamics is usually a group (or a semigroup) of automorphisms of the algebra α^t. Therefore, following [12], we simply put $\mathcal{S} = \{(\mathcal{A},\mathcal{A}_0),\Sigma,\alpha^t\}$.

The system $\mathcal{S}$ is now *regularized*: we introduce some cutoff L, (e.g., a volume or an occupation number cutoff), belonging to a certain set Λ, so that $\mathcal{S}$ is replaced by a sequence or, more generally, a net of systems $\mathcal{S}_L$, one for each value of $L \in \Lambda$. This cutoff is chosen in such a way that all the observables of $\mathcal{S}_L$ belong to a certain *-algebra $\mathcal{A}_L$ contained in $\mathcal{A}_0$: $\mathcal{A}_L \subset \mathcal{A}_0 \subset \mathcal{A}$. As for the states, we choose $\Sigma_L = \Sigma$, that is, the set of states over $\mathcal{A}_L$ is taken to coincide with the set of states over $\mathcal{A}$. This is a common choice, see [4], even if also different possibilities are considered in the literature. For instance, in [5], also the states depend on L. Finally, since the dynamics is related to a hamiltonian operator H (or to the Lindblad generator of a semigroup), and since H has to be replaced with H_L, because of the cutoff, α^t is replaced by the family $\alpha_L^t(\cdot) = e^{iH_Lt} \cdot e^{-iH_Lt}$. Therefore,

$$\mathcal{S} = \{(\mathcal{A},\mathcal{A}_0),\Sigma,\alpha^t\} \longrightarrow \{\mathcal{S}_L = \{\mathcal{A}_L,\Sigma,\alpha_L^t\},L \in \Lambda\}. \tag{1.1}$$

2. The mathematical framework

Let $\mathcal{A}$ be a linear space and $\mathcal{A}_0$ a *-algebra contained in $\mathcal{A}$ as a subspace. We say that $\mathcal{A}$ is a quasi *-algebra over $\mathcal{A}_0$ if (i) the right and left multiplications of an element of $\mathcal{A}$ and an element of $\mathcal{A}_0$ are always defined and linear; (ii) $x_1(x_2a) = (x_1x_2)a$, $(ax_1)x_2 = a(x_1x_2)$, and $x_1(ax_2) = (x_1a)x_2$, for each $x_1, x_2 \in \mathcal{A}_0$ and $a \in \mathcal{A}$; (iii) an involution $*$ (which extends the involution of $\mathcal{A}_0$) is defined in $\mathcal{A}$ with the property $(ab)^* = b^*a^*$ whenever the multiplication is defined.

In this paper, we will always assume that the quasi $*$-algebra under consideration has a unit, that is, an element $\mathbb{1} \in \mathcal{A}_0$ such that $a\mathbb{1} = \mathbb{1}a = a$, for all $a \in \mathcal{A}$.

A quasi $*$-algebra $(\mathcal{A}, \mathcal{A}_0)$ is said to be a locally convex quasi $*$-algebra if in $\mathcal{A}$, a locally convex topology τ is defined such that (a) the involution is continuous and the multiplications are separately continuous; and (b) $\mathcal{A}_0$ is dense in $\mathcal{A}[\tau]$. We indicate with $\{p_\alpha\}$ a directed set of seminorms which defines τ. Throughout this paper, we will always suppose, without loss of generality, that a locally convex quasi $*$-algebra $(\mathcal{A}[\tau], \mathcal{A}_0)$ is *complete*.

In the following, we also need the concept of $*$-*representation*.

Let $\mathcal{D}$ be a dense subspace in some Hilbert space $\mathcal{H}$. We denote with $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ the set of all closable operators X in $\mathcal{H}$ such that $D(X) = \mathcal{D}$ and $D(X^*) \supset \mathcal{D}$ which is a *partial* $*$-*algebra*, see [1], with the usual operations $X + Y$, λX, the involution $X^\dagger = X^* \mid \mathcal{D}$, and the weak product $X \square Y \equiv X^{\dagger *} Y$ whenever $Y\mathcal{D} \subset D(X^{\dagger *})$ and $X^\dagger \mathcal{D} \subset D(Y^*)$. We also denote with $\mathcal{L}^\dagger(\mathcal{D})$ the $*$-algebra consisting of the elements $A \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ such that both A and its adjoint A^* map $\mathcal{D}$ into itself (in this case, the weak multiplication reduces to the ordinary multiplication of operators).

Let $(\mathcal{A}, \mathcal{A}_0)$ be a quasi $*$-algebra, $\mathcal{D}_\pi$ a dense domain in a certain Hilbert space $\mathcal{H}_\pi$, and π a linear map from $\mathcal{A}$ into $\mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)$ such that

(i) $\pi(a^*) = \pi(a)^\dagger$, for all $a \in \mathcal{A}$;

(ii) if $a \in \mathcal{A}$, $x \in \mathcal{A}_0$, then $\pi(a) \square \pi(x)$ is well defined and $\pi(ax) = \pi(a) \square \pi(x)$.

We say that such a map π is a $*$-*representation of* $\mathcal{A}$. Moreover, if

(iii) $\pi(\mathcal{A}_0) \subset \mathcal{L}^\dagger(\mathcal{D}_\pi)$, then π is said to be a $*$-*representation of the quasi* $*$-*algebra* $(\mathcal{A}, \mathcal{A}_0)$.

The $*$-representation π is called *ultra-cyclic* if there exists $\xi_0 \in \mathcal{D}_\pi$ such that $\pi(\mathcal{A}_0)\xi_0 = \mathcal{D}_\pi$.

Let π be a $*$-representation of $\mathcal{A}$. The strong topology τ_s on $\pi(\mathcal{A})$ is the locally convex topology defined by the following family of seminorms: $\{p_\xi(\cdot);\ \xi \in \mathcal{D}_\pi\}$, where $p_\xi(\pi(a)) \equiv \|\pi(a)\xi\|$, where $a \in \mathcal{A}$, $\xi \in \mathcal{D}_\pi$.

For an overview on partial $*$-algebras and related topics, we refer to [1].

Definition 2.1. Let $(\mathcal{A}, \mathcal{A}_0)$ be a quasi $*$-algebra. A $*$-*derivation of* $\mathcal{A}_0$ is a map $\delta : \mathcal{A}_0 \to \mathcal{A}$ with the following properties:

(i) $\delta(x^*) = \delta(x)^*$, for all $x \in \mathcal{A}_0$;

(ii) $\delta(\alpha x + \beta y) = \alpha\delta(x) + \beta\delta(y)$, for all $x, y \in \mathcal{A}_0$, for all $\alpha, \beta \in \mathbb{C}$;

(iii) $\delta(xy) = x\delta(y) + \delta(x)y$, for all $x, y \in \mathcal{A}_0$.

Let $(\mathcal{A}, \mathcal{A}_0)$ be a quasi $*$-algebra and let δ be a $*$-derivation of $\mathcal{A}_0$. Let π be a $*$-representation of $(\mathcal{A}, \mathcal{A}_0)$. We will always assume that whenever $x \in \mathcal{A}_0$ is such that $\pi(x) = 0$, then $\pi(\delta(x)) = 0$. Under this assumption, the linear map

$$\delta_\pi(\pi(x)) = \pi(\delta(x)), \quad x \in \mathcal{A}_0, \tag{2.1}$$

is well defined on $\pi(\mathscr{A}_0)$ with values in $\pi(\mathscr{A})$ and it is a *-derivation of $\pi(\mathscr{A}_0)$. We call δ_π the *-derivation *induced* by π.

