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RADICAL APPROACH IN BCH-ALGEBRAS

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We define the notion of radical in BCH-algebra and investigate the structure of $[X;k]$, a viewpoint of radical in BCH-algebras.

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1. Introduction. In 1966, Imai and Iséki [8] and Iséki [9] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In 1983, Hu and Li [5, 6] introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. They have studied some properties of these algebras.

As we know, the primary aim of the theory of BCH-algebras is to determine the structure of all BCH-algebras. The main task of a structure theorem is to find a complete system of invariants describing the BCH-algebra up to isomorphism, or to establish some connection with other mathematics branches. In addition, the ideal theory plays an important role in studying BCI-algebras, and some interesting results have been obtained by several authors [1, 2, 3, 4, 11, 14, 15]. In 1992, Huang [7] introduced nil ideals in BCI-algebras. In 1999, Roh and Jun [13] introduced nil ideals in BCH-algebras. They introduced the concept of nil subsets by using nilpotent elements, and investigated some related properties.

In this note, we define the notion of radical in BCH-algebra, and some fundamental results concerning this notion are proved.

2. Preliminaries. A BCH-*algebra* is a nonempty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(1) $x*x=0$,
(2) $x*y=0$ and $y*x=0$ imply $x=y$,
(3) $(x*y)*z=(x*z)*y$

for all $x,y,z\in X$. A BCH-algebra X satisfying the identity $((x*y)*(x*z))*(z*y)=0$ and $0*x=0$ for all $x,y,z\in X$ is called a BCK-*algebra*. We define the relation $\le$ by $x\le y$ if and only if $x*y=0$.

In any BCH-algebra X, the following hold: for all $x,y\in X$,

(4) $(x*(x*y))\le y$,
(5) $x\le 0$ implies $x=0$,
(6) $0*(x*y)=(0*x)*(0*y)$,

(7) $x*0=x$,
(8) $0*(0*(0*x))=0*x$.

A nonempty subset S of BCH-algebra X is called a *subalgebra* of X if $x*y\in S$ whenever $x,y\in S$.

A nonempty subset I of BCH-algebra X is called an *ideal* of X if $0\in I$ and if $x*y, y\in I$ imply that $x\in I$. It is possible that an ideal of a BCH-algebra may not be a subalgebra.

3. Main results. In what follows, the letter X denotes a BCH-algebra unless otherwise specified.

Definition 3.1. For any $x\in X$ and any positive integer n, the *nth power* x^n of x is defined by

$$x^1=x,\quad x^n=x*(0*x^{n-1}). \tag{3.1}$$

Clearly $0^n=0$.

Theorem 3.2. *For any $x\in X$ and any positive integer n,*

$$(0*x)^n=0*x^n. \tag{3.2}$$

Proof. We argue by induction on the positive integer n. For $n=1$ there is nothing to prove. Assume that the theorem is true for some positive integer n. Then using (6) we have

$$\begin{aligned}(0*x)^{n+1}&=(0*x)*(0*(0*x)^n)\\&=(0*x)*(0*(0*x^n))\\&=0*(x*(0*x^n))=0*x^{n+1}.\end{aligned} \tag{3.3}$$

□

Definition 3.3. [10] In a BCH-algebra X, the set $A^+:=\{x\in X\mid 0\le x\}$ is called a *positive part* of X and the set $A(X):=\{x\in X\mid 0*(0*x)=x\}$ is called an *atom part* of X. Further an element of $A(X)$ will be called an *atom* of X.

In the following theorem we give some properties of BCK-algebras.

Theorem 3.4. *If X is a BCH-algebra, then the positive part A^+ of X is a subset of the set $\{x\in X\mid x^2=x\}$.*

Proof. Let $x\in A^+$. Then we have $x^2=x*(0*x)=x*0=x$, and hence $A^+\subseteq\{x\in X\mid x^2=x\}$. □

Corollary 3.5. *If X is a BCK-algebra, then $X=\{x\in X\mid x^2=x\}$.*

In [10], Kim and Roh proved $A(X)=\{0*(0*x)\mid x\in X\}=\{0*x\mid x\in X\}$.

