ON FUNCTIONS WITH THE CAUCHY DIFFERENCE BOUNDED BY A FUNCTIONAL. PART II

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We are going to consider the functional inequality $f(x+y)-f(x)-f(y)\ge \phi(x,y)$, $x,y\in X$, where $(X,+)$ is an abelian group, and $\phi: X\times X\to\mathbb{R}$ and $f: X\to\mathbb{R}$ are unknown mappings. In particular, we will give conditions which force biadditivity and symmetry of ϕ and the representation $f(x)=(1/2)\phi(x,x)+a(x)$ for $x\in X$, where a is an additive function. In the present paper, we continue and develop our earlier studies published by the author (2004).

Let $(X,+)$ be an abelian group. We consider the functional inequality

$$f(x+y)-f(x)-f(y)\ge \phi(x,y),\quad x,y\in X, \tag{1}$$

where $\phi: X\times X\to\mathbb{R}$ and $f: X\to\mathbb{R}$ are unknown mappings.

First, we quote [3, Proposition].

Proposition 1. *If $f: X\to\mathbb{R}$, $\phi: X\times X\to\mathbb{R}$ satisfy (1) and*

$$\phi(x,-x)\ge -\phi(x,x),\quad x\in X, \tag{2}$$

then, (a) $f(0)\le 0$*;* (b) $f(x)+f(-x)\le\phi(x,x)$ *for* $x\in X$*;* (c) $f(2x)\ge 3f(x)+f(-x)$ *for* $x\in X$.

One can see that an even function $f: X\to\mathbb{R}$ which fulfills assumptions of Proposition 1 satisfies $f(2x)\ge 4f(x)$ for $x\in X$. This observation was used in [3], where a new function $Q: X\to\mathbb{R}$ was defined by the formula $Q(x):=\lim_{k\to+\infty} f(2^k x)/4^k$ for $x\in X$. The resulted equality $Q(2x)=4Q(x)$ for $x\in X$ played a crucial role.

The main idea of the present paper is to drop the assumption that f is even and use Proposition 1(c) to get a limit function $\varphi: X\to\mathbb{R}$ satisfying the equality $\varphi(2x)=3\varphi(x)+\varphi(-x)$ for $x\in X$ (see Theorems 14 and 16).

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It is assumed that $\mathbb{N} = \{1,2,\dots\}$ and $\mathbb{N}_0 = \{0,1,2,\dots\}$.

Let us quote here [3, Lemma 1].

Lemma 2. *Assume that $f : X \to \mathbb{R}$ and $\phi : X \times X \to \mathbb{R}$ satisfy (1). If*

$$\phi(x,-y) \geq -\phi(x,y), \quad x,y \in X, \tag{3}$$

$$f(2x) \leq 4f(x), \quad x \in X, \tag{4}$$

then

$$f(x) = \frac{1}{2}\phi(x,x), \quad x \in X. \tag{5}$$

Moreover, ϕ is biadditive and symmetric.

The foregoing result was the main tool in [3]. In fact, this lemma, in slightly different version, was first proved by K. Baron (see [4]). In the present paper, we need to state a more general lemma, which works for maps satisfying $f(2x) \leq 3f(x) + f(-x)$ for $x \in X$.

Lemma 3. *Assume that $f : X \to \mathbb{R}$ and $\phi : X \times X \to \mathbb{R}$ satisfy (1) and (3). If*

$$f(2x) \leq 3f(x) + f(-x), \quad x \in X, \tag{6}$$

then there exists an additive function $a : X \to \mathbb{R}$ such that

$$f(x) = \frac{1}{2}\phi(x,x) + a(x), \quad x \in X. \tag{7}$$

Moreover, ϕ is biadditive and symmetric.

