EXPANSION OF α-OPEN SETS AND DECOMPOSITION OF α-CONTINUOUS MAPPINGS

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We introduce the notions of expansion $\mathcal{A}_\alpha$ of α-open sets and $\mathcal{A}_\alpha$-expansion α-continuous mappings in topological spaces. The main result of this paper is that a map f is α-continuous if and only if it is $\mathcal{A}_\alpha$-expansion α-continuous and $\mathcal{B}_\alpha$-expansion α-continuous, where $\mathcal{A}_\alpha$, $\mathcal{B}_\alpha$ are two mutually dual expansions.

1. Introduction

In 1965, Njastad [2] introduced the notion of α-sets in topological space. In 1983, Mashhour et al. [1] introduced, with the help of α-sets, a weak form of continuity which they termed as α-continuity. Noiri [3] introduced the same concept, but under the name strong semicontinuity. Noiri [4] defined with the aid of α-sets a new weakened form of continuous mapping called weakly α-continuous mapping. Sen and Bhattacharyya [5] introduced another new weakened form of continuity called weak $*\alpha$-continuity and proved that a mapping is α-continuous if and only if it is weakly α-continuous and weak $*\alpha$-continuous.

In this paper, we give a general setting for such decompositions of α-continuity by using expansion of α-open sets, whereas in [6], Tong used expansion of open sets to give a general setting for the decomposition of continuous mapping into weakly continuous and weak * continuous mappings.

2. Preliminaries

Throughout this paper, (X,τ), (Y,σ), and so forth (or simply X, Y, etc.) will always denote topological spaces. The family of all α-open sets in X is denoted by τ_α.

We recall the definition of weakly α-continuous and weak $*\alpha$-continuous mappings.

Definition 2.1 [1]. A mapping $f: X \to Y$ is said to be α-continuous if for each open set V in Y, $f^{-1}(V) \in \tau_\alpha$.

Definition 2.2 [3]. A mapping $f: X \to Y$ is said to be weakly α-continuous if for each x in X and for each open set V in Y containing $f(x)$, there exists a set $U \in \tau_\alpha$ containing x such that $f(U) \subseteq \operatorname{Cl} V$, where $\operatorname{Cl} V$ means the closure of V.

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PROPOSITION 2.3 [5]. *A mapping $f : X \to Y$ is weakly α-continuous if and only if $f^{-1}(V) \subseteq \alpha\operatorname{int}[f^{-1}(\operatorname{Cl} V)]$, for every open set V in Y, where $\alpha\operatorname{int}(A)$ means α-interior of A.*

Definition 2.4 [5]. A mapping $f : X \to Y$ is said to be weak $^*\alpha$-continuous if and only if for every open set $V \subseteq Y$, f^{-1} $(\operatorname{Fr} V)$ is α-closed in X, where $\operatorname{Fr} V = \operatorname{Cl} V \setminus V$ is the boundary operator for open sets.

3. Decompositions of α-continuity

Definition 3.1. Let (X,τ) be a topological space, let 2^x be the set of all subsets in X. A mapping $\mathcal{A}_\alpha : \tau_\alpha \to 2^x$ is said to be an expansion on X if $U \subseteq \mathcal{A}_\alpha U$ for each $U \in \tau_\alpha$.

Remark 3.2. If γ_α is the identity expansion, then γ_α is defined by $\gamma_\alpha U = U$. μ_α defined by $\mu_\alpha U = (\alpha\operatorname{int} U \cap U^c)^c$ is an expansion. Cl_α defined by $\operatorname{Cl}_\alpha U = \operatorname{Cl} U$ and $\mathcal{F}_\alpha U$ defined by $\mathcal{F}_\alpha U = (\operatorname{Fr} U)^c$ are expansions.

Definition 3.3 [6]. Let (X,τ) be a topological space. A pair of expansions $\mathcal{A}$ and $\mathcal{B}$ on X is said to be mutually dual if $\mathcal{A}U \cap \mathcal{B}U = U$ for each $U \in \tau$.

Remark 3.4. Let (X,τ) be a topological space. Then Cl_α and $\mathcal{F}_\alpha$ are mutually dual. This follows from [6, Proposition 2].

