sn-METRIZABLE SPACES AND RELATED MATTERS

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We give a mapping theorem on *sn*-metrizable spaces, discuss relationships among spaces with point-countable *sn*-networks, spaces with uniform *sn*-networks, spaces with locally countable *sn*-networks, spaces with σ-locally countable *sn*-networks, and *sn*-metrizable spaces, and obtain some related results.

1. Introduction and definitions

sn-networks were first introduced by Lin [12], which are the concept between weak bases and *cs*-networks. *sn*-metrizable spaces [6] (i.e., spaces with σ-locally finite *sn*-networks) are one class of generalized metric spaces, and they play an important role in metrization theory, see [6, 13]. In this paper, we give a mapping theorem on *sn*-metrizable spaces, discuss relationships among spaces with point-countable *sn*-networks, spaces with uniform *sn*-networks, spaces with locally countable *sn*-networks, spaces with σ-locally countable *sn*-networks, and *sn*-metrizable spaces, and obtain some related results.

In this paper, all spaces are regular and T_1, all mappings are continuous and surjective. $\mathbb{N}$ denotes the set of all natural numbers. ω denotes $\mathbb{N} \cup \{0\}$. For a family $\mathcal{P}$ of subsets of a space X and $x \in X$, denote $(\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\}$. For two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of X, denote $\mathcal{A} \wedge \mathcal{B} = \{A \cap B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$.

Definition 1.1. Let $f : X \to Y$ be a mapping.

(1) f is called a σ-mapping [1] if there exists a base $\mathcal{B}$ for X such that $f(\mathcal{B})$ is a σ-locally finite family of subsets of Y.

(2) f is called a sequence-covering mapping [19] if each convergent sequence (including its limit point) of Y is the image of some convergent sequence (including its limit point) of X.

(3) f is called a 1-sequence-covering mapping [12] if for each $y \in Y$, there exists $x \in f^{-1}(y)$ satisfying the following condition. Whenever $\{y_n\}$ is a sequence of Y converging to a point y in Y, there exists a sequence $\{x_n\}$ of X converging to a point x in X such that each $x_n \in f^{-1}(y_n)$.

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Definition 1.2. Let $\mathcal{P}$ be a cover of a space X.

(1) $\mathcal{P}$ is called a k-network [18] for X if for each compact subset K of X and its open neighborhood V, there exists a finite subfamily $\mathcal{P}'$ of $\mathcal{P}$ such that $K \subset \cup\mathcal{P}' \subset V$.

(2) $\mathcal{P}$ is called a *cs*-network for X if for each $x \in X$, its open neighborhood V, and a sequence $\{x_n\}$ converging to x, there exists $P \in \mathcal{P}$ such that $\{x_n : n \geq m\} \cup \{x\} \subset P \subset V$ for some $m \in \mathbb{N}$.

(3) $\mathcal{P}$ is called a cs^*-network for X if for each $x \in X$, its open neighborhood V, and a sequence $\{x_n\}$ converging to x, there exist $P \in \mathcal{P}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k} : k \in \mathbb{N}\} \cup \{x\} \subset P \subset V$.

(4) X is called an $\aleph$-space if X has a σ-locally finite k-network.

Definition 1.3 [5]. Let X be a space, and $P \subset X$. Then, the following hold.

(1) A sequence $\{x_n\}$ in X is called eventually in P, if $\{x_n\}$ converges to x, and there exists $m \in \mathbb{N}$ such that $\{x\} \cup \{x_n : n \geq m\} \subset P$.
(2) P is called a sequential neighborhood of x in X, if whenever a sequence $\{x_n\}$ in X converges to x, then $\{x_n\}$ is eventually in P.
(3) P is called sequential open in X if P is a sequential neighborhood of each of its points.
(4) X is called a sequential space if any sequential open subset of X is open in X.

Definition 1.4. Let $\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$ be a family of subsets of a space X satisfying that for each $x \in X$, the following exist.