Given such a representation π and its dense domain $\mathscr{D}_\pi$, we consider the usual graph topology $t_\dagger$ generated by the seminorms

$$\xi \in \mathscr{D}_\pi \longrightarrow \|A\xi\|, \quad A \in \mathscr{L}^\dagger(\mathscr{D}_\pi). \tag{2.2}$$

If $\mathscr{D}'_\pi$ denotes the conjugate dual space of $\mathscr{D}_\pi$, we get the usual rigged Hilbert space $\mathscr{D}_\pi[t_\dagger] \subset \mathscr{H}_\pi \subset \mathscr{D}'_\pi[t'_\dagger]$, where $t'_\dagger$ denotes the strong dual topology of $\mathscr{D}'_\pi$. Let $\mathscr{L}(\mathscr{D}_\pi,\mathscr{D}'_\pi)$ denote the space of all continuous linear maps from $\mathscr{D}_\pi[t_\dagger]$ into $\mathscr{D}'_\pi[t'_\dagger]$. Then one has

$$\mathscr{L}^\dagger(\mathscr{D}_\pi) \subset \mathscr{L}(\mathscr{D}_\pi,\mathscr{D}'_\pi). \tag{2.3}$$

Each operator $A \in \mathscr{L}^\dagger(\mathscr{D}_\pi)$ can be extended to an operator $\hat{A}$ on the whole $\mathscr{D}'_\pi$ in the following way:

$$\langle \hat{A}\xi',\eta\rangle = \langle \xi',A^\dagger\eta\rangle, \quad \forall \xi' \in \mathscr{D}'_\pi,\ \eta \in \mathscr{D}_\pi. \tag{2.4}$$

Therefore, the left and right multiplications of $X \in \mathscr{L}(\mathscr{D}_\pi,\mathscr{D}'_\pi)$ and $A \in \mathscr{L}^\dagger(\mathscr{D}_\pi)$ can always be defined:

$$(X \circ A)\xi = X(A\xi), \quad (A \circ X)\xi = \hat{A}(X\xi), \quad \forall \xi \in \mathscr{D}_\pi. \tag{2.5}$$

With these definitions, it is known that $(\mathscr{L}(\mathscr{D}_\pi,\mathscr{D}'_\pi),\mathscr{L}^\dagger(\mathscr{D}_\pi))$ is a quasi *-algebra.

Let δ be a *-derivation of $\mathscr{A}_0$ and π a *-representation of $(\mathscr{A},\mathscr{A}_0)$. Then $\pi(\mathscr{A}_0) \subset \mathscr{L}^\dagger(\mathscr{D}_\pi)$. We say that the *-derivation δ_π induced by π is *spatial* if there exists $H = H^\dagger \in \mathscr{L}(\mathscr{D}_\pi,\mathscr{D}'_\pi)$ such that

$$\delta_\pi(\pi(x)) = i\{H \circ \pi(x) - \widehat{\pi(x)} \circ H\}, \quad \forall x \in \mathscr{A}_0, \tag{2.6}$$

where $\widehat{\pi(x)}$ denotes the extension of $\pi(x)$ defined as in (2.4) (from now on, whenever no confusion may arise, we use the same notation for $\pi(x)$ and for its extension).

Let now $(\mathscr{A},\mathscr{A}_0)$ be a locally convex quasi *-algebra with topology τ. Necessary and sufficient conditions for the existence of a $(\tau - \tau_s)$-continuous, ultra-cyclic *-representation π of $\mathscr{A}$, with ultra-cyclic vector ξ_0 such that the *-derivation δ_π induced by π is spatial have been given in [3, Theorem 4.1]. We now suppose that these conditions occur, so that there exists an ultra-cyclic $(\tau - \tau_s)$-continuous *-representation π of $\mathscr{A}$ in Hilbert space $\mathscr{H}_\pi$, with ultra-cyclic vector ξ_0. Furthermore, we assume that a family of *-derivations (in the sense of Definition 2.1) $\{\delta_n : n \in \mathbb{N}\}$ of the *-algebra with identity $\mathscr{A}_0$ is given. As done in [3], we consider the related family of *-derivations $\delta_\pi^{(n)}$ induced by π defined on $\pi(\mathscr{A}_0)$ and with values in $\pi(\mathscr{A})$:

$$\delta_\pi^{(n)}(\pi(x)) = \pi(\delta_n(x)), \quad x \in \mathscr{A}_0. \tag{2.7}$$

Suppose that each $\delta_\pi^{(n)}$ is spatial and let $H_n \in \mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi)$ be the corresponding implementing operator. Assume, in addition, that

$$\sup_n \|H_n \xi_0\| =: L < \infty. \tag{2.8}$$

Then, as shown in [3, Proposition 4.3], if $\{\delta_n(x)\}$ τ-converges to $\delta(x)$, for every $x \in \mathcal{A}_0$, it turns out that δ is a $*$-derivation of $\mathcal{A}_0$ and the $*$-derivation δ_π induced by π is well defined and spatial. The relation between H_n and the operator H implementing δ_π has also been discussed.

The above statements appear to be crucial for the discussion of the existence of the dynamics of systems where a *cutoff* has been introduced, as we will see later. Examples for which these conditions are satisfied have been discussed in [3, Examples 4.4 and 4.5].

3. Applications to regularized systems

As we have discussed in the introduction, given a physical system $\mathcal{S}$, the first step in dealing with it consists in replacing $\mathcal{S}$ with a whole family of *regularized* systems $\{\mathcal{S}_L = \{\mathcal{A}_L, \Sigma, \alpha_L^t\},\ L \in \Lambda\}$, obtained by introducing some cutoff which is related to $\mathcal{S}$ itself. We suppose that the dynamics α_L^t is generated by a $*$-derivation δ_L. The procedure of the previous section suggests to introduce the following definition.

Definition 3.1. A family $\{\mathcal{S}_L,\ L \in \Lambda\}$ is said to be *c*-representable if there exists a $*$-representation π of $(\mathcal{A}, \mathcal{A}_0)$ such that

(i) π is $(\tau - \tau_s)$-continuous;
(ii) π is ultra-cyclic with ultra-cyclic vector ξ_0;
(iii) if π is such that $\pi(x) = 0$, then $\pi(\delta_L(x)) = 0$, for all $L \in \Lambda$.

Any such representation π is said to be a *c*-representation.

Proposition 3.2. *Let $\{\mathcal{S}_L, L \in \Lambda\}$ be a c-representable family and π a c-representation. Let $h_L = h_L^* \in \mathcal{A}_L$ be the element which implements δ_L: $\delta_L(x) = i[h_L, x]$, for all $x \in \mathcal{A}_0$. Suppose that the following conditions are satisfied:*

(1) *$\delta_L(x)$ is τ-Cauchy for all $x \in \mathcal{A}_0$;*
(2) $\sup_L \|\pi(h_L)\xi_0\| < \infty$.

Then,

(a) *$\delta(x) = \tau - \lim_L \delta_L(x)$ exists in $\mathcal{A}$ and is a $*$-derivation of $\mathcal{A}_0$;*
(b) *δ_π, the $*$-derivation induced by π, is well defined and spatial.*

Proof. The proof of the first statement is trivial.