Example 3.5. Let $X = \{a,b,c\}$ with topologies $\tau = \{\phi,\{a\},X\}$, $\tau_\alpha = \{\phi,\{a\},\{a,b\},\{a,c\},X\}$. $\mathcal{A}_\alpha(\phi) = \phi$, $\mathcal{A}_\alpha\{a\} = \{a\}$, $\mathcal{A}_\alpha\{a,b\} = \{a,b\}$, $\mathcal{A}_\alpha\{a,c\} = X$, and $\mathcal{A}_\alpha(X) = X$. Then $\mathcal{A}_\alpha$ is an expansion. Let $\mathcal{B}_1(\phi) = X$, $\mathcal{B}_1\{a\} = \{a,c\}$, $\mathcal{B}_1\{a,b\} = X$, $\mathcal{B}_1\{a,c\} = \{a,c\}$, and $\mathcal{B}_1(X) = X$. $\mathcal{B}_2(\phi) = X$, $\mathcal{B}_2\{a\} = \{a,b\}$, $\mathcal{B}_2\{a,b\} = \{a,b\}$, $\mathcal{B}_2\{a,c\} = \{a,c\}$, and $\mathcal{B}_2(X) = X$. Then $\mathcal{B}_1$, $\mathcal{B}_2$ are both mutually dual to $\mathcal{A}_\alpha$.

PROPOSITION 3.6. *Let (X,τ) be a topological space. Then γ_α and μ_α are mutually dual.*

Proof.

$$\begin{aligned}
(\gamma_\alpha U) \cap (\mu_\alpha U) &= U \cap (\alpha\operatorname{int} U \cap U^c)^c \\
&= (\alpha\operatorname{int} U) \cap (\alpha\operatorname{int} U \cap U^c)^c \\
&= (\alpha\operatorname{int} U) \cap ((\alpha\operatorname{int} U)^c \cup U) \\
&= ((\alpha\operatorname{int} U) \cap (\alpha\operatorname{int} U)^c) \cup (\alpha\operatorname{int} U \cap U) \\
&= \phi \cup U = U.
\end{aligned} \tag{3.1}$$

□

Definition 3.7. Let (X,τ) and (Y,σ) be two topological spaces and let A_α be an expansion on Y. Then the mapping $f : X \to Y$ is said to be $\mathcal{A}_\alpha$-expansion α-continuous if $f^{-1}(V) \subseteq \alpha\operatorname{int}[f^{-1}(\mathcal{A}_\alpha V)]$, for each $V \in \sigma$.

Remark 3.8. A weakly α-continuous mapping $f : X \to Y$ can be renamed as Cl_α-expansion α-continuous mapping.

THEOREM 3.9. *Let (X,τ) and (Y,σ) be two topological spaces and $\mathcal{A}_\alpha$, $\mathcal{B}_\alpha$ are two mutually dual expansions on Y. Then the mapping $f : X \to Y$ is α-continuous if and only if f is $\mathcal{A}_\alpha$-expansion α-continuous and $\mathcal{B}_\alpha$-expansion α-continuous.*

Proof. Necessity. Suppose that f is α-continuous. Since $\mathcal{A}_\alpha$, $\mathcal{B}_\alpha$ are mutually dual on Y, $\mathcal{A}_\alpha V \cap \mathcal{B}_\alpha V = V$ for each $V \in \sigma$.

Then

$$\begin{aligned} f^{-1}(V) &= f^{-1}(\mathcal{A}_\alpha V \cap \mathcal{B}_\alpha V) \\ &= f^{-1}(\mathcal{A}_\alpha V) \cap f^{-1}(\mathcal{B}_\alpha V). \end{aligned} \tag{3.2}$$

Since f is α-continuous, $f^{-1}(V) = \alpha \operatorname{int} f^{-1}(V)$.

Therefore,

$$\begin{aligned} f^{-1}(V) = \alpha \operatorname{int} f^{-1}(V) &= \alpha \operatorname{int} (f^{-1})(\mathcal{A}_\alpha V \cap \mathcal{B}_\alpha V) \\ &= \alpha \operatorname{int} f^{-1}(\mathcal{A}_\alpha V) \cap \alpha \operatorname{int} f^{-1}(\mathcal{B}_\alpha V). \end{aligned} \tag{3.3}$$

This implies that $f^{-1}(V) \subseteq \alpha \operatorname{int} f^{-1}(\mathcal{A}_\alpha V)$ and $f^{-1}(V) \subseteq \alpha \operatorname{int} f^{-1}(\mathcal{B}_\alpha V)$. This shows that f is $\mathcal{A}_\alpha$-expansion α-continuous and $\mathcal{B}_\alpha$-expansion α-continuous.

