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MULTIVARIATE FRÉCHET COPULAS AND CONDITIONAL VALUE-AT-RISK

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Based on the method of copulas, we construct a parametric family of multivariate distributions using mixtures of independent conditional distributions. The new family of multivariate copulas is a convex combination of products of independent and comonotone subcopulas. It fulfills the four most desirable properties that a multivariate statistical model should satisfy. In particular, the bivariate margins belong to a simple but flexible one-parameter family of bivariate copulas, called linear Spearman copula, which is similar but not identical to the convex family of Fréchet. It is shown that the distribution and stop-loss transform of dependent sums from this multivariate family can be evaluated using explicit integral formulas, and that these dependent sums are bounded in convex order between the corresponding independent and comonotone sums. The model is applied to the evaluation of the economic risk capital for a portfolio of risks using conditional value-at-risk measures. A multivariate conditional value-at-risk vector measure is considered. Its components coincide for the constructed multivariate copula with the conditional value-at-risk measures of the risk components of the portfolio. This yields a “fair” risk allocation in the sense that each risk component becomes allocated to its coherent conditional value-at-risk.

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1. Introduction. A natural framework for the construction of multivariate nonnormal distributions is the method of copulas, justified by the theorem of Sklar [48]. It permits a separate study and modeling of the marginal distributions and the dependence structure. According to Joe [30, Section 4.1], a parametric family of distributions should satisfy four desirable properties.

(a) There should exist an interpretation like a mixture or other stochastic representation.

(b) The margins, at least the univariate and bivariate ones, should belong to the same parametric family and numerical evaluation should be possible.

(c) The bivariate dependence between the margins should be described by a parameter and cover a wide range of dependence.

(d) The multivariate distribution and density should preferably have a closed-form representation; at least numerical evaluation should be possible.

In general, these desirable properties cannot be fulfilled simultaneously. For example, multivariate normal distributions satisfy properties (a), (b), and (c) but not (d). The method of copulas satisfies property (c) but implies only partial closedness under the taking of margins, and can lead to computational complexity as the dimension increases. In fact, it is an open problem to find parametric families of copulas that satisfy all of the desirable properties. In the present paper, such a parametric family, called *multivariate linear Spearman copula*, is constructed (formula (4.9)). It is based on the method of mixtures of independent conditional distributions.

A growing need for and interest in suitable multivariate nonnormal distributions stem from applications in actuarial science and finance, especially in risk management. Given a risk or portfolio of risks, represented by a random variable X or random vector $\mathbf{X} = (X_1, \dots, X_n)$ with distribution $F_x(x)$, one looks for risk measures suitable to model the economic risk capital of the risk X or aggregate risk $\sum_{i=1}^{n} X_i$. Two simple measures are the value-at-risk and the conditional value-at-risk. Given a random variable X, one considers the *value-at-risk* (VaR) to the confidence level α, defined as the lower α-quantile:

$$\mathrm{VaR}_\alpha[X] = Q_X(\alpha) = \inf\{x : F_X(x) \geq \alpha\}, \tag{1.1}$$

and the *upper conditional value-at-risk* (CVaR^+) to the confidence level α, defined by

$$\mathrm{CVaR}_\alpha^+[X] = E[X \mid X > \mathrm{VaR}_\alpha[X]]. \tag{1.2}$$

The VaR quantity represents the maximum possible loss, which is not exceeded with the probability α. The CVaR^+ quantity is the conditional expected loss given that the loss strictly exceeds its value-at-risk. Next, consider the *α-tail transform* X^a of X with distribution

$$F_{X^\alpha}(x) = \begin{cases} 0, & x < \mathrm{VaR}_\alpha[X], \\ \dfrac{F_X(x) - \alpha}{1 - \alpha}, & x \geq \mathrm{VaR}_\alpha[X]. \end{cases} \tag{1.3}$$

Rockafellar and Uryasev [44] define *conditional value-at-risk* (CVaR) to the confidence level α as the expected value of the α-tail transform, that is, by

$$\mathrm{CVaR}_\alpha[X] = E[X^\alpha]. \tag{1.4}$$

The obtained measure is a coherent risk measure in the sense of Artzner et al. [4, 5] and coincides with CVaR^+ in the case of continuous distributions. It is well known that the VaR measure is not coherent. For simplicity, we restrict throughout the attention to the case of continuous distributions and identify CVaR with CVaR^+. For portfolios of risks, we define a *multivariate conditional value-at-risk vector measure*, whose components coincide for the multivariate linear Spearman copula with the CVaR measures of the risk components of the portfolio (Theorem 6.1). This yields a "fair" risk allocation in the sense that each risk component becomes allocated to its coherent univariate conditional value-at-risk measure.