We define $\delta_L^{(\pi)}(\pi(x)) = \pi(\delta_L(x))$, $x \in \mathcal{A}_0$, and $H_L^{(\pi)} = \pi(h_L)$. Then we have $\delta_L^{(\pi)}(\pi(x)) = i[h_L^{(\pi)}, \pi(x)]$, which means that $\delta_L^{(\pi)}(\pi(x))$ is spatial and it is implemented by $H_L^{(\pi)}$. In order to apply [3, Proposition 4.3], we have to check that $H_L^{(\pi)}$ satisfies the following requirements: (a) $H_L^{(\pi)} = H_L^{(\pi)\dagger}$; (b) $H_L^{(\pi)} \in \mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi)$; (c) $H_L^{(\pi)}\xi_0 \in \mathcal{H}_\pi$; (d) $\delta_L^{(\pi)}(\pi(x)) = i\{H_L^{(\pi)} \circ \pi(x) - \pi(x) \circ H_L^{(\pi)}\}$; (e) $\sup_L \|\pi(H_L)\xi_0\| < \infty$.

Condition (a) follows from the selfadjointness of h_L and from the fact that π is a $*$-representation. Condition (b) holds in an even stronger form. In fact, since h_L belongs to $\mathcal{A}_L \subset \mathcal{A}_0$, then $H_L^{(\pi)} \in \mathcal{L}^\dagger(\mathcal{D}_\pi) \subset \mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi)$. For this reason, we also have that

$H_L^{(\pi)}\xi_0 \in \mathscr{D}_\pi \subset \mathscr{H}_\pi$ while condition (d) is satisfied without even the need of using the "$\circ$" multiplication. Finally, condition (e) coincides with assumption (2). Proposition 4.3 of [3] implies therefore the statement and, in particular, it says that the implementing operator of $\delta^{(\pi)}$, $H^{(\pi)}$, satisfies the following properties: $H^{(\pi)} = H^{(\pi)\dagger}$; $H^{(\pi)} \in \mathscr{L}(\mathscr{D}_\pi, \mathscr{D}'_\pi)$; $H^{(\pi)}\xi_0 \in \mathscr{H}_\pi$; and $\delta^{(\pi)}(\pi(x)) = i\{H^{(\pi)} \circ \pi(x) - \pi(x) \circ H^{(\pi)}\}$, for all $x \in \mathscr{A}_0$. □

Remarks 3.3. (1) It is clear that if the sequence $\{h_L\}$ is τ-convergent, then assumption (2) of the above proposition is automatically satisfied, at least if L is a discrete index.

(2) It is interesting to observe also that the outcome of this proposition is that any physical system $\mathscr{S}$ whose related family $\{\mathscr{S}_L, L \in \Lambda\}$ is c-representable *admits an effective Hamiltonian in the sense of* [7].

Now we show how to use the previous results, together with some statement contained in [7], to define the time evolution of $\mathscr{S}$. We will use here quite a special strategy, which is suggested by our previous result on the existence of an effective Hamiltonian. Many other possibilities could be considered as well, and we will discuss some of them in Section 4.

First of all, we will assume that the h_L introduced in the previous section can be written in terms of some (intensive) elements s_L^α, $\alpha = 1,2,\dots,N$, which are assumed to be Hermitian (this is not a big constraint, of course), and τ-converging to some elements $s^\alpha \in \mathscr{A}$ commuting with all elements of $\mathscr{A}_0$:

$$s^\alpha = \tau - \lim_L s_L^\alpha, \qquad [s^\alpha, x] = 0, \quad \forall x \in \mathscr{A}_0. \tag{3.1}$$

It is worth remarking here that this is what happens, for instance, in all mean-field spin models, where the elements s_L^α are nothing but the mean magnetization $s_V^\alpha = (1/|V|)\sum_{i\in V}\sigma_i^\alpha$, see [4].

In order to ensure that all the powers of these elements converge, which is what happens in many concrete applications, see [4, 7] and references therein, we introduce here the following definition, which is suggested by [3].

Definition 3.4. The sequence $\{s_L^\alpha\}$ is said to be uniformly τ-continuous if, for each continuous seminorm p of τ and for all $\alpha = 1,2,\dots,N$, there exist another continuous seminorm q of τ and a positive constant $c_{p,q,\alpha}$ such that

$$p(s_L^\alpha a) \le c_{p,q,\alpha} q(a), \quad \forall a \in \mathscr{A},\ \forall L \in \Lambda. \tag{3.2}$$

Because of the properties of τ, it is easily checked that (3.2) also implies that $p(a s_L^\alpha) \le c_{p,q,\alpha} q(a)$, for all $a \in \mathscr{A}$, and that the same inequalities can be extended to s^α.

It is now straightforward to prove the following lemma.

Lemma 3.5. *If $\{s_L^\alpha\}$ is a uniformly τ-continuous sequence and if $\tau - \lim_L s_L^\alpha = s^\alpha$ for $\alpha = 1,2,\dots,N$, then $\tau - \lim_L (s_L^\alpha)^k = (s^\alpha)^k$ for $\alpha = 1,2,\dots,N$ and $k = 1,2,\dots$.*

This lemma has the following consequence. If we define the multiple commutators $[x,y]_k$ as usual ($[x,y]_1 = [x,y]$, $[x,y]_k = [x,[x,y]_{k-1}]$), then one has the following proposition.

Proposition 3.6. *Suppose that*

(1) *for all $x \in \mathcal{A}_0$, $[h_L, x]$ depends on L only through s_L^α;*

(2) *$s_L^\alpha \xrightarrow{\tau} s^\alpha$ and $\{s_L^\alpha\}$ is a uniformly τ-continuous sequence.*

Then, for each $k \in \mathbb{N}$, the following limit exists:

$$\tau - \lim_L i^k [h_L, x]_k = \tau - \lim_L \delta_L^k(x), \quad \forall x \in \mathcal{A}_0, \tag{3.3}$$

and defines an element of $\mathcal{A}$ which is called $\delta^{(k)}(x)$.

Remarks 3.7. (1) The proof is an easy extension of that given in [7] and will be omitted.

(2) Of course, we could replace condition (2) above directly with the requirement that the following limits exist: $\tau - \lim_L (s_L^\alpha)^k = (s^\alpha)^k$ for $\alpha = 1, 2, \dots, N$ and $k = 1, 2, \dots$.

(3) It is worth noticing that we have used the notation $\delta^{(k)}(x)$ instead of the more natural $\delta^k(x)$ since this last quantity could not be well defined because of domain problems, since we are not working with algebras, in general. In other words, we cannot claim that $\pi(\tau - \lim_L \delta_L^k(x)) = [H^{(\pi)}, \pi(x)]_k$, since the right-hand side could be not well defined.

In order to go on, it is convenient to introduce the following definition, see [7].

Definition 3.8. Say that $x \in \mathcal{A}_0$ is a generalized analytic element of δ if, for all t, the series $\sum_{k=0}^{\infty} (t^k/k!) \pi(\delta^{(k)}(x))$ is τ_s-convergent. The set of all generalized elements is denoted with $\mathcal{G}$.

We can now prove the following proposition.