Sufficiency. Since f is $\mathcal{A}_\alpha$-expansion α-continuous, $f^{-1}(V) \subseteq \alpha \operatorname{int} f^{-1}(\mathcal{A}_\alpha V)$ for each $V \in \sigma$. Since f is $\mathcal{B}_\alpha$-expansion α-continuous, $f^{-1}(V) \subseteq \alpha \operatorname{int} f^{-1}(\mathcal{B}_\alpha V)$ for each $V \in \sigma$. As $\mathcal{A}_\alpha$ and $\mathcal{B}_\alpha$ are two mutually dual expansions on Y, $\mathcal{A}_\alpha V \cap \mathcal{B}_\alpha V = V$,

$$\begin{gathered} f^{-1}(V) = f^{-1}(\mathcal{A}_\alpha V \cap \mathcal{B}_\alpha V) = f^{-1}(\mathcal{A}_\alpha V) \cap f^{-1}(\mathcal{B}_\alpha V), \\ \alpha \operatorname{int} f^{-1}(V) = \alpha \operatorname{int} f^{-1}(\mathcal{A}_\alpha V) \cap \alpha \operatorname{int} f^{-1}(\mathcal{B}_\alpha V) \supseteq f^{-1}(V) \cap f^{-1}(V) = f^{-1}(V). \end{gathered} \tag{3.4}$$

This implies that $f^{-1}(V) \subseteq \alpha \operatorname{int} f^{-1}(V)$. Always, $\alpha \operatorname{int} f^{-1}(V) \subseteq f^{-1}(V)$. So $f^{-1}(V) = \alpha \operatorname{int} f^{-1}(V)$. Therefore, $f^{-1}(V)$ is an α-open set in X for each $V \in \sigma$. Hence f is α-continuous. □

Definition 3.10. Let (X,τ) and (Y,σ) be two topological spaces, $\mathcal{B}_\alpha$ an expansion on Y. Then a mapping $f : X \to Y$ is said to be α-closed $\mathcal{B}_\alpha$-continuous if $f^{-1}((\mathcal{B}_\alpha V)^c)$ is an α-closed set in X for each $V \in \sigma$.

Remark 3.11. A weak $^*\alpha$-continuous mapping can be renamed as α-closed $\mathcal{F}_\alpha$-continuous mapping since $(\mathcal{F}_\alpha V)^c = (\operatorname{Fr} V)^c = \operatorname{Fr} V$.

Proposition 3.12. *An α-closed $\mathcal{B}_\alpha$-continuous mapping is $\mathcal{B}_\alpha$-expansion α-continuous.*

Proof. First, we prove that $(f^{-1}((\mathcal{B}_\alpha V)^c))^c = f^{-1}(\mathcal{B}_\alpha V)$.

Let $x \in (f^{-1}((\mathcal{B}_\alpha V)^c))^c$. Then $x \notin (f^{-1}(\mathcal{B}_\alpha V)^c)$. Hence $f(x) \notin (\mathcal{B}_\alpha V)^c$, $f(x) \in \mathcal{B}_\alpha V$, and $x \in f^{-1}(\mathcal{B}_\alpha V)$.

Conversely, if $x \in f^{-1}(\mathcal{B}_\alpha V)$, then $f(x) \in \mathcal{B}_\alpha V$. Hence $f(x) \notin (\mathcal{B}_\alpha V)^c$, $x \notin f^{-1}(\mathcal{B}_\alpha V)^c$, $x \in (f^{-1}((\mathcal{B}_\alpha V)^c))^c$. Therefore, $(f^{-1}((\mathcal{B}_\alpha V)^c))^c = f^{-1}(\mathcal{B}_\alpha V)$.

Since $f^{-1}((\mathcal{B}_\alpha V)^c)$ is an α-closed set in X, $(f^{-1}((\mathcal{B}_\alpha V)^c))^c$ is an α-open set in X. Hence $f^{-1}(\mathcal{B}_\alpha V)$ is an α-open in X and $f^{-1}(\mathcal{B}_\alpha V) = \alpha \operatorname{int} f^{-1}(\mathcal{B}_\alpha V)$.

Since $\mathcal{B}_\alpha$ is an expansion on Y, $V \subseteq \mathcal{B}_\alpha V$, we have $f^{-1}(V) \subseteq f^{-1}(\mathcal{B}_\alpha V) = \alpha \operatorname{int} f^{-1}(\mathcal{B}_\alpha V)$. Therefore, f is $\mathcal{B}_\alpha$-expansion α-continuous. □

By Theorem 3.9 and Proposition 3.12, we have the following corollary.

Corollary 3.13. *Let (X,τ) and (Y,σ) be two topological spaces and $\mathcal{A}_\alpha$, $\mathcal{B}_\alpha$ are two mutually dual expansions on Y. Then a mapping $f : X \to Y$ is α-continuous if and only if f is $\mathcal{A}_\alpha$-expansion α-continuous, and α-closed $\mathcal{B}_\alpha$-continuous.*

By Remarks 3.8, 3.11, and by the above corollary, we have the following corollary.

Corollary 3.14 [5]. *A mapping is α-continuous if and only if it is weakly α-continuous and weak $^*\alpha$-continuous.*

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