(a) $\mathcal{P}_x$ is a network of x in X (i.e., $x \in \bigcap\mathcal{P}_x$ and for each neighborhood U of x in X, $P \subset U$ for some $P \in \mathcal{P}_x$).
(b) If $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.
(1) $\mathcal{P}$ is called a weak base [3] for X if $G \subset X$ such that for each $x \in G$, there exists $P \in \mathcal{P}_x$ satisfying $P \subset G$, then G is open in X, here $\mathcal{P}_x$ is called a weak base of x in X.
(2) $\mathcal{P}$ is called an *sn*-network [12] for X if each element of $\mathcal{P}_x$ is a sequential neighborhood of x in X, here $\mathcal{P}_x$ is called an *sn*-network of x in X.
(3) X is called *sn*-metrizable [6] (resp., g-metrizable [20]) if X has a σ-locally finite *sn*-network (resp., weak-base).
(4) X is called *sn*-first countable [13] (resp., g-first countable) if X has an *sn*-network $\mathcal{P}$ (resp., weak-base) such that each $\mathcal{P}_x$ is countable.

Definition 1.5. Let $\mathcal{P}$ be a cover of a space X.

(1) $\mathcal{P}$ is called a uniform cover for X [2], if for each $x \in X$, whenever $\mathcal{P}'$ is a countable infinite subset of $(\mathcal{P})_x$, then $\mathcal{P}'$ is a network of x in X (i.e., $x \in \bigcap\mathcal{P}'$ and for each neighborhood U of x in X, $P \subset U$ for some $P \in \mathcal{P}'$).

(2) $\mathcal{P}$ is called a uniform *sn*-network (resp., weak base, *cs*-network) for X if $\mathcal{P}$ is both a uniform cover and *sn*-network (resp., weak base, *cs*-network) for X.

Remark 1.6. (1) For a space, weak base $\Rightarrow$ *sn*-network $\Rightarrow$ *cs*-network $\Rightarrow$ cs^*-network. An *sn*-network for a sequential space is a weak base [12].

(2) g-metrizable spaces $\Rightarrow$ *sn*-metrizable spaces $\Rightarrow$ $\aleph$-spaces $\Leftrightarrow$ spaces with σ-locally finite *cs*-networks $\Leftrightarrow$ spaces with σ-locally finite cs^*-networks [4, 11].

(3) g-first countable spaces ⇔ sequential, sn-first countable spaces.

(4) Spaces with uniform weak-bases ⇔ sequential spaces with uniform sn-networks [12].

2. The characterization of spaces with uniform *sn*-networks

Lemma 2.1 [15]. *The following are equivalent for a space X.*

(1) *X is a 1-sequence-covering compact image of a metric space.*

(2) *X is a sequence-covering compact image of a metric space.*

(3) *X has a uniform sn-network.*

(4) *X has a uniform cs-network.*

From Lemma 2.1 and [12, Proposition 2.3], we have the following theorem.

Theorem 2.2. *Let X be a space with a uniform sn-network. Then X has a point-countable sn-network.*

Theorem 2.3. *The following are equivalent for a space X.*

(1) *X has a uniform base.*

(2) *X is a Fréchet space with a uniform sn-network.*

(3) *X is a sequential space with a uniform sn-network and contains no closed copy of S_2.*

Proof. (1)⇒(2) is clear.

(2)⇒(3) holds by [14, Corollary 2.1.11] and the fact that a space with a uniform sn-network has a point-countable sn-network.