Proposition 3.9. *Let x_γ be a net of elements of $\mathcal{A}_0$ and suppose that whenever $\pi(x_\gamma) \xrightarrow{\tau_s} \pi(x)$, then $x_\gamma \xrightarrow{\tau} x$. Then, for all $x \in \mathcal{G}$ and for all $t \in \mathbb{R}$, the series $\sum_{k=0}^{\infty} (t^k/k!) \delta^{(k)}(x)$ converges in the τ-topology to an element of $\mathcal{A}$ which is called $\alpha^t(x)$. Moreover, α^t can be extended to the τ-closure $\overline{\mathcal{G}}$ of $\mathcal{G}$.*

Proof. Because of the assumption, given a seminorm p of τ, there exist a positive constant c' and some vectors $\{\eta_j,\ j = 1, \dots, n\}$ in $\mathcal{D}_\pi$ such that $p(x) \le c' \sum_{j=1}^n p_{\eta_j}(\pi(x))$, for all $x \in \mathcal{A}$. Then we have the following. If $x \in \mathcal{G}$, $L, M \in \mathbb{N}$ with $M > L$,

$$p\left(\sum_{k=L}^{M} \frac{t^k}{k!} \delta^{(k)}(x)\right) \le c' \sum_{j=1}^{n} p_{\eta_j}\left(\sum_{k=L}^{M} \frac{t^k}{k!} \pi\left(\delta^{(k)}(x)\right)\right) \xrightarrow{L,M} 0, \tag{3.4}$$

because $x \in \mathcal{G}$. Therefore, we can put $\alpha^t(x) = \tau - \sum_{k=0}^{\infty} (t^k/k!) \delta^{(k)}(x)$.

The extension of α^t to $\overline{\mathcal{G}}$ is simply a consequence of its τ-continuity. □

Remark 3.10. It is worth remarking that the assumptions of this proposition are rather strong. In particular, for instance, the fact that for all $x \in \mathcal{A}_0$ the following estimate holds, $p(x) \le c \sum_{j=1}^n p_{\eta_j}(\pi(x))$, implies that the representation π is faithful and that π^{-1} is continuous. Moreover, the nontriviality of the set $\mathcal{G}$ must be proven case by case. It is also in view of these facts that in the next section, we further specify our algebraic setup in order to avoid the use of these strong assumptions in the analysis of the existence of α^t.

We will now discuss an approach, different from the one considered so far, to what we have called *the exponentiation problem*, that is, the possibility of deducing the existence of the time evolution for certain elements of the *-algebra $\mathcal{A}_0$ (actually a C^*-algebra in many applications) starting from some given *-derivation. In what follows, π is assumed to be a faithful *-representation of the quasi *-algebra $(\mathcal{A},\mathcal{A}_0)$ and δ a *-derivation on $\mathcal{A}_0$. As always, we will assume that the *-derivation induced by π, δ_π, is well defined on $\pi(\mathcal{A}_0)$ with values in $\pi(\mathcal{A})$.

We define the following subset of $\mathcal{A}_0$:

$$\mathcal{A}_0(\delta) := \{x \in \mathcal{A}_0 : \delta^k(x) \in \mathcal{A}_0,\ \forall k \in \mathbb{N}_0\}. \tag{3.5}$$

It is clear that $\mathcal{A}_0(\delta)$ depends on δ: the more regular δ is, the larger the set $\mathcal{A}_0(\delta)$ turns out to be. For example, if δ is inner in $\mathcal{A}_0$ and the implementing element h belongs to $\mathcal{A}_0$, then $\mathcal{A}_0(\delta) = \mathcal{A}_0$. For general δ, we can surely say that $\mathcal{A}_0(\delta)$ is not empty since it contains, at least, all the multiples of the identity $\mathbf{1}$ of $\mathcal{A}_0$.

It is straightforward to check that $\mathcal{A}_0(\delta)$ is a *-algebra which is mapped into itself by δ. Moreover, we also find that $\pi(\delta^k(x)) = \delta_\pi^k(\pi(x))$, for all $x \in \mathcal{A}_0(\delta)$ and for all $k \in \mathbb{N}_0$. This also implies that for all $k \in \mathbb{N}_0$ and for all $x \in \mathcal{A}_0(\delta)$, $\delta_\pi^k(\pi(x)) \in \pi(\mathcal{A}_0)$. This suggests to introduce the following subset of $\pi(\mathcal{A}_0)$, $\mathcal{A}_0(\delta)^\pi := \{\pi(x) \in \pi(\mathcal{A}_0) : \delta_\pi^k(\pi(x)) \in \pi(\mathcal{A}_0),\ \forall k \in \mathbb{N}_0\}$, and it is clear that $x \in \mathcal{A}_0 \Leftrightarrow \pi(x) \in \pi(\mathcal{A}_0)$.

We now introduce on $\mathcal{A}$ the topology σ_s defined via τ_s in the following way:

$$\mathcal{A} \ni a \longrightarrow q_\xi(a) = p_\xi(\pi(a)) = \|\pi(a)\xi\|, \quad \xi \in \mathcal{D}_\pi. \tag{3.6}$$

It is worth noticing that σ_s does not make of $(\mathcal{A},\mathcal{A}_0)$ a locally convex quasi *-algebra, since the multiplication is not separately continuous. We can now state the following theorem.

Theorem 3.11. *Let $(\mathcal{A},\mathcal{A}_0)$ be a quasi *-algebra with identity, δ a *-derivation on $\mathcal{A}_0$, and π a faithful *-representation of $(\mathcal{A},\mathcal{A}_0)$ such that the induced derivation δ_π is well defined. Then, the following statements hold.*

(1) *Suppose that*

$$\forall \eta \in \mathcal{D}_\pi\ \exists c_\eta > 0 : p_\eta(\delta_\pi(\pi(x))) \le c_\eta p_\eta(\pi(x)), \quad \forall x \in \mathcal{A}_0(\delta), \tag{3.7}$$

then $\sum_{k=0}^{\infty}(t^k/k!)\delta^k(x)$ converges for all t in the topology σ_s to an element of $\overline{\mathcal{A}_0(\delta)}^{\sigma_s}$ which is called $\alpha^t(x)$; α^t can be extended to $\overline{\mathcal{A}_0(\delta)}^{\sigma_s}$.

(2) *Suppose that, instead of (3.7), the following inequality holds:*

$$\begin{gathered}\exists c > 0 : \forall \eta_1 \in \mathcal{D}_\pi\ \exists A_{\eta_1} > 0, \quad n \in \mathbb{N},\ \eta_2 \in \mathcal{D}_\pi,\\ p_{\eta_1}(\delta_\pi^k(\pi(x))) \le A_{\eta_1} c^k k! k^n p_{\eta_2}(\pi(x)), \quad \forall x \in \mathcal{A}_0(\delta),\ \forall k \in \mathbb{N}_0,\end{gathered} \tag{3.8}$$

then $\sum_{k=0}^{\infty}(t^k/k!)\delta^k(x)$ *converges, for* $t<1/c$ *in the topology* σ_s *to an element of* $\overline{\mathscr{A}_0(\delta)}^{\sigma_s}$ *which is called* $\alpha^t(x)$*;* α^t *can be extended to* $\overline{\mathscr{A}_0(\delta)}^{\sigma_s}$.