A more detailed outline of the content follows. Based on the method of copulas summarized in Section 2.1, we recall in Section 2.2 the construction of parametric families of multivariate copulas using mixtures of independent conditional distributions. Following this approach, it is first necessary to focus on a simple but sufficiently flexible one-parameter family of bivariate copulas, called linear Spearman copula, which is similar but not identical to the convex family of Fréchet [23] and is introduced in Section 3.1. The analytical evaluation of the distribution and stop-loss transform of bivariate sums following a linear Spearman copula, required in conditional value-at-risk calculations, is presented in Section 3.2. Section 4 is devoted to the construction of the new multivariate family of copulas that satisfies the four desirable properties (a), (b), (c), and (d). Two important features of the multivariate linear Spearman copula are presented in Section 5. First, we show that the distribution and stop-loss transform of dependent sums following a multivariate linear Spearman copula can be evaluated using explicit integral formulas (Theorem 5.1). Then, we establish that these dependent sums are bounded in convex order between the corresponding independent and comonotone sums (Theorem 5.2). Finally, Section 6 presents our application to conditional value-at-risk.

2. Multivariate models with arbitrary marginals. Our view of multivariate statistical modeling is that of Joe [30, Section 1.7]: "Models should try to capture important characteristics, such as the appropriate density shapes for the univariate margins and the appropriate dependence structure, and otherwise be as simple as possible." To fulfill this, a parametric family of multivariate distributions should satisfy the desirable properties (a), (b), (c), and (d) mentioned in Section 1. It is an open problem to find parametric families of copulas that satisfy all these desirable properties (Joe [30, Section 4.13, page 138]). In the present paper, such a parametric family is constructed. It is based on the method of mixtures of independent conditional distributions, discussed in Section 2.2.

2.1. The method of copulas. Though copulas have been introduced since Sklar [48], their use in insurance and finance is more recent. Textbooks treating copulas include those by Hutchinson and Lai [29], Joe [30], Nelsen [41], and Drouet Mari and Kotz [17].

Recall that the copula representation of a continuous multivariate distribution allows for a separate modeling of the univariate margins and the dependence structure. Denote by $M_n := M_n(F_1,\dots,F_n)$ the class of all continuous multivariate random variables $(X_1,\dots,X_n)$ with given marginals F_i of X_i. If F denotes the multivariate distribution of $(X_1,\dots,X_n)$, then the *copula associated* with F is a distribution function $C:[0,1]^n \to [0,1]$ that satisfies

$$F(x) = C(F_1(x_1),\dots,F_n(x_n)), \quad x = (x_1,\dots,x_n) \in R^n. \tag{2.1}$$

Reciprocally, if $F \in M_n$ and F_i^{-1} are quantile functions of the margins, then

$$C(u) = F(F_1^{-1}(u_1),\dots,F_n^{-1}(u_n)), \quad u = (u_1,\dots,u_n) \in [0,1]^n, \tag{2.2}$$

is the unique copula satisfying (2.1) (theorem of Sklar [48]).

Copulas are especially useful for the modeling and measurement of bivariate dependence. For an axiomatic definition, one needs the important notion of concordance ordering. A copula $C_1(u,v)$ is said to be smaller than a copula $C_2(u,v)$ in *concordance order*, written $C_1 \prec C_2$, if one has

$$C_1(u,v) \le C_2(u,v), \quad (u,v) \in [0,1]^2. \tag{2.3}$$

Definition 2.1 (Scarsini [46]). A numeric measure κ, written $\kappa_{X,Y}$ or κ_C, of association between two continuous random variables X and Y with copula $C(u,v)$ is a *measure of concordance* if it satisfies the following properties:

(C1) $\kappa_{X,Y}$ is defined for every couple (X,Y) of continuous random variables;
(C2) $-1 \le \kappa_{X,Y} \le 1$, and $\kappa_{X,-X} = -1$, $\kappa_{X,X} = 1$;
(C3) $\kappa_{X,Y} = \kappa_{Y,X}$;
(C4) if X and Y are independent, then $\kappa_{X,Y} = 0$;
(C5) $\kappa_{-X,Y} = \kappa_{X,-Y} = -\kappa_{X,Y}$;
(C6) if $C_1 \prec C_2$, then $\kappa_{C_1} \le \kappa_{C_2}$;
(C7) if $\{(X_n,Y_n)\}$ is a sequence of continuous random variables with copulas C_n and if $\{C_n\}$ converges pointwise to C, then $\lim_{n\to\infty} \kappa_{C_n} = \kappa_C$.