(3)⇒(1). Suppose that X is a Fréchet space with a uniform sn-network. From Lemma 2.1, X is a sequence-covering compact image of a metric space. Let f be a sequence-covering compact map from the metric space M onto X. Then, by [11, Proposition 2.1.16(2)], f is quotient. Since X is Fréchet, then f is pseudo-open (see [11, Proposition 2.1.16(3)]). Hence X has a uniform base (see [11, Theorem 2.9.18]). □

3. The characterization of *sn*-metrizable spaces

Lemma 3.1 [6]. *The following are equivalent for a space X.*

(1) *X is sn-metrizable.*

(2) *X has a σ-discrete sn-network.*

(3) *X is an sn-first countable and $\aleph$-space.*

Theorem 3.2. *The following are equivalent for a space X.*

(1) *X is sn-metrizable.*

(2) *X is a sequence-covering, compact, and σ-image of a metric space.*

(3) *X is a 1-sequence-covering and σ-image of a metric space.*

Proof. (1)⇒(2). Suppose X is sn-metrizable. From Lemma 3.1, X has a σ-discrete sn-network $\mathcal{F}$. Since X is regular, we can assume that each element of $\mathcal{F}$ is closed in X. Put $\mathcal{F} = \cup\{\mathcal{B}_i : i \in \mathbb{N}\} = \cup\{\mathcal{F}_x : x \in X\}$, where $\mathcal{B}_i$ is a discrete family of closed sets of X, and $\mathcal{F}_x$ is a weak base of x in X. For each $i \in \mathbb{N}$, let $Q_i = \{x \in X : \mathcal{F}_x \cap \mathcal{B}_i = \phi\}$, $\mathcal{P}_i = \mathcal{B}_i \cup \{Q_i, X\}$, $\mathcal{P} = \cup\{\mathcal{P}_i : i \in \mathbb{N}\}$. Then $\mathcal{P}_i$ is a locally finite cover of X, and $\mathcal{P}$ is a σ-locally finite *cs*-network for X. Let $\mathcal{P}_i = \{P_\alpha : \alpha \in A_i\}$, where $\mathcal{P}_i$ is closed under finite intersections and $X \in \mathcal{P}_i \subset \mathcal{P}_{i+1}$. For each $i \in \mathbb{N}$, endow A_i with discrete topology, then A_i is a metric space. Put

$$M = \left\{\alpha = (\alpha_i) \in \prod_{i\in\mathbb{N}} A_i : \{P_{\alpha_i} : i \in \mathbb{N}\} \subset \mathcal{P} \text{ forms a network at some point } x(\alpha) \in X\right\}, \tag{3.1}$$

and endow M with the subspace topology induced from the usual product topology of the family $\{A_i : i \in \mathbb{N}\}$ of metric spaces, then M is a metric space. Since X is Hausdorff, $x(\alpha)$ is unique in X for each $\alpha \in M$. We define $f : M \to X$ by $f(\alpha) = x(\alpha)$ for each $\alpha \in M$. Because $\mathcal{P}$ is a σ-locally finite *cs*-network for X, then f is surjective. For each $\alpha = (\alpha_i) \in M$, $f(\alpha) = x(\alpha)$. Suppose that V is an open neighborhood of $x(\alpha)$ in X. Then there exists $n \in \mathbb{N}$ such that $x(\alpha) \in P_{\alpha_n} \subset V$. Set $W = \{c \in M :$ the nth coordinate of c is $\alpha_n\}$. Then W is an open neighborhood of α in M, and $f(W) \subset P_{\alpha_n} \subset V$. Hence f is continuous. We will show that f is a sequence-covering, compact, and σ-mapping.

(i) f is sequence-covering.

For each sequence $\{x_n\}$ converging to x_0, we can assume that all $x_n's$ are distinct, and that $x_n \neq x_0$ for each $n \in \mathbb{N}$. Set $K = \{x_m : m \in \omega\}$. Suppose that V is an open neighborhood of K in X. A subfamily $\mathcal{A}$ of $\mathcal{P}_i$ is called to hold the following property, which is denoted by $F(K,V)$:

(a) $\mathcal{A}$ is finite;
(b) for each $P \in \mathcal{A}$, $\phi \neq P \cap K \subset P \subset V$;
(c) for each $z \in K$, exists unique $P_z \in \mathcal{A}$ such that $z \in \mathcal{P}_z$;
(d) if $x_0 \in P \in \mathcal{A}$, then $K \setminus P$ is finite.