Proof. (1) Iterating (3.7), we find that $p_\eta(\delta_\pi^k(\pi(x)))\le c_\eta^k p_\eta(\pi(x))$, for all natural k. It is easy to prove now the following inequality, for $x\in\mathscr{A}_0(\delta)$ and $t>0$:

$$q_\eta\left(\sum_{k=0}^{N}\frac{t^k}{k!}\delta^k(x)\right)\le\sum_{k=0}^{N}\frac{t^k}{k!}q_\eta(\delta^k(x))\le\sum_{k=0}^{N}\frac{(tc_\eta)^k}{k!}q_\eta(x)\longrightarrow e^{tc_\eta}q_\eta(x). \tag{3.9}$$

This proves the existence of $\alpha^t(x)=\sigma_s-\lim_{N,\infty}\sum_{k=0}^{N}(t^k/k!)\delta^k(x)$ for all t. The extension of α^t to $\overline{\mathscr{A}_0(\delta)}^{\sigma_s}$ by continuity is straightforward.

(2) The proof is based on analogous estimates. □

Remarks 3.12. (1) The first remark is related to the different conditions (3.7) and (3.8). The second condition is much lighter, but the price we have to pay is that α^t can be defined only on a finite interval.

(2) Condition (3.7) could be changed by requiring that the seminorms on the left- and the right-hand sides of the inequality are not necessarily the same. In this case, however, we also have to require that the constant c_η is independent of η and that $\sup_{\varphi\in\mathscr{D}_\pi}p_\varphi(\pi(x))<\infty$.

Corollary 3.13. *Under the general assumptions of Theorem 3.11, the following statements hold.*

(1) *If condition (3.7) is satisfied, then* α^t *maps* $\overline{\mathscr{A}_0(\delta)}^{\sigma_s}$ *into* $\overline{\mathscr{A}_0(\delta)}^{\sigma_s}$ *and*

$$\alpha^{t+\tau}(x)=\alpha^t(\alpha^\tau(x)),\quad \forall t,\tau,\ \forall x\in\mathscr{A}_0(\delta). \tag{3.10}$$

(2) *If condition (3.8) is satisfied, then* α^t *maps* $\overline{\mathscr{A}_0(\delta)}^{\sigma_s}$ *into* $\overline{\mathscr{A}_0(\delta)}^{\sigma_s}$ *for* $t<1/c$ *and*

$$\alpha^{t+\tau}(x)=\alpha^t(\alpha^\tau(x)),\quad \forall t,\tau,\ \textit{with}\ t+\tau<\frac{1}{c},\ \forall x\in\mathscr{A}_0(\delta). \tag{3.11}$$

Proof. (1) The proof of the first statement is a trivial consequence of the definition of α^t as given in Theorem 3.11. For all $y\in\overline{\mathscr{A}_0(\delta)}^{\sigma_s}$, then $\alpha^t(y)\in\overline{\mathscr{A}_0(\delta)}^{\sigma_s}$.

In order to prove (3.10), we begin by fixing $x\in\mathscr{A}_0(\delta)$. We have for all $\eta\in\mathscr{D}_\pi$,

$$\begin{aligned}q_\eta(\alpha^{t+\tau}(x)-\alpha^t(\alpha^\tau(x)))&\le q_\eta(\alpha^{t+\tau}(x)-\alpha_N^{t+\tau}(x))+q_\eta(\alpha_N^{t+\tau}(x)-\alpha_N^t(\alpha_N^\tau(x)))\\&\quad+q_\eta(\alpha_N^t(\alpha_N^\tau(x))-\alpha^t(\alpha^\tau(x))),\end{aligned} \tag{3.12}$$

where $\alpha_N^t(x)=\sum_{k=0}^{N}(t^k/k!)\delta^k(x)$. First we observe that, because of Theorem 3.11, $q_\eta(\alpha^{t+\tau}(x)-\alpha_N^{t+\tau}(x))\to 0$ for all t,τ and $\eta\in\mathscr{D}_\pi$. The proof of the convergence to zero of the third contribution $q_\eta(\alpha_N^t(\alpha_N^\tau(x))-\alpha^t(\alpha^\tau(x)))\to 0$ follows from the fact that for all

$\eta \in \mathcal{D}_\pi$ and $\forall \epsilon > 0$, there exists $N(\epsilon,\eta) > 0$ such that, for all $N > N(\epsilon,\eta)$,

$$\begin{aligned} q_\eta(\alpha_N^t(\alpha_N^\tau(x)) - \alpha^t(\alpha^\tau(x))) &\le q_\eta((\alpha_N^t - \alpha^t)(\alpha_N^\tau(x))) + q_\eta(\alpha^t(\alpha_N^\tau(x) - \alpha^\tau(x))) \\ &\le \sum_{k=N+1}^{\infty} \frac{(tc_\eta)^k}{k!} q_\eta(\alpha_N^t(x)) + \sum_{k=N+1}^{\infty} \frac{\tau^k}{k!} q_\eta(\alpha^t(\delta^k(x))) \\ &\le \sum_{k=N+1}^{\infty} \frac{(tc_\eta)^k}{k!} (\epsilon + q_\eta(\alpha^t(x))) + q_\eta(x) e^{tc_\eta} \sum_{k=N+1}^{\infty} \frac{(\tau c_\eta)^k}{k!} \longrightarrow 0, \end{aligned} \tag{3.13}$$

as $N \to \infty$, for all t, τ (which we are assuming to be positive here) and $x \in \mathcal{A}_0(\delta)$.

The conclusion follows from the fact that we also have

$$q_\eta(\alpha_N^{t+\tau}(x) - \alpha_N^t(\alpha_N^\tau(x))) \longrightarrow 0, \tag{3.14}$$

for all t, τ and $x \in \mathcal{A}_0(\delta)$. This can be proved by a direct estimate on the difference $\alpha_N^{t+\tau}(x) - \alpha_N^t(\alpha_N^\tau(x))$ which can be written as $\sum_{n=0}^N \sum_{l+k=n} A_{lk} - \sum_{l=0}^N \sum_{k=0}^N A_{lk}$, where we have introduced $A_{lk} = (t^l \tau^k / l! k\,!)\delta^{l+k}(x)$ for shortness. Equation (3.14) can now be proved using the same estimate as for the third contribution. The extension to $\overline{\mathcal{A}_0(\delta)}^{\sigma_s}$ is now straightforward.

(2) The proof is only a minor modification of the one above. □

Remark 3.14. In general, for fixed t, α^t is not an automorphism of $\overline{\mathcal{A}_0(\delta)}^{\sigma_s}$. This is essentially due to the fact that the multiplication is not continuous with respect to the topology σ_s.

4. The case of proper CQ^*-algebras

As discussed in the introduction, a standard assumption in the algebraic approach to quantum systems is that the *-algebra $\mathcal{A}_0$ of local observables is a C^*-algebra. For this reason, in this section, we will specialize our discussion to a particular class of quasi *-algebras, named *proper CQ^*-algebras*, that arise when completing a C^*-algebra $\mathcal{A}_0$ with respect to a weaker norm. More precisely, a proper CQ^*-algebra $(\mathcal{A}, \mathcal{A}_0)$ is constructed in the following way. Assume that $\mathcal{A}_0[\|\ \|_0]$ is a C^*-algebra and let $\|\cdot\|$ be another norm on $\mathcal{A}_0$ satisfying the following two conditions:

(i) $\|x^*\| = \|x\|$, for all $x \in \mathcal{A}_0$;

(ii) $\|xy\| \le \|x\|_0 \|y\|$, for all $x, y \in \mathcal{A}_0$.