Proof. Setting $-y$ instead of y in (1), we obtain

$$f(x-y) - f(x) - f(-y) \geq \phi(x,-y) \geq -\phi(x,y), \quad x,y \in X. \tag{8}$$

Adding this to (1) leads to

$$f(x+y) + f(x-y) \geq 2f(x) + f(y) + f(-y), \quad x,y \in X. \tag{9}$$

Fix arbitrarily $u,v \in X$. Applying this inequality with $x = u+v$ and $y = u-v$ and using (6), we infer that

$$\begin{aligned} &3f(u) + f(-u) + 3f(v) + f(-v) \\ &\quad \geq f(2u) + f(2v) \geq 2f(u+v) + f(u-v) + f(v-u), \quad u,v \in X. \end{aligned} \tag{10}$$

The last two inequalities imply that f satisfies the equality

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y), \quad x,y \in X. \tag{11}$$

Now, define $q : X \to \mathbb{R}$ and $a : X \to \mathbb{R}$ by the formulas

$$a(x) := \frac{f(x) - f(-x)}{2}, \qquad q(x) := \frac{f(x) + f(-x)}{2}, \quad x \in X. \tag{12}$$

It is clear that

$$a(x+y)+a(x-y)=2a(x), \quad x,y\in X, \tag{13}$$

thus a is additive. Moreover,

$$q(x+y)+q(x-y)=2q(x)+2q(y), \quad x,y\in X, \tag{14}$$

that is, q is quadratic. There exists a biadditive and symmetric functional $B: X\times X\to\mathbb{R}$ such that $q(x)=B(x,x)$ for $x\in X$ (see, e.g., Aczél and Dhombres [1, Chapter 11, Proposition 1]). Moreover, we have

$$q(x+y)-q(x)-q(y)=2B(x,y), \quad x,y\in X. \tag{15}$$

This implies that $2B(x,y)\geq\phi(x,y)$ for $x,y\in X$. By the use of this, (3), and the biadditivity of B, we get that $\phi(x,y)\geq-\phi(x,-y)\geq-2B(x,-y)=2B(x,y)$ for $x,y\in X$. So $2B=\phi$. This completes the proof. □

Our next step is to drop the assumption of the evenness of function f in [3, Lemma 3]. We have the following generalization of this result.

Recall that a group X is called uniquely 2-divisible if and only if the map $X\ni x\to x+x\in X$ is bijective.

Lemma 4. *Assume X to be uniquely 2-divisible, $f: X\to\mathbb{R}$, $\phi: X\times X\to\mathbb{R}$ satisfy (1), (2), and*

$$\phi(2x,2x)\leq 4\phi(x,x), \quad x\in X. \tag{16}$$

If f is nonnegative, then f is even and $f(x)=(1/2)\phi(x,x)$ for $x\in X$.

Proof. By Proposition 1(c) and nonnegativity of f, we get that for $x\in X$, the sequence $(2^n f(x/2^n))_{n\in\mathbb{N}}$ is nonincreasing and nonnegative and thus convergent. So, the formula

$$A(x):=\lim_{n\to+\infty}2^n f\left(\frac{x}{2^n}\right), \quad x\in X, \tag{17}$$

correctly defines a map $A: X\to\mathbb{R}$. Moreover, $A(x)\geq 0$ and $A(2x)=2A(x)$ for $x\in X$.

Proposition 1(c) implies that

$$2^n f\left(\frac{x}{2^{n-1}}\right)\geq 3\cdot 2^n f\left(\frac{x}{2^n}\right)+2^n f\left(\frac{-x}{2^n}\right), \quad x\in X,\ n\in\mathbb{N}. \tag{18}$$

So

$$2A(x)=A(2x)\geq 3A(x)+A(-x), \quad x\in X, \tag{19}$$

and we can easily observe that $A=0$.

Now, we will follow the original proof of [3, Lemma 3]. Fix an $x\in X$. From (1) and (16), we derive inductively the estimations

$$2^k f\left(\frac{x}{2^{k-1}}\right)-2^{k+1}f\left(\frac{x}{2^k}\right)\geq 2^k\phi\left(\frac{x}{2^k},\frac{x}{2^k}\right)\geq\frac{1}{2^k}\phi(x,x), \tag{20}$$

for all $k \in \mathbb{N}$. Summing up these inequalities side by side for $k \in \{1,\dots,n\}$, we get that

$$2f(x) - 2^{n+1} f\left(\frac{x}{2^n}\right) \geq \sum_{k=1}^{n} \frac{1}{2^k}\phi(x,x), \quad n \in \mathbb{N}. \tag{21}$$

Letting n tend to $+\infty$ yields the inequality $2f(x) \geq \phi(x,x)$.