Since $\mathcal{P}$ is a σ-locally finite *cs*-network for X, then the above construction can be realized, and we can assume that $\{\mathcal{A} \subset \mathcal{P}_i : \mathcal{A}$ holds the property $F(K,X)\} = \{\mathcal{A}_{ij} : j \in \mathbb{N}\}$.

For each $n \in \mathbb{N}$, put

$$\mathcal{P}'_n = \bigwedge_{i,j\leq n} \mathcal{P}_{ij}, \tag{3.2}$$

then $\mathcal{P}'_n \subset \mathcal{P}_n$ and $\mathcal{P}'_n$ also holds the property $F(K,X)$.

For each $i \in \mathbb{N}$, $m \in \omega$, and $x_m \in K$, there is $\alpha_{im} \in A_i$ such that $x_m \in P_{\alpha_{im}} \in \mathcal{P}'_i$. Let $\beta_m = (\alpha_{im}) \in \prod_{i\in\mathbb{N}} A_i$. It is easy to prove that $\{P_{\alpha_{im}} : i \in \mathbb{N}\}$ is a network of x_m in X. Then there is a $\beta_m \in M$ such that $f(\beta_m) = x_m$ for each $m \in \omega$. For each $i \in \mathbb{N}$, there is $n(i) \in \mathbb{N}$ such that $\alpha_{in} = \alpha_{io}$ when $n \geq n(i)$. Hence the sequence $\{\alpha_{in}\}$ converges to α_{io} in A_i. Thus the sequence $\{\beta_n\}$ converges to β_0 in M. This implies that f is sequence-covering.

(ii) f is a compact mapping.

For any $x \in X$, since $\{\alpha \in A_i : x \in P_\alpha\}$ is finite, put

$$L = \left(\prod_{n\in\mathbb{N}} \{\alpha \in A_i : x \in P_\alpha\}\right) \cap X. \tag{3.3}$$

Then L is a compact subspace of X. In view of $f^{-1}(x) = L$, then f is a compact mapping.

(iii) f is a σ-mapping.

For each $n \in \mathbb{N}$ and $\alpha_n \in A_n$, put

$$V(\alpha_1,\dots,\alpha_n) = \{\beta \in M : \text{ for each } i \le n, \text{ the } i\text{th coordinate of } \beta \text{ is } \alpha_i\}. \tag{3.4}$$

Let $\mathcal{B} = \{V(\alpha_1,\dots,\alpha_n) : \alpha_i \in A_i\ (i \le n) \text{ and } n \in \mathbb{N}\}$. Then $\mathcal{B}$ is a base for M.

To prove that f is a σ-mapping, we only need to check that for each $n \in \mathbb{N}$ and $\alpha_n \in A_n$, $f(V(\alpha_1,\dots,\alpha_n)) = \bigcap_{i\le n} P_{\alpha_i}$ because $f(\mathcal{B})$ is σ-locally finite in X by this result.

For each $n \in \mathbb{N}$, $\alpha_n \in A_n$, and $i \le n$, $f(V(\alpha_1,\dots,\alpha_n)) \subset P_{\alpha_i}$, then $f(V(\alpha_1,\dots,\alpha_n)) \subset \bigcap_{i\le n} P_{\alpha_i}$. On the other hand, for each $x \in \bigcap_{i\le n} P_{\alpha_i}$, there is $\beta = (\beta_j) \in M$ such that $f(\beta) = x$. For each $j \in \mathbb{N}$, $P_{\beta_j} \in \mathcal{P}_j \subset \mathcal{P}_{j+n}$, then there is $\alpha_{j+n} \in A_{j+n}$ such that $P_{\alpha_{j+n}} = P_{\beta_j}$. Set $\alpha = (\alpha_j)$. Then $\alpha \in V(\alpha_1,\dots,\alpha_n)$ and $f(\alpha) = x$. Thus $\bigcap_{i\le n} P_{\alpha_i} \subset f(V(\alpha_1,\dots,\alpha_n))$. Hence $f(V(\alpha_1,\dots,\alpha_n)) = \bigcap_{i\le n} P_{\alpha_i}$. Therefore, f is a σ-mapping.