Let $\mathcal{A}$ be the $\|\cdot\|$-completion of $\mathcal{A}_0$. The quasi *-algebra $(\mathcal{A}, \mathcal{A}_0)$ is then a proper CQ^*-algebra. For details, we refer to [1]. We remark here that the construction outlined above does not yield the most general type of proper CQ^*-algebra, but it produces the right object needed in our discussion.

The advantage of considering proper CQ^*-algebras relies on the fact that this makes it easier to use some known results which hold for bounded operators. This is convenient mainly because, as we have seen in the previous section, the fact that the implementing operator $H^{(\pi)}$ belongs to $\mathcal{L}(\mathcal{D}_\pi,\mathcal{D}'_\pi)$ makes it impossible, in general, to consider powers of $H^{(\pi)}$. For this reason, we have proposed in Section 3 a different strategy, which may appear rather peculiar. In this section, we show that some standard result can be used easily if we add an extra assumption to the sesquilinear forms which produce (and are produced by) the *-representations of CQ^*-algebras we are going to work with.

For proper CQ^*-algebras, [3, Theorem 4.1] gives the following theorem.

Theorem 4.1. *Let $(\mathcal{A},\mathcal{A}_0)$ be a proper CQ^*-algebra with unit and let δ be a *-derivation on $\mathcal{A}_0$. Then the following statements are equivalent.*

(i) *There exists a positive sesquilinear form φ on $\mathcal{A}\times\mathcal{A}$ such that φ is invariant, that is,*

$$\varphi(ax,y)=\varphi(x,a^*y),\quad \forall a\in\mathcal{A},\ x,y\in\mathcal{A}_0; \tag{4.1}$$

φ is $\|\cdot\|$-continuous, that is,

$$|\varphi(a,b)|\le\|a\|\|b\|,\quad \forall a,b\in\mathcal{A}, \tag{4.2}$$

φ satisfies the following inequalities:

$$|\varphi(\delta(x),\mathbf{1})|\le C\Big(\sqrt{\varphi(x,x)}+\sqrt{\varphi(x^*,x^*)}\Big),\quad \forall x\in\mathcal{A}_0, \tag{4.3}$$

for some positive constant C, and for all $a\in\mathcal{A}$, there exists some positive constant γ_a^2 such that

$$|\varphi(ax,ax)|\le\gamma_a^2\varphi(x,x),\quad \forall x\in\mathcal{A}_0. \tag{4.4}$$

(ii) *There exists a $(\|\cdot\|-\tau_s)$-continuous, ultra-cyclic, and bounded *-representation π of $\mathcal{A}$, with ultra-cyclic vector ξ_0, such that the *-derivation δ_π induced by π is s-spatial, that is, there exists a symmetric operator $\widehat{H}$ on the Hilbert space of the representation $\mathcal{H}_\pi$ such that*

$$\begin{gathered}D(\widehat{H})=\pi(\mathcal{A}_0)\xi_0,\\ \delta_\pi(\pi(x))\Psi=i[\widehat{H},\pi(x)]\Psi,\quad \forall x\in\mathcal{A}_0,\ \forall\Psi\in D(\widehat{H}).\end{gathered} \tag{4.5}$$

The proof of this theorem is not significantly different from that given in [3] and will be omitted here. It is worth remarking that condition (4.4) implies that the representation π_φ, constructed starting from φ as in [3], is bounded, that is, $\pi_\varphi(a)\in B(\mathcal{H}_\varphi)$ for all elements $a\in\mathcal{A}$. However, since $\mathcal{A}$ is not an algebra, ab is not defined for general $a,b\in\mathcal{A}$, while $\pi_\varphi(a)\pi_\varphi(b)$ turns out to be a bounded operator. Therefore, it has no meaning asking whether $\pi_\varphi(ab)=\pi_\varphi(a)\pi_\varphi(b)$, since the left-hand side is not well defined, unless a and/or b belongs to $\mathcal{A}_0$. In fact in this case, we can check that

$$\pi_\varphi(ax)=\pi_\varphi(a)\pi_\varphi(x),\quad \forall a\in\mathcal{A},\ \forall x\in\mathcal{A}_0. \tag{4.6}$$

We also want to remark that the proof of the implication (i)⇒(ii) is mainly based on condition (4.4) which makes it possible to use the well-known result stated, for instance, in [8, Proposition 3.2.28]. Another remark which may be of some help in concrete applications is the following. Suppose that our sesquilinear form satisfies the following modified version of (4.4): $|\varphi(yx,yx)| \leq \gamma^2\varphi(x,x)$, for all $x,y \in \mathcal{A}_0$, with γ independent of y. In this case, due to the normcontinuity of φ, condition (4.4) easily follows.

It is very easy to construct examples of positive sesquilinear forms on $\mathcal{A} \times \mathcal{A}$ satisfying (4.1), (4.2), and (4.3); in fact, the results in [6] suggest to define, on the abelian proper CQ^*-algebra $(L^p(X,\mu), C(X))$, where X a compact interval of the real line, μ the Lebesgue measure, and $p \geq 2$, a sesquilinear form as, for example, $\varphi(f,g) := \int_X f(x)\overline{g(x)}\Psi(x)d\mu$, where we take here $\Psi(x) = Ne^{\gamma x}$, and $N,\gamma > 0$ are such that $\|\Psi\|_{p/(p-2)} \leq 1$ (we put $p/(p-2) = \infty$ if $p = 2$). More difficult is to find examples of sesquilinear forms satisfying also condition (4.4). We refer to [16] for a general analysis on this (and the other) requirements, while we construct here an example in the context of Hilbert algebras, which are relevant, for example, in the Tomita-Takesaki theory.

Let $\mathcal{A}$ be an achieved Hilbert algebra with identity and $\mathcal{H}_{\mathcal{A}}$ the Hilbert space obtained by the completion of $\mathcal{A}$, see [13, 14]. For any $a \in \mathcal{A}$, we put $L_a^o b = ab$, $R_a^o b = ba$, $b \in \mathcal{A}$. Then L_a^o and R_a^o can be extended to bounded linear operators L_a and R_a on $\mathcal{H}_{\mathcal{A}}$, respectively. The sets $L_{\mathcal{A}}$ and $R_{\mathcal{A}}$ are von Neumann algebras on $\mathcal{H}_{\mathcal{A}}$, and $JL_aJ = R_{a^*}$ for all $a \in \mathcal{A}$, so that $JL_{\mathcal{A}}J = R_{\mathcal{A}}$. Here J is the isometric involution on $\mathcal{H}_{\mathcal{A}}$ which extends the involution$*$ of $\mathcal{A}$. Furthermore, for any $x \in \mathcal{H}_{\mathcal{A}}$, we define two operators on $\mathcal{A}$ as $L_x a = R_a x$ and $R_x a = L_a x$, for $a \in \mathcal{A}$ (we use the same symbol L and R since no confusion can arise). It is known, see [11], that L_x and R_x are closable operators and $L_x^* = \overline{L}_{Jx}$, $LR_x^* = \overline{R}_{Jx}$, for all $x \in \mathcal{H}_{\mathcal{A}}$.