Two famous measures of concordance are *Kendall's tau*,

$$\tau = 1 - 4 \cdot \int_0^1 \int_0^1 \frac{\partial}{\partial u} C(u,v) \cdot \frac{\partial}{\partial v} C(u,v)\,du\,dv, \tag{2.4}$$

and *Spearman's rho*,

$$\rho_S = 12 \cdot \int_0^1 \int_0^1 [C(u,v) - uv]\,du\,dv. \tag{2.5}$$

The latter parameter will completely describe the bivariate dependence in our construction. When extreme values are involved, tail dependence should also be measured.

Definition 2.2. The *coefficient of (upper) tail dependence* of a couple (X,Y) of continuous random variables is defined by

$$\lambda = \lambda_{X,Y} = \lim_{u\to 1^-} \Pr\left(Y > F_Y^{-1}(u) \mid X > F_X^{-1}(u)\right), \tag{2.6}$$

provided a limit $\lambda \in [0,1]$ exists. If $\lambda \in (0,1]$, this defines the *asymptotic dependence* (in the upper tail), while if $\lambda = 0$, this defines the *asymptotic independence*.

Tail dependence is an asymptotic property. Its calculation follows easily from the relation

$$\lambda = \lambda_{X,Y} = \lim_{u\to 1^-} \frac{1 - 2u + C(u,u)}{1-u}. \tag{2.7}$$

2.2. Mixtures of independent conditional distributions. Our goal is the construction of a parametric family of n-dimensional copulas that satisfies the desirable properties (a), (b), (c), and (d). It uses a simple variant of the method of mixtures of conditional distributions described by Joe [30, Section 4.5]. To satisfy property (b), we focus on the n Fréchet classes $FC_i := FC_i(F_{ij},\ j \ne i)$, $i = 1,\dots,n$, of n-variate distributions for which the bivariate margins $F_{ij}(x_i,x_j) = F_{(X_i,X_j)}(x_i,x_j) = C_{ij}\lfloor F_i(x_i), F_j(x_j)\rfloor$, $j \ne i$, belong to a given parametric family of copulas $C_{ij}\lfloor u_i,u_j\rfloor$. Assume that the conditional distributions

$$F_{j|i}(x_j \mid x_i) = \frac{\partial C_{ij}}{\partial u_i}[F_i(x_i), F_j(x_j)] \tag{2.8}$$

are well defined. The n-variate distribution such that the random variables X_j, $j \neq i$, are conditionally independent, given X_i, is contained in FC_i and is defined by

$$F^{(i)}(x) = \int_{-\infty}^{x_i} \left[\prod_{j \neq i} F_{j|i}(x_j \mid t) \right] \cdot dF_i(t). \tag{2.9}$$

Choosing appropriately the bivariate copulas $C_{ij}[u_i, u_j]$, it is possible to construct n-variate copulas $C^{(i)}(u_1, \dots, u_n)$, $i = 1, \dots, n$, such that $F^{(i)}$ belongs to $C^{(i)}$ and the bivariate margins F_{ij}, $j \neq i$, belong to C_{ij}. Moreover, any convex combination of the $C^{(i)}$'s, that is,

$$C(u_1, \dots, u_n) = \sum_{i=1}^{n} \lambda_i C^{(i)}(u_1, \dots, u_n), \quad 0 \leq \lambda_i \leq 1, \ \sum_{i=1}^{n} \lambda_i = 1, \tag{2.10}$$

is again an n-variate copula, which, by appropriate choice, may satisfy the desirable properties.

3. A bivariate model with arbitrary marginals. Our aim is the construction of a parametric family of n-variate copulas satisfying the four desirable properties in Section 2. Following the approach through mixtures of independent conditional distributions described in Section 2.2, it is first necessary to focus on a simple but flexible one-parameter family of bivariate copulas, called linear Spearman copula, which is introduced in Section 3.1. The analytical evaluation of the distribution and stop-loss transform of bivariate sums following a linear Spearman copula, often required in actuarial and financial calculations, is presented in Section 3.2.

The dependence parameter of the linear Spearman copula is Spearman's grade correlation coefficient. In practice, however, often only Pearson's linear correlation coefficient is available. Stochastic relationships between these two parameters, which allow parameter estimation from each other, have been derived by Hürlimann [25].