(2)⇒(3). It is clear that every sequence-covering and compact mapping on a metric space is 1-sequence-covering (see [16, Theorem 4.4]).

(3)⇒(1). Suppose that $f : M \to X$ is a 1-sequence-covering σ-mapping, where M is a metric space. Since f is a σ-mapping, then $f(\mathcal{B})$ is σ-locally finite in X for some base $\mathcal{B}$ for X. For each $x \in X$, there exists $\beta_x \in f^{-1}(x)$ satisfying Definition 1.1(3). Put

$$\mathcal{P}_x = \{f(B) : \beta_x \in B \in \mathcal{B}\}, \qquad \mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}, \tag{3.5}$$

it is easy to prove that $\mathcal{P}$ is a *sn*-network for X. Thus $\mathcal{P}$ is a σ-locally finite *sn*-network. This implies that X is *sn*-metrizable. □

From Lemma 2.1 and Theorem 3.2, we have the following corollary.

Corollary 3.3. *Let X be sn-metrizable, then X has a uniform sn-network.*

4. The characterization of spaces with locally countable *sn*-networks

Lemma 4.1 [9]. *The following are equivalent for a space X.*

(1) *X has a locally countable k-network.*

(2) *X has a locally countable cs-network.*

(3) *X has a locally countable cs*-network.*

Theorem 4.2. *The following are equivalent for a space X.*

(1) *X has a locally countable sn-network.*

(2) *X is an sn-first countable space with a locally countable cs-network (k-network, cs*-network).*

Proof. (1)⇒(2) is clear. We show that (2)⇒(1). Suppose that X is an *sn*-first countable space with a locally countable *cs*-network. Let $\mathcal{P}$ be a locally countable *cs*-network for X which is closed under finite intersections. For each $x \in X$, let $\{B(n,x) : n \in \mathbb{N}\}$ be a decrease *sn*-network at x in X. Put

$$\mathcal{F}_x = \{P \in \mathcal{P} : B(n,x) \subset P \text{ for some } n \in \mathbb{N}\}, \qquad \mathcal{F} = \cup\{\mathcal{F}_x : x \in X\}. \tag{4.1}$$

Obviously, $x \in \cap \mathcal{F}_x$ and $\mathcal{F}_x$ is closed under finite intersections. Then $\mathcal{F}$ satisfies Definition 1.4(a), (b). We claim that each element of $\mathcal{F}_x$ is a sequential neighborhood at x in X. Otherwise, there exists $P \in \mathcal{F}_x$ such that P is not a sequential neighborhood at x in X. Then there exists a sequence $\{x_n\}$ converging to x such that for each $k \in \mathbb{N}$, $\{x_n : n > k\} \not\subset P$. Take $x_{n_1} \in \{x_n : n > 1\} \setminus P$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that each $x_{n_{k+1}} \in \{x_n : n > n_k\} \setminus P$. Obviously, x_{n_k} converges to x. Since $P \in \mathcal{F}_x$, then $B(m,x) \subset P$ for some $m \in \mathbb{N}$. Because $B(m,x)$ is a sequential neighborhood at x in X, then $\{x\} \cup \{x_{n_k} : k \geq j\} \subset B(m,x)$ for some $j \in \mathbb{N}$, and so $\{x_{n_k} : k \geq j\} \subset P$, a contradiction. Hence $\mathcal{F}$ is an *sn*-network for X. Obviously, $\mathcal{F} \subset \mathcal{P}$. Therefore $\mathcal{F}$ is a locally countable *sn*-network for X. □