On the other hand, Proposition 1(b) states that $f(x) + f(-x) \leq \phi(x,x)$ for $x \in X$. So, f is even and $f(x) = (1/2)\phi(x,x)$ for $x \in X$. This completes the proof. □

In the next lemma, we will provide a certain property of the inequality from Proposition 1(c).

Lemma 5. *Assume X to be uniquely 2-divisible. If $f : X \to \mathbb{R}$ satisfies*

$$f(2x) \geq 3f(x) + f(-x), \quad x \in X, \tag{22}$$

$$\forall_{x \in X}\ \exists_{k_0 \in \mathbb{N}}\ \forall_{k \geq k_0} f\left(\frac{x}{2^k}\right) \geq 0, \tag{23}$$

then $f \geq 0$.

Proof. Define a sequence $(\varphi_k)_{k \in \mathbb{N}_0}$ of real mappings on X by the formula

$$\varphi_k(x) := \frac{4^k + 2^k}{2} f\left(\frac{x}{2^k}\right) + \frac{4^k - 2^k}{2} f\left(-\frac{x}{2^k}\right), \quad x \in X,\ k \in \mathbb{N}_0. \tag{24}$$

We will show that this sequence is nonincreasing. Fix an $x \in X$ and $k \in \mathbb{N}_0$. We have

$$\begin{aligned}
\varphi_k(x) &= \frac{4^k + 2^k}{2} f\left(\frac{x}{2^k}\right) + \frac{4^k - 2^k}{2} f\left(-\frac{x}{2^k}\right) \\
&\geq \frac{4^k + 2^k}{2}\left[3f\left(\frac{x}{2^{k+1}}\right) + f\left(-\frac{x}{2^{k+1}}\right)\right] \\
&\quad + \frac{4^k - 2^k}{2}\left[3f\left(-\frac{x}{2^{k+1}}\right) + f\left(\frac{x}{2^{k+1}}\right)\right] \\
&= \frac{4^{k+1} + 2^{k+1}}{2} f\left(\frac{x}{2^{k+1}}\right) + \frac{4^{k+1} - 2^{k+1}}{2} f\left(-\frac{x}{2^{k+1}}\right) \\
&= \varphi_{k+1}(x).
\end{aligned} \tag{25}$$

The assumption (23) implies that the sequence $(\varphi_k(x))_{k \in \mathbb{N}_0}$ is nonnegative for $x \in X$. In particular $f(x) = \varphi_0(x) \geq 0$ for $x \in X$. This completes the proof. □

Now, we may join this lemma with our Lemmas 4, 2 and Proposition 1(c) to get the following result.

Corollary 6. *Assume X to be uniquely 2-divisible, $f : X \to \mathbb{R}$, $\phi : X \times X \to \mathbb{R}$ satisfy (1), (2), (16), and (23). Then f is nonnegative, even, and $f(x) = (1/2)\phi(x,x)$ for $x \in X$. Moreover, if (3) is also satisfied, then ϕ is biadditive and symmetric.*

Next, we will quote [3, Theorem 2].

THEOREM 7. *Assume X to be uniquely 2-divisible and that* $f: X \to \mathbb{R}$, $\phi: X \times X \to \mathbb{R}$ *satisfy (1), (3), (16) jointly with*

$$f(x) + f(-x) \geq 0, \quad x \in X. \tag{26}$$

Then there exists an additive function $a: X \to \mathbb{R}$ *such that*

$$f(x) = \frac{1}{2}\phi(x,x) + a(x), \quad x \in X. \tag{27}$$

Moreover, ϕ *is biadditive and symmetric.*

This result together with Lemma 5 applied for a map $x \mapsto f(x) + f(-x)$ leads to the following corollary.