It is also known that the Hilbert space $\mathcal{H}_{\mathcal{A}}$ over the C^*-algebra $\mathcal{A}$ with the norm $\|x\|_\flat = \|R_x\|$ ($\|\cdot\|$ is the operator norm) and with the involution $J = *$ is a proper CQ^*-algebra, see [2]. Here we consider a family $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ of Hilbert algebras. The direct sum $\bigoplus_{\lambda\in\Lambda}\mathcal{H}_{\mathcal{A}_\lambda}$ of the Hilbert spaces $\mathcal{H}_{\mathcal{A}_\lambda}$ is a proper CQ^*-algebra under the usual operations. Now we assume that one $\mathcal{A}_{\lambda_0}$ is a H*-algebra, that is, $\mathcal{A}_{\lambda_0} = \mathcal{H}_{\mathcal{A}_{\lambda_0}}$. Then we consider a *-derivation δ of $\bigoplus_{\lambda\in\Lambda}\mathcal{A}_\lambda$ satisfying $\delta P_\lambda = P_\lambda\delta$ for all $\lambda \in \Lambda$, that is, $\delta : \mathcal{A}_\lambda \to \mathcal{H}_{\mathcal{A}_\lambda}$, for all $\lambda \in \Lambda$, where P_λ is the projection of $\bigoplus_{\lambda\in\Lambda}\mathcal{H}_{\mathcal{A}_\lambda}$ onto $\mathcal{H}_{\mathcal{A}_\lambda}$. If we finally define

$$\varphi_{P_\lambda}((x_\lambda),(y_\lambda)) \equiv \langle x_{\lambda_0}, y_{\lambda_0}\rangle, \quad \forall (x_\lambda),(y_\lambda) \in \bigoplus_{\lambda\in\Lambda}\mathcal{H}_{\mathcal{A}_\lambda}, \tag{4.7}$$

then φ_{P_λ} satisfies all conditions required in Theorem 4.1.

Once the symmetric operator $\widehat{H}$ has been defined by means of this theorem, it is clear that if $\widehat{H}$ is also selfadjoint, then $e^{i\widehat{H}t}$ exists as a unitary operator in $B(\mathcal{H}_\pi)$ and $\widehat{H}$ is the generator of a one-parameter group of unitary operators on $\mathcal{H}_\pi$.

We now assume that the representation of the proper CQ^*-algebra π satisfies all the requirement Theorem 4.1(ii), so that to have an implementing $\widehat{H}$ operator for δ_π. Then we define the set

$$\mathcal{B}_0{}^\pi := \{\pi(x) \in \pi(\mathcal{A}_0) : [\widehat{H},\pi(x)]_k \in \mathcal{A}_0,\ \forall k \in \mathbb{N}_0\}, \tag{4.8}$$

which surely contains all the elements $\lambda\pi(\mathbf{1})$, $\lambda \in \mathbb{C}$. Also, if $\widehat{H}$ belongs to $\pi(\mathcal{A}_0)$, then $\mathcal{B}_0{}^\pi = \pi(\mathcal{A}_0)$. We want to show now that $\mathcal{B}_0{}^\pi = \mathcal{A}_0(\delta)^\pi$. Using a simple extension argument, we can first check that for all $x \in \mathcal{A}_0(\delta)$ for which $\pi(x) \in \mathcal{B}_0{}^\pi$, we have

$$\delta_\pi(\pi(x)) = \pi(\delta(x)) = i[\widehat{H},\pi(x)]. \tag{4.9}$$

Also, it is evident that if $\pi(x) \in \mathcal{B}_0{}^{\pi}$, then $[\hat{H},\pi(x)] \in \mathcal{B}_0{}^{\pi}$. With this in mind, we can now prove the following lemma.

Lemma 4.2. *$x \in \mathcal{A}_0(\delta)$ if and only if $\pi(x) \in \mathcal{B}_0{}^{\pi}$. For any such element,*

$$\delta_\pi^k(\pi(x)) = i^k[\hat{H},\pi(x)]_k, \quad \forall k \in \mathbb{N}_0. \tag{4.10}$$

Proof. We first take $x \in \mathcal{A}_0(\delta)$. Then (a) $x \in \mathcal{A}_0 \Rightarrow \pi(x) \in \pi(\mathcal{A}_0)$ and (b) $\delta(x) \in \mathcal{A}_0 \Rightarrow \delta_\pi(\pi(x)) = \pi(\delta(x)) \in \pi(\mathcal{A}_0)$. Since π is bounded, this implies that $\delta_\pi(\pi(x)) = i[\hat{H},\pi(x)]$ and, as a consequence, that $[\hat{H},\pi(x)] \in \pi(\mathcal{A}_0)$. The same argument, applied to $\delta(x)$ which, as we know, is still in $\mathcal{A}_0(\delta)$, produces $\delta_\pi(\pi(\delta(x))) = i[\hat{H},\pi(\delta(x))]$ which, after few computations, produces $\delta_\pi^2(\pi(x)) = i^2[\hat{H},\pi(x)]_2$ which still belongs to $\pi(\mathcal{A}_0)$ since $\delta_\pi^2(\pi(x)) = \pi(\delta^2(x)) \in \pi(\mathcal{A}_0)$. Iterating this procedure, we find that $\delta_\pi^k(\pi(x)) = i^k[\hat{H},\pi(x)]_k$ and $[\hat{H},\pi(x)]_k \in \pi(\mathcal{A}_0)$ for all $k \in \mathbb{N}_0$. Therefore $\pi(x) \in \mathcal{B}_0{}^{\pi}$.

In a similar way, we can also prove that if $\pi(x) \in \mathcal{B}_0{}^{\pi}$, then $x \in \mathcal{A}_0(\delta)$ and $\delta_\pi^k(\pi(x)) = i^k[\hat{H},\pi(x)]_k$ for all $k \in \mathbb{N}_0$. □

Remark 4.3. It may be worth recalling that Lemma 4.2 also implies that $\mathcal{A}_0(\delta)^{\pi} = \mathcal{B}_0{}^{\pi}$.

It is now straightforward to use this lemma and Theorem 3.11 to find conditions under which the sequence $\alpha_N^t(x) = \sum_{k=0}^N (t^k/k!)\delta^k(x)$ is σ_s-convergent and defines the time evolution of x, $\alpha^t(x)$, for $x \in \mathcal{A}_0(\delta)$. Each one of the following conditions can be used to deduce the existence of $\alpha^t(x)$.

Condition 4.4. For all $\eta \in \mathcal{D}_\pi$, there exists $c_\eta > 0$ such that

$$p_\eta([\hat{H},\pi(x)]) \le c_\eta p_\eta(\pi(x)) = c_\eta q_\eta(x), \quad \forall x \in \mathcal{A}_0(\delta). \tag{4.11}$$

In this case, $\alpha^t(x)$ exists for all values of t.

Condition 4.5. There exists $c > 0$: for all $\eta_1 \in \mathcal{D}_\pi$, there exist A_{η_1}, $n \in \mathbb{N}$ and $\eta_2 \in \mathcal{D}_\pi$ such that

$$p_{\eta_1}([\hat{H},\pi(x)]_k) \le A_{\eta_1} c^k k! k^n q_{\eta_2}(x), \quad \forall x \in \mathcal{A}_0(\delta),\ \forall k \in \mathbb{N}. \tag{4.12}$$

In this case, $\alpha^t(x)$ exists for all values of $t < 1/c$.

As we have already shown before, α^t can be extended to $\overline{\mathcal{A}_0(\delta)}^{\sigma_s}$ and is a semigroup.