Theorem 4.3. *A space with a locally countable sn-network is sn-metrizable.*

Proof. Suppose that a space X has a locally countable *sn*-network. Then X is an *sn*-first countable space with a locally countable k-network by Theorem 4.2, and so X is a k-space with a locally countable k-network. By [10, Theorem 1], X is an $\aleph$-space. Thus X is *sn*-metrizable by Lemma 3.1. □

5. The characterization of spaces with σ-locally countable *sn*-networks

Theorem 5.1. *For a space X,* (1)⇔(2) ⇒ (3) *below hold.*

(1) *X has a σ-locally countable sn-network.*

(2) *X is an sn-first countable space with a σ-locally countable cs-network.*

(3) *X is an sn-first countable space with a σ-locally countable k-network.*

Proof. (1)⇒(2) is obvious.

(2)⇒(3). Suppose that X is an *sn*-first countable space with a σ-locally countable *cs*-network. Let $\mathcal{P} = \cup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-locally countable *cs*-network for X, where each $\mathcal{P}_n$ is locally countable in X. We will show that $\mathcal{P}$ is a k-network for X. Suppose that $K \subset V$ with K nonempty compact and V open in X. For each $n \in \mathbb{N}$, put

$$\mathcal{A}_n = \{P \in \mathcal{P}_n : P \cap K \neq \Phi \text{ and } P \subset V\}, \tag{5.1}$$

then $\mathcal{A}_n$ is countable, and so $\mathcal{A} = \cup\{\mathcal{A}_n : n \in \mathbb{N}\}$ is countable. Denoting $\mathcal{A} = \{P_i : i \in \mathbb{N}\}$, then $K \subset \bigcup_{i \leq n} P_i$ for some $n \in \mathbb{N}$. Otherwise, $K \not\subset \bigcup_{i \leq n} P_i$ for each $n \in \mathbb{N}$, so choose $x_n \in K \setminus \bigcup_{i \leq n} P_i$. Because $\{P \cap K : P \in \mathcal{P}\}$ is a countable *cs*-network for a subspace K and a compact space with a countable network is metrizable, then K is a compact metrizable space. Thus $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$, where $x_{n_k} \to x$. Obviously $x \in K$. Since $\mathcal{P}$ is a *cs*-network for X, then there exist $m \in \mathbb{N}$ and $P \in \mathcal{P}$ such that $\{x_{n_k} : k \geq m\} \cup \{x\} \subset P \subset V$. Now, $P = P_j$ for some $j \in \mathbb{N}$. Take $l \geq m$ such that $n_l \geq j$, then $x_{n_l} \in P_j$. This is a contradiction. Therefore, (2)⇒(3) holds.

(2)⇒(1). Suppose that X is an *sn*-first countable space with σ-locally countable *cs*-network. Let $\mathcal{P} = \cup\{\mathcal{P}_m : m \in \mathbb{N}\}$ be a σ-locally countable *cs*-network for X, where each $\mathcal{P}_m$ is locally countable in X which is closed under finite intersections and $X \in \mathcal{P}_m \subset \mathcal{P}_{m+1}$, and for each $x \in X$, let $\{B(n,x) : n \in \mathbb{N}\}$ be a decreasing *sn*-network of x in X. Put

$$\begin{gathered}\mathcal{F}_{m,x} = \{P \in \mathcal{P}_m : B(n,x) \subset P \text{ for some } n \in \mathbb{N}\}, \\ \mathcal{F}_x = \cup\{\mathcal{F}_{m,x} : m \in \mathbb{N}\}, \\ \mathcal{F}_m = \cup\{\mathcal{F}_{m,x} : x \in X\}, \\ \mathcal{F} = \cup\{\mathcal{F}_x : x \in X\}.\end{gathered} \tag{5.2}$$

Similar to the proof of Theorem 4.2, we can show that $\mathcal{F}$ is an *sn*-network for X.