COROLLARY 8. *Assume X to be uniquely 2-divisible and that* $f: X \to \mathbb{R}$, $\phi: X \times X \to \mathbb{R}$ *satisfy (1), (3), (16) jointly with*

$$\forall_{x \in X}\ \exists_{k_0 \in \mathbb{N}}\ \forall_{k \geq k_0} f\left(\frac{x}{2^k}\right) + f\left(-\frac{x}{2^k}\right) \geq 0. \tag{28}$$

Then there exists an additive function $a: X \to \mathbb{R}$ *such that*

$$f(x) = \frac{1}{2}\phi(x,x) + a(x), \quad x \in X. \tag{29}$$

Moreover, ϕ *is biadditive and symmetric.*

Now, we quote [2, Corollary 2].

COROLLARY 9. *Assume X to be a real linear space and that* $f: X \to \mathbb{R}$, $\phi: X \times X \to \mathbb{R}$ *satisfy (1), f is nonnegative, and* $\phi(x,\cdot)$ *is homogeneous for* $x \in X$. *Then* ϕ *is bilinear and symmetric and* $f(x) = (1/2)\phi(x,x)$ *for* $x \in X$.

In the light of Lemma 5, we get then the following corollary.

COROLLARY 10. *Assume X to be a real linear space,* $f: X \to \mathbb{R}$, $\phi: X \times X \to \mathbb{R}$ *satisfy (1), (23), and* $\phi(x,\cdot)$ *is homogeneous for* $x \in X$. *Then* ϕ *is bilinear and symmetric and* $f(x) = (1/2)\phi(x,x) \geq 0$ *for* $x \in X$.

We recall also the following corollary.

COROLLARY 11 [2, Corollary 1]. *Assume X to be a real linear space and that* $f: X \to \mathbb{R}$, $\phi: X \times X \to \mathbb{R}$ *satisfy (1). If for every* $x \in X$ *the function* $\mathbb{R} \ni t \mapsto f(tx) \in \mathbb{R}$ *has the property that its Jensen convexity implies its convexity and f satisfies (26) with* $\phi(x,\cdot)$ *being homogeneous for* $x \in X$, *then there exists a linear functional* $L: X \to \mathbb{R}$ *such that*

$$f(x) = \frac{1}{2}\phi(x,x) + L(x), \quad x \in X. \tag{30}$$

Moreover, ϕ *is bilinear and symmetric.*

A similar reasoning as above allows us to derive the following fact.

Corollary 12. *Assume X to be a real linear space and that $f : X \to \mathbb{R}$, $\phi : X \times X \to \mathbb{R}$ satisfy (1). If for every $x \in X$ the function $\mathbb{R} \ni t \mapsto f(tx) \in \mathbb{R}$ has the property that its Jensen convexity implies its convexity and f satisfies (28) with $\phi(x,\cdot)$ being homogeneous for $x \in X$, then there exists a linear functional $L : X \to \mathbb{R}$ such that*

$$f(x) = \frac{1}{2}\phi(x,x) + L(x), \quad x \in X. \tag{31}$$

Moreover, ϕ is bilinear and symmetric.

Remark 13. If X is a real linear topological Hausdorff space, then (23) is satisfied if f is nonnegative in a certain neighborhood of zero.

Now, we state and prove our next result.

Theorem 14. *Assume X to be uniquely 2-divisible, $f : X \to \mathbb{R}$, $\phi : X \times X \to \mathbb{R}$ satisfy (1), (3),*

$$\phi(2x,2y) \le 4\phi(x,y), \quad x,y \in X, \tag{32}$$

$$\begin{gathered} \forall_{x\in X}\left(\liminf_{k\to+\infty}\left[4^k f\left(\frac{x}{2^k}\right) + 4^k f\left(\frac{-x}{2^k}\right)\right] > -\infty\right), \\ \forall_{x\in X}\left(\liminf_{k\to+\infty} 2^k f\left(\frac{x}{2^k}\right) > -\infty \vee \limsup_{k\to+\infty} 2^k f\left(\frac{-x}{2^k}\right) < +\infty\right). \end{gathered} \tag{33}$$

Then there exists an additive function $a : X \to \mathbb{R}$ such that

$$f(x) = \frac{1}{2}\phi(x,x) + a(x), \quad x \in X. \tag{34}$$

Moreover, ϕ is biadditive and symmetric.