3.1. The linear Spearman copula. We consider a one-parameter family of copulas $C_\theta(u, v)$, which is able to model continuously a whole range of dependence between the lower Fréchet bound $C_{-1}(u, v) = \max(u + v - 1, 0)$, the independent copula $C_0(u, v) = uv$, and the upper Fréchet bound $C_1(u, v) = \min(u, v)$. Such families are called *inclusive* or *comprehensive* (Devroye [14, page 581]). A number of inclusive families of copulas are well known, namely, those by Fréchet [23], Plackett [42], Mardia [39], Clayton [8], and Frank [21]. Another one, which is similar but not identical to the convex family of Fréchet [23], is the *linear Spearman copula* defined by

$$C_\theta(u, v) = (1 - |\theta|) \cdot C_0(u, v) + |\theta| \cdot C_{\mathrm{sgn}(\theta)}(u, v). \tag{3.1}$$

For $\theta \in [0, 1]$, this copula is family B11 in Joe [30, page 148]. It represents a mixture of perfect dependence and independence. If X and Y are uniform $(0, 1)$, $Y = X$ with probability θ, and Y is independent of X with probability $1 - \theta$, then (X, Y) has the linear Spearman copula. This distribution has been first considered by Konijn [34] and motivated by Cohen [9] along Cohen's kappa statistic (see Hutchinson and Lai [29, Section 10.9]). For the extended copula, the chosen nomenclature *linear* refers to the piecewise linear sections of this copula, and *Spearman* refers to the fact that the *grade correlation coefficient* ρ_S by Spearman [49] coincides with the parameter θ. This follows from the calculation

$$\rho_S = 12 \cdot \int_0^1 \int_0^1 [C_\theta(u,v) - uv]\, du\, dv = \theta. \tag{3.2}$$

The linear Spearman copula, which leads to the *linear Spearman bivariate distribution*, has a singular component, which, according to Joe, should limit its field of applicability. Despite this, it has many interesting and important properties, and is suitable for computation. Moreover, it is a good competitor in fitting bivariate cumulative returns, as shown by Hürlimann [28].

For the reader's convenience, we describe first two extremal properties. Kendall's tau for this copula is defined as follows:

$$\begin{aligned} \tau &= 1 - 4 \cdot \int_0^1 \int_0^1 \frac{\partial}{\partial u} C_\theta(u,v) \cdot \frac{\partial}{\partial v} C_\theta(u,v)\, du\, dv \\ &= \frac{1}{3}\rho_S \cdot [2 + \operatorname{sgn}(\rho_S)\rho_S]. \end{aligned} \tag{3.3}$$

Invert this to get

$$\rho_S = \begin{cases} -1 + \sqrt{1 + 3\tau}, & \tau \geq 0, \\ 1 - \sqrt{1 - 3\tau}, & \tau \leq 0. \end{cases} \tag{3.4}$$

Relate this to the convex two-parameter copula by Fréchet [23] defined by

$$\begin{aligned} C_{\alpha,\beta}(u,v) = \beta \cdot C_{-1}(u,v) &+ (1 - \alpha - \beta) \cdot C_0(u,v) \\ &+ \alpha \cdot C_1(u,v), \quad \alpha, \beta \geq 0,\ \alpha + \beta \leq 1. \end{aligned} \tag{3.5}$$

Since $\rho_S = \alpha - \beta$ and $\tau = ((\alpha - \beta)/3)(2 + \alpha + \beta)$ for this copula, one has the inequalities

$$\begin{aligned} \tau \leq \rho_S \leq -1 + \sqrt{1 + 3\tau}, \quad \tau \geq 0, \\ 1 - \sqrt{1 - 3\tau} \leq \rho_S \leq \tau, \quad \tau \leq 0. \end{aligned} \tag{3.6}$$

The linear Spearman copula satisfies the following extremal property. For $\tau \geq 0$, the upper bound for ρ_S in Fréchet's copula is attained by the linear Spearman copula, and for $\tau \leq 0$, it is the lower bound, which is attained.