For each $m \in \mathbb{N}$, $\mathcal{F}_m \subset \mathcal{P}_m$, then $\mathcal{F}_m$ is locally countable in X. Thus $\mathcal{F} = \cup\{\mathcal{F}_m : m \in \mathbb{N}\}$ is σ-locally countable in X. Therefore, (2)⇒(1) holds. □

Lemma 5.2. *A paracompact space with a σ-locally countable k-network is an $\aleph$-space.*

Proof. Suppose that X is a paracompact space with a σ-locally countable k-network $\mathcal{P}$. Let $\mathcal{P} = \cup\{\mathcal{P}_i : i \in \mathbb{N}\}$, where each $\mathcal{P}_i$ is locally countable in X. Since locally countable families are closed under finite unions, we can assume that each $\mathcal{P}_i \subset \mathcal{P}_{i+1}$. For each $i \in \mathbb{N}$, since $\mathcal{P}_i$ is locally countable in X, then there exists an open cover $\mathcal{U}_i$ of X such that any element of $\mathcal{U}_i$ only intersects many countable elements of $\mathcal{P}_i$. Because X is paracompact, then $\mathcal{U}_i$ has a locally finite open refinement $\mathcal{V}_i$. We will show that $\bigcup_{i\in\mathbb{N}}(\mathcal{P}_i \wedge \mathcal{V}_i)$ is a σ-locally finite k-network for X. For each $V \in \mathcal{V}_i$, let $\{P \in \mathcal{P}_i : V \cap P \neq \phi\} = \{P(V,n) : n \in \mathbb{N}\}$. Put $\mathcal{H}_{i,n} = \{P(V,n) \cap V : V \in \mathcal{V}_i\}$. Since $\mathcal{V}_i$ is locally finite in X, then $\mathcal{H}_{i,n}$ also is. Now, $\mathcal{P}_i \wedge \mathcal{V}_i = \bigcup_{n\in\mathbb{N}} \mathcal{H}_{i,n}$, thus $\bigcup_{i\in\mathbb{N}}(\mathcal{P}_i \wedge \mathcal{V}_i)$ is σ-locally finite in X. Suppose that $K \subset W$ with K nonempty compact and W open in X. Then, there are $i \in \mathbb{N}$ and finite $\mathcal{P}_i^* \subset \mathcal{P}_i$ such that $K \subset \bigcup \mathcal{P}_i^* \subset W$. So $K \subset \bigcup \mathcal{V}_i^*$ for some finite $\mathcal{V}_i^* \subset \mathcal{V}_i$. As $\mathcal{P}_i^* \wedge \mathcal{V}_i^*$ is a finite family of $\mathcal{P}_i \wedge \mathcal{V}_i$, and $K \subset \bigcup(\mathcal{P}_i^* \wedge \mathcal{V}_i^*) \subset W$, then $\bigcup_{i\in\mathbb{N}}(\mathcal{P}_i \wedge \mathcal{V}_i)$ is a k-network for X. This implies that X is an $\aleph$-space. □

From Theorem 5.1 and Lemmas 5.2 and 3.1, we have the following theorem.

Theorem 5.3. *A paracompact space with a σ-locally countable sn-network is sn-metrizable.*

6. Examples

Example 6.1. A space X has a point-countable *sn*-network $\not\Rightarrow$ X has a uniform *sn*-network.

For each $n \in \mathbb{N}$, let C_n be a convergent sequence which includes a limit point p_n, and $C_n \cap C_m = \phi$ if $n \neq m$. And let $S = \bigoplus_{n\in\mathbb{N}} C_n$, and $M = S \bigoplus \mathbb{R}$. Then M is a separable, locally compact metric space. Put $Q = \{q_n : n \in \mathbb{N}\}$, and let X be the quotient space obtained from M by identifying p_n in S with q_n in $\mathbb{R}$ for each $n \in \mathbb{N}$. Then X is a regular, non-Cauchy space, which has a point-countable weak base (see [21, Example 2.14(3)] or [14, Example 3.1.13(2)]). Obviously, X has a point-countable *sn*-network. By [17, Corollary 2], X is not a sequence-covering, quotient, and π-image of a metric space. Note that X is sequential, X is not a sequence-covering π-image of a metric space (see [11, Proposition 2.1.16(2)]). Thus X is not a sequence-covering compact image of a metric space. By Lemma 2.1, X has not any uniform *sn*-network.