Proof. Define a sequence $(\varphi_k)_{k\in\mathbb{N}_0}$ of real mappings on X by the formula (24). We have already checked (proof of Lemma 5) that this sequence is nonincreasing. We will show that it is pointwise bounded. Fix an $x \in X$ and observe that

$$\begin{aligned} \varphi_k(x) &= \frac{4^k + 2^k}{2\cdot 4^k}\left[4^k f\left(\frac{x}{2^k}\right) + 4^k f\left(\frac{-x}{2^k}\right)\right] - 2^k f\left(\frac{-x}{2^k}\right), \quad k \in \mathbb{N}_0, \\ \varphi_k(x) &= \frac{4^k - 2^k}{2\cdot 4^k}\left[4^k f\left(\frac{x}{2^k}\right) + 4^k f\left(\frac{-x}{2^k}\right)\right] + 2^k f\left(\frac{x}{2^k}\right), \quad k \in \mathbb{N}_0. \end{aligned} \tag{35}$$

So, by (33), the sequence $(\varphi_k)_{k\in\mathbb{N}_0}$ is pointwise convergent. Define $\varphi : X \to \mathbb{R}$ by $\varphi(x) := \lim_{k\to+\infty}\varphi_k(x)$ for $x \in X$. Observe that

$$\varphi_{k+1}(2x) = 3\varphi_k(x) + \varphi_k(-x), \quad x \in X,\ k \in \mathbb{N}_0, \tag{36}$$

and thus

$$\varphi(2x) = 3\varphi(x) + \varphi(-x), \quad x \in X. \tag{37}$$

Next, by the definition of φ and φ_k, (1) and (32), we have

$$\begin{aligned}\varphi(x+y)-\varphi(x)-\varphi(y) &= \lim_{k\to+\infty}\left[\varphi_k(x+y)-\varphi_k(x)-\varphi_k(y)\right]\\ &\geq \limsup_{k\to+\infty}\frac{4^k+2^k}{2}\phi\left(\frac{x}{2^k},\frac{y}{2^k}\right)+\limsup_{k\to+\infty}\frac{4^k-2^k}{2}\phi\left(\frac{-x}{2^k},\frac{-y}{2^k}\right)\\ &\geq \frac{1}{2}\phi(x,y)+\frac{1}{2}\phi(-x,-y), \quad x,y\in X.\end{aligned} \tag{38}$$

Define $\phi_1 : X\times X \to \mathbb{R}$ by $\phi_1(x,y) := (1/2)[\phi(x,y)+\phi(-x,-y)]$ for $x,y\in X$. Now, we may apply Lemma 3 with φ and ϕ_1 to get that ϕ_1 is biadditive and symmetric and $\varphi = q+a$, where q is a quadratic mapping and a is an additive one. Moreover,

$$\varphi(x+y)-\varphi(x)-\varphi(y) = q(x+y)-q(x)-q(y) = \phi_1(x,y), \quad x,y\in X. \tag{39}$$

Now, put $f_1 := f-\varphi$ and $\phi_2 := \phi-\phi_1$. We have $f_1 \geq 0$ and

$$f_1(x+y)-f_1(x)-f_1(y) \geq \phi_2(x,y), \quad x,y\in X. \tag{40}$$

Lemma 4 applied for $f = f_1$ and $\phi = \phi_2$ implies that f_1 is even and $f_1(2x) = 4f_1(x)$ for $x\in X$. By Proposition 1(c), we have

$$\begin{aligned}3\varphi(x)+\varphi(-x)+4f_1(x) &= \varphi(2x)+f_1(2x) = f(2x)\\ &\geq 3f(x)+f(-x) = 3\varphi(x)+\varphi(-x)+4f_1(x), \quad x\in X.\end{aligned} \tag{41}$$

So $f(2x) = 3f(x)+f(-x)$ for $x\in X$. This means that $f=\varphi$, and as a consequence $\phi_2 = 0$. This completes the proof. □

Remark 15. The assumption (33) is fulfilled if f satisfies the condition (26), which appears (among others) in Theorem 7. But Theorem 14 does not generalize Theorem 7 or Corollary 8, unless we are able to replace the assumption (32) by (16) in Theorem 14 (note that (32) in its whole strength was used only to prove that $\varphi(x+y)-\varphi(x)-\varphi(y) \geq \phi_1(x,y)$ for $x,y\in X$).