Note that $A(X)$ is a subalgebra of X and $((x*y)*(x*z))*(z*y)=0$ for all $x,y,z\in A(X)$, and hence $A(X)$ is a p-semisimple BCI-algebra. Thus by [12] we have the following property: for any $a,b\in A(X)$ and any positive integer n, we have $(a*b)^n=a^n*b^n$.

Hence the following corollary is an immediate consequence of Theorem 3.2.

TABLE 3.1

$*$	0	a	b	c
0	0	c	0	a
a	a	0	a	c
b	b	c	0	a
c	c	a	c	0

COROLLARY 3.6. *For any x in a BCH-algebra X and any positive integer n,*
(i) $0 * x^n \in A(X)$,
(ii) $0 * (x * y)^n = (0 * x^n) * (0 * y^n)$.

DEFINITION 3.7. Let R be a nonempty subset of a BCH-algebra X and k a positive integer. Then define

$$[R;k] := \{x \in R \mid x^k = 0\}, \tag{3.4}$$

which is called the *radical* of R.

We know that, in general, the radical of an ideal in X may not be an ideal.

EXAMPLE 3.8. Let $X = \{0, a, b, c\}$ be a BCH-algebra in which $*$-operation is defined as in Table 3.1. Taking an ideal $R = X$, then $[R;3] = \{0, a, c\}$ is not an ideal of X since $b * a = c \in [R;3]$ and $b \notin [R;3]$.

THEOREM 3.9. *Let S be a subalgebra of a BCH-algebra X and k a positive integer. If $x \in [S;k]$, then $0 * x \in [S;k]$.*

PROOF. Let $x \in [S;k]$. Then $x^k = 0$ and $x \in S$. Thus by Theorem 3.2 we have

$$(0 * x)^k = 0 * x^k = 0, \quad 0 * x \in S, \tag{3.5}$$

and hence $0 * x \in [S;k]$. □

This leaves open question, if R is a subalgebra of X and $0 * x \in [R;k]$, then is x in $[R;k]$? The answer is negative. In Example 3.8, $[X;3]$ is a subalgebra of X and $0 * b \in [X;3]$, but $b \notin [X;3]$.

DEFINITION 3.10 [10]**.** For $e \in A(X)$, the set $\{x \in X \mid e * x = 0\}$ is called the *branch* of X determined by e and is denoted by $A(e)$.

THEOREM 3.11. *Let k be a positive integer and $A(X) = X$. Then*

$$A(e) \cap [X;k] \neq \varnothing \Longrightarrow A(e) \subseteq [X;k]. \tag{3.6}$$

PROOF. Suppose that $A(e) \cap [X;k] \neq \varnothing$, then there exists $x \in A(e) \cap [X;k]$. Thus by Theorem 3.9, we have

$$e = 0 * (0 * x) \in [X;k]. \tag{3.7}$$

Let $y \in A(e)$, then $y \in [X;k]$ since $0 * (0 * y) = e \in [X;k]$, and hence $A(e) \subseteq [X;k]$. □

Theorem 3.12. *For any positive integer k and $A(X)=X$,*

$$[X;k]=\bigcup_{x\in[X;k]}A(0*(0*x))=\bigcup_{e\in A(X)\cap[X;k]}A(e)=\bigcup_{e\in[A(X);k]}A(e). \tag{3.8}$$

Proof. $A(X)\cap[X;k]=[A(X);k]$ is obvious. By Theorem 3.11, we have $x\in A(0*(0*x))\subseteq[X;k]$ for all $x\in[X;k]$, and so there exists $e=0*(0*x)\in A(X)\cap[X;k]$ such that $x\in A(e)\subseteq[X;k]$. Therefore we obtain

$$[X;k]=\bigcup_{x\in[X;k]}A(0*(0*x))=\bigcup_{e\in A(X)\cap[X;k]}A(e)=\bigcup_{e\in[A(X);k]}A(e). \tag{3.9}$$

□

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