Example 6.2. A space X has a uniform *sn*-network $\not\Rightarrow$ X is *sn*-metrizable.

Let

$$S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}, \qquad X = [0,1] \times S. \tag{6.1}$$

And let

$$Y = [0,1] \times \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \tag{6.2}$$

have the usual Euclidean topology as a subspace of $[0,1] \times S$. Define a typical neighborhood of $(t,0)$ in X to be of the form

$$\{(t,0)\} \cup \left(\bigcup_{k \geq n} V\left(t, \frac{1}{k}\right)\right), \quad n \in \mathbb{N}, \tag{6.3}$$

where $V(t,1/k)$ is a neighborhood of $(t,1/k)$ in $[0,1] \times \{1/k\}$. Put

$$M = \left(\bigoplus_{n \in \mathbb{N}} [0,1] \times \left\{\frac{1}{n}\right\}\right) \oplus \left(\bigoplus_{t \in [0,1]} \{t\} \times S\right), \tag{6.4}$$

and define f from M onto X such that f is an obvious mapping.

Then f is a compact-covering, quotient, two-to-one mapping from the locally compact metric space M onto separable, regular, non-meta-Lindelöf space X (see [11, Example 2.8.16] or [8, Example 9.3]). It is easy to check that f is a 1-sequence-covering mapping. From Lemma 2.1, X has a uniform *sn*-network.

Because X is a sequential space, and a regular sequential space with a σ-locally countable k-network is meta-Lindelöf (see [10, Proposition 1]), then X has not any σ-locally countable k-network. So X is not an $\aleph$-space. By Lemma 3.1, X is not *sn*-metrizable.

Example 6.3. Let Y be a subset of $\mathbb{R}$ such that $Q \subset Y \subset \mathbb{R}$ and $|Y| > \omega$. Let $X = Y \cup (\bigcup_{n \in \mathbb{N}} Q \times \{1/n\})$, and define a base $\mathcal{B}$ for the desired topology on X as follows:

(1) if $x \in X - Y$, let $\{x\} \in \mathcal{B}$,

(2) if $x \in Y$, then $\{\{x\} \cup (\bigcup_{n \geq m}([a_{x,n}, x) \cup Q) \times \{1/m\}) : m \in \mathbb{N}, x > a_{x,n} \in \mathbb{R}\} \subset \mathcal{B}$.

Then X is a separable, *sn*-metrizable space, which has not any countable *sn*-network (see [7, Example 2.3]). Thus the following holds:

X is *sn*-metrizable $\not\Rightarrow$ X has a countable *sn*-network.

Example 6.4. Let $S = \{1/n : n \in \mathbb{N}\} \cup \{0\}$. Let $X = \omega_1 \times S$ and define a base $\mathcal{B}$ for the desired topology on X as follows:

(1) $\{\{x\} : x \in X \setminus \omega_1 \times \{0\}\} \subset \mathcal{B}$,

(2) if $\alpha < \omega_1$, $\{\{(\alpha,0)\} \cup (\bigcup_{n \geq m}(V(\alpha,n) \times \{1/n\})) : m \in \mathbb{N}, V(\alpha,n)$ is an open neighborhood α in $\omega_1$which has the order topology$\} \subset \mathcal{B}$.

Then X has a locally countable k-network, which is not an $\aleph$-space (see [11, Example 2.8.17]). From Lemma 4.1, X has a locally countable *cs*-network. Since X is not *sn*-metrizable, then X has not any locally countable *sn*-network by Theorem 4.3. Thus the following holds.

(1) X has a locally countable cs-network $\not\Rightarrow$ X has a σ-locally finite cs-network.

(2) X has a locally countable cs-network $\not\Rightarrow$ X has a locally countable sn-network.

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