Now, we will state and prove our last result, which yields a generalization to [3, Theorem 1].

Theorem 16. *Assume that $f : X\to\mathbb{R}$ and $\phi : X\times X\to\mathbb{R}$ satisfy (1), (3) and*

$$\begin{aligned}\limsup_{k\to+\infty}\frac{1}{4^k}\phi(2^kx,2^kx) &< +\infty, \quad x\in X,\\ \liminf_{k\to+\infty}\frac{1}{4^k}\phi(2^kx,2^ky) &\geq \phi(x,y), \quad x,y\in X.\end{aligned} \tag{42}$$

If the sequence $(2^{-k}[f(2^kx)-f(-2^kx)])_{k\in\mathbb{N}}$ is pointwise convergent to a superadditive function, then there exists a subadditive function $A : X\to\mathbb{R}$ such that

$$f(x) = \frac{1}{2}\phi(x,x)-A(x), \quad x\in X. \tag{43}$$

Moreover, ϕ is biadditive and symmetric.

Proof. Define a sequence $(\widehat{\varphi}_k)_{k\in\mathbb{N}_0}$ of real mappings on X by the formula

$$\widehat{\varphi}_k(x) := \frac{4^{-k}+2^{-k}}{2} f(2^k x) + \frac{4^{-k}-2^{-k}}{2} f(-2^k x), \quad x \in X,\ k \in \mathbb{N}_0. \tag{44}$$

We will show that this sequence is convergent. Fix an $x \in X$. We have

$$\widehat{\varphi}_k(x) = \frac{f(2^k x) + f(-2^k x)}{2\cdot 4^k} + \frac{f(2^k x) - f(-2^k x)}{2^{k+1}}, \quad k \in \mathbb{N}_0. \tag{45}$$

Observe that by Proposition 1(c), the first summand is nondecreasing and (by Proposition 1(b)) pointwise upper bounded by $4^{-k}\phi(2^k x, 2^k x)$, whereas the second one is convergent by the assumption. Thus the sequence $(\widehat{\varphi}_k)_{k\in\mathbb{N}}$ is convergent. Therefore, the formula

$$\widehat{\varphi}(x) := \lim_{k\to+\infty} \widehat{\varphi}_k(x), \quad x \in X, \tag{46}$$

correctly defines a map $\widehat{\varphi}: X \to \mathbb{R}$. Moreover, $\widehat{\varphi}(2x) = 3\widehat{\varphi}(x) + \widehat{\varphi}(-x)$ for $x \in X$ and the following inequality is satisfied:

$$\begin{aligned}
&\widehat{\varphi}(x+y) - \widehat{\varphi}(x) - \widehat{\varphi}(y)\\
&\quad= \lim_{k\to+\infty} \frac{1}{2}\cdot 4^{-k}\left[f(2^k x + 2^k y) - f(2^k x) - f(2^k y)\right]\\
&\qquad + \frac{1}{2}\cdot 4^{-k}\left[f(-2^k x - 2^k y) - f(-2^k x) - f(-2^k y)\right]\\
&\qquad + 2^{-k-1}\left[f(2^k x + 2^k y) - f(2^k x) - f(2^k y)\right]\\
&\qquad - 2^{-k-1}\left[f(-2^k x - 2^k y) - f(-2^k x) - f(-2^k y)\right]\\
&\quad\geq \liminf_{k\to+\infty} \frac{1}{2}\cdot 4^{-k}\left[\phi(2^k x, 2^k y) + \phi(-2^k x, -2^k y)\right]\\
&\qquad + \frac{1}{2}\left[p(x+y) - p(x) - p(y)\right]\\
&\quad\geq \frac{1}{2}\left[\phi(x,y) + \phi(-x,-y)\right], \quad x, y \in X,
\end{aligned} \tag{47}$$

where $p: X \to \mathbb{R}$ is defined by

$$p(x) := \lim_{k\to+\infty} \frac{1}{2^k}\left[f(2^k x) - f(-2^k x)\right], \quad x \in X. \tag{48}$$