We adapt now [3, Proposition 4.3] to the present setting. For that, we again consider a family of $*$-derivations of $\mathcal{A}_0$ and a single representation π with the properties required in Theorem 4.1(ii). We remind that this is the most common situation in physical applications.

Proposition 4.6. *Let $\{\mathcal{S}_L, L \in \Lambda\}$ be a c-representable family such that the corresponding c-representation π is bounded. Also, suppose that the following conditions hold.*

(1) *$\delta_n(x)$ is $\|\,\|$-Cauchy for all $x \in \mathcal{A}_0$.*

(2) *For all $n \in \mathbb{N}$, the induced *-derivation $\delta_\pi^{(n)}$ is s-spatial, that is, a symmetric operator $\widehat{H}_n$ on $\mathcal{H}_\pi$ exists such that*

$$D(\widehat{H}_n) = \pi(\mathcal{A}_0)\xi_0, \\ \delta_\pi^{(n)}(\pi(x))\Psi = i[\widehat{H}_n, \pi(x)]\Psi, \quad \forall x \in \mathcal{A}_0, \ \forall \Psi \in D(\widehat{H}_n). \tag{4.13}$$

(3) $\sup_n \|\widehat{H}_n\xi_0\| = L < \infty$.

Then,

(a) $\delta(x) = \|\cdot\| - \lim_n \delta_n(x)$ *exists in $\mathcal{A}$ and is a *-derivation of $\mathcal{A}_0$;*

(b) δ_π, *the *-derivation induced by π, is well defined and s-spatial. There exists a symmetric operator $\widehat{H}$ on $\mathcal{H}_\pi$ such that*

$$D(\widehat{H}) = \pi(\mathcal{A}_0)\xi_0, \\ \delta_\pi(\pi(x))\Psi = i[\widehat{H}, \pi(x)]\Psi, \quad \forall x \in \mathcal{A}_0, \ \forall \Psi \in D(\widehat{H}); \tag{4.14}$$

(c) *if* $\langle \widehat{H}_n\xi_0, \pi(y)\xi_0\rangle \to \langle \widehat{H}\xi_0, \pi(y)\xi_0\rangle$ *for all* $y \in \mathcal{A}_0$, *then* $\langle \widehat{H}_n\pi(x)\xi_0, \pi(y)\xi_0\rangle \to \langle \widehat{H}\pi(x)\xi_0, \pi(y)\xi_0\rangle$ *for all* $x, y \in \mathcal{A}_0$;

(d) *if* $\|(\widehat{H}_n - \widehat{H})\xi_0\| \to 0$ *for all* $y \in \mathcal{A}_0$, *then* $\|(\widehat{H}_n - \widehat{H})\pi(x)\xi_0\| \to 0$ *for all* $x \in \mathcal{A}_0$.

Proof. The first three statements can be proven in quite the same way as in [3].

The proof of the statement (d) is a consequence of the definition of the implementing operator of an *s*-spatial derivation as it can be deduced by [8, Proposition 3.2.8]. We have

$$\widehat{H}_n\pi(x)\xi_0 = \frac{1}{i}\delta_\pi^{(n)}(\pi(x))\xi_0 + \pi(x)\widehat{H}_n\xi_0, \\ \widehat{H}\pi(x)\xi_0 = \frac{1}{i}\delta_\pi(\pi(x))\xi_0 + \pi(x)\widehat{H}\xi_0, \quad \forall x \in \mathcal{A}_0. \tag{4.15}$$

Therefore,

$$||(\widehat{H}_n - \widehat{H})\pi(x)\xi_0|| \le ||(\delta_\pi^{(n)}(\pi(x)) - \delta_\pi(\pi(x)))\xi_0|| + ||(\widehat{H}_n - \widehat{H})\xi_0|| \longrightarrow 0 \tag{4.16}$$

because of the assumptions on $\widehat{H}_n$ and π. □

5. Concluding remarks

As we have discussed in the introduction, in this paper, we have chosen to regularize only the algebra related to a physical system, leaving the set of states unchanged. However, in some approaches discussed in the literature, see [12] for instance, a *cutoff* is introduced for both the states and the algebra. If we consider for a moment this point of view here, we wonder what can be said if we have a family of positive sesquilinear forms φ_n on $\mathcal{A}_0 \times \mathcal{A}_0$ instead of a single one on $\mathcal{A} \times \mathcal{A}$. The simplest situation, which is the only one we will consider here, is when the family $\{\varphi_n, n \in \mathbb{N}\}$ satisfies the following requirements:

(i) $\varphi_n(xy, z) = \varphi_n(y, x^*z)$ for all $x, y, z \in \mathcal{A}_0$, for all $n \in \mathbb{N}$;

(ii) $|\varphi_n(x, y)| \le \|x\|\|y\|$ for all $x, y \in \mathcal{A}_0$, for all $n \in \mathbb{N}$;

(iii) the sequence $\{\varphi_n(x, y)\}_{n\in\mathbb{N}}$ is Cauchy for all $x, y \in \mathcal{A}_0$.

The second condition allows us to extend each φ_n to the whole $\mathcal{A} \times \mathcal{A}$. This extension $\tilde{\varphi}_n$ satisfies the same conditions as above. Furthermore, since $\{\tilde{\varphi}_n(a,b)\}_n$ is a Cauchy sequence for all $a,b \in \mathcal{A}$, we can also define a new positive sesquilinear form Φ on $\mathcal{A} \times \mathcal{A}$:

$$\Phi(a,b) = \lim_n \tilde{\varphi}_n(a,b), \quad \forall a,b \in \mathcal{A}. \tag{5.1}$$

It is clear that also Φ is $*$-invariant and $\|\cdot\|$-continuous. If we also have that

(i) $|\tilde{\varphi}_n(\delta(x),\mathbf{1})| \le C\sqrt{\varphi_n(x,x)+\varphi_n(x^*,x^*)}$ for all $x \in \mathcal{A}_0$, for all $n \in \mathbb{N}$, and for some positive C;

(ii) for all $a \in \mathcal{A}$, there exists $\gamma_a > 0 : |\tilde{\varphi}_n(ax,ax)| \le \gamma_a^2 \varphi_n(x,x)$ for all $x \in \mathcal{A}_0$, for all $n \in \mathbb{N}$,

then we easily extend these properties to Φ so that we get a positive sesquilinear form on $\mathcal{A} \times \mathcal{A}$ satisfying all the requirements of Theorem 4.1(ii). Therefore we have two different possibilities, at least if we are dealing with a single derivation δ.

First possibility. we use each $\tilde{\varphi}_n$ to construct, using Theorem 4.1, a $*$-representation π_n and an induced derivation $\delta_{\pi_n}(\pi_n(x)) = \pi_n(\delta(x))$, $x \in \mathcal{A}_0$, which turns out to be s-spatial. Therefore we find a sequence of symmetric operators $\widehat{H}_n$ acting in possibly different Hilbert spaces $\mathcal{H}_n$.

Second possibility. we use Φ to construct, using again Theorem 4.1, a single $*$-representation π and an induced derivation $\delta_\pi(\pi(x)) = \pi(\delta(x))$, $x \in \mathcal{A}_0$, which is s-spatial. Therefore we get a symmetric operator $\widehat{H}$ acting on the Hilbert space of the representation $\mathcal{H}$.

Both of these possibilities have a certain interest. We will analyze in a forthcoming paper the details of these constructions and the relations between $\widehat{H}$ and $\widehat{H}_n$.

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