Lemma 3 states that the map $\phi_1 : X \times X \to \mathbb{R}$, defined by $\phi_1(x,y) = (1/2)[\phi(x,y) + \phi(-x,-y)]$ for $x, y \in X$, is biadditive and symmetric and $\widehat{\varphi}(x) = (1/2)\phi_1(x,x) + a(x)$ for $x \in X$, where a is an additive mapping. It implies that

$$\widehat{\varphi}(x+y) - \widehat{\varphi}(x) - \widehat{\varphi}(y) = \phi_1(x,y), \quad x, y \in X, \tag{49}$$

that is, the foregoing estimation holds with the equality. In particular,

$$\lim_{k\to+\infty} 4^{-k}\left[f(2^k x + 2^k y) - f(2^k x) - f(2^k y)\right] = \phi(x,y), \quad x, y \in X. \tag{50}$$

Moreover, observe that $\widehat{\varphi}_k(x) - \widehat{\varphi}_k(-x) = (1/2^k)(f(2^k x) - f(-2^k x))$ for $x \in X$ and $k \in \mathbb{N}_0$, whence $2a = p$.

Now, put $f_1 := f - \widehat{\varphi}$ and $\phi_2 := \phi - \phi_1$. Clearly, ϕ_2 satisfies (3), (42), and

$$f_1(x+y) - f_1(x) - f_1(y) \geq \phi_2(x,y), \quad x, y \in X. \tag{51}$$

Moreover, one has

$$\lim_{k\to+\infty} 4^{-k}[f_1(2^k x + 2^k y) - f_1(2^k x) - f_1(2^k y)] = \phi_2(x,y), \quad x, y \in X, \tag{52}$$

$$\begin{aligned} &\lim_{k\to+\infty} \frac{1}{2^k}[f_1(2^k x) - f_1(-2^k x)] \\ &\quad = \lim_{k\to+\infty} \frac{1}{2^k}[f(2^k x) - f(-2^k x)] - \lim_{k\to+\infty} \frac{1}{2^k}[\widehat{\varphi}(2^k x) - \widehat{\varphi}(-2^k x)] \\ &\quad = p(x) - 2a(x) = 0, \quad x \in X. \end{aligned} \tag{53}$$

Split f_1 into its even and odd parts, that is, define $P, g : X \to \mathbb{R}$ by $P(x) := (1/2)[f_1(x) + f_1(-x)]$ and $g(x) := (1/2)[f_1(x) - f_1(-x)]$ for $x \in X$. Next, fix $x, y \in X$ and apply (51) twice: for x and y and then for $-x$ and $-y$. Summing up side by side the two inequalities obtained and using the definition of ϕ_1 and ϕ_2, we get

$$f_1(x+y) + f_1(-x-y) - f_1(x) - f_1(-x) - f_1(y) - f_1(-y) \geq 0, \tag{54}$$

that is, P is superadditive. In particular, due to its evenness, P is nonpositive and $P(2x) \geq 2P(x)$ for $x \in X$. Thus, the sequence $(2^{-k}P(2^k x))_{k\in\mathbb{N}}$ is convergent, whence

$$\lim_{k\to+\infty} 4^{-k}P(2^k x) = 0, \quad x \in X. \tag{55}$$

This, jointly with (52), implies that

$$\lim_{k\to+\infty} 4^{-k}[g(2^k x + 2^k y) - g(2^k x) - g(2^k y)] = \phi_2(x,y), \quad x, y \in X. \tag{56}$$

On the other hand, we have

$$\lim_{k\to+\infty} 2^{-k}g(2^k x) = \lim_{k\to+\infty} \frac{1}{2^{k+1}}[f_1(2^k x) - f_1(-2^k x)] = 0, \quad x \in X. \tag{57}$$

From the last two equalities, it follows that $\phi_2 = 0$. So $\phi = \phi_1$ is biadditive and symmetric. It remains to define $A : X \to \mathbb{R}$ by $A(x) := (1/2)\phi(x,x) - f(x)$ for $x \in X$. This completes the proof. □

Remark 17. The convergence assumption spoken of in Theorem 16 is weaker than the supposition of the evenness of f, used in [3, Theorem 1]. However, we do not know definitely whether or not it could be omitted.

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