In case $\tau \geq 0$, a second more important extremal property holds, which is related to a conjectural statement. Recall that Y is *stochastically increasing* on X, written $\mathrm{SI}(Y|X)$, if $\Pr(Y > y \mid X = x)$ is a nondecreasing function of x for all y. Similarly, X is *stochastically increasing* on Y, written $\mathrm{SI}(X|Y)$, if $\Pr(X > x \mid Y = y)$ is a nondecreasing function of y for all x. (Note that Lehmann [36] speaks instead of *positive regression dependence*.) If X and Y are continuous random variables with copula $C(u,v)$, then one has the equivalences (Nelsen [41, Theorem 5.2.10])

$$\begin{aligned} \mathrm{SI}(Y|X) &\Longleftrightarrow \frac{\partial}{\partial u}C(u,v) \text{ is nonincreasing in } u \text{ for all } v, \\ \mathrm{SI}(X|Y) &\Longleftrightarrow \frac{\partial}{\partial v}C(u,v) \text{ is nonincreasing in } v \text{ for all } u. \end{aligned} \tag{3.7}$$

The *Hutchinson-Lai conjecture* consists of the following statement. If (X,Y) satisfies the properties (3.7), then ρ_S satisfies the inequalities

$$-1+\sqrt{1+3\tau} \le \rho_S \le \min\left\{\frac{3}{2}\tau, 2\tau-\tau^2\right\}. \tag{3.8}$$

The upper bound $2\tau-\tau^2$ is attained for the one-parameter copula introduced by Kimeldorf and Sampson [33] (see also Hutchinson and Lai [29, Section 13.7]). The lower bound is attained by the linear Spearman copula, as shown already by Konijn [34, page 277]. Alternatively, if the conjecture holds, the maximum value of Kendall's tau given by ρ_S is attained for the linear Spearman copula. Note that the upper bound $\rho_S \le (3/2)\tau$ has been disproved recently by Nelsen [41, Exercise 5.36]. The remaining conjecture $-1+\sqrt{1+3\tau} \le \rho_S \le 2\tau-\tau^2$ is still unsettled (however, see Hürlimann [27] for the case of bivariate extreme value copulas).

As an important modeling characteristic, we show that the linear Spearman copula leads to a simple tail dependence structure. Using (2.7), one obtains

$$\lambda(X,Y) = \lim_{u\to 1^-}\frac{1-2u+C_\theta(u,u)}{1-u} = \lim_{u\to 1^-}(1-u+\theta u) = \theta. \tag{3.9}$$

Therefore, unless X and Y are independent, a linear Spearman couple is always asymptotically dependent. This is a desirable property in insurance and financial modeling, where data tend to be dependent in their extreme values. In contrast to this, the ubiquitous Gaussian copula always yields asymptotic independence, unless perfect correlation holds (Sibuya [47], Resnick [43, Chapter 5], and Embrechts et al. [20, Section 4.4]).

3.2. Distribution and stop-loss transform of bivariate sums. For several purposes in actuarial science and finance, it is of interest to have analytical expressions for the distribution and stop-loss transform of dependent sums $S = X+Y$, denoted respectively by $F_S(x) = \Pr(S \le x)$ and $\pi_S(x) = E[(S-x)_+]$. If (X,Y) follows a linear Spearman bivariate distribution, we show in Theorem 3.4 that the evaluation of these quantities depends on the knowledge of the quantiles and stop-loss transform of the independent sum of X and Y, denoted by $S^\perp = X^\perp + Y^\perp$, where $(X^\perp, Y^\perp)$ represents an independent version of (X,Y) such that $X^\perp$ and $Y^\perp$ are independent and $X^\perp$ and $Y^\perp$ are identically distributed as X and Y. Similarly, if (X^+,Y^+) is a comonotone version of (X,Y) with bivariate distribution $F_{(X^+,Y^+)}(x,y) = \min\{F_X(x), F_Y(y)\}$, the sum is denoted by $S^+ = X^+ + Y^+$, while if (X^-,Y^-) is a countercomonotone version such that $(X^-,-Y^-)$ is a comonotone couple, the sum is denoted by $S^- = X^- + Y^-$. We assume throughout that the margins have continuous and strictly increasing distribution functions, hence the quantile functions are uniquely defined. A linear Spearman random couple (X,Y) with Spearman coefficient θ is denoted by $LS_\theta(X,Y)$.

Lemma 3.1. *For each $LS_\theta(X,Y)$, $\theta \in [-1,1]$, the distribution and stop-loss transform of the sum $S = X+Y$ satisfy the relationships*

$$\begin{aligned} F_S(x) &= (1-|\theta|)\cdot F_{S^\perp}(x) + |\theta|\cdot F_{S^{\mathrm{sgn}(\theta)}}(x), \\ \pi_S(x) &= (1-|\theta|)\cdot \pi_{S^\perp}(x) + |\theta|\cdot \pi_{S^{\mathrm{sgn}(\theta)}}(x). \end{aligned} \tag{3.10}$$

Proof. This follows without difficulty from the representation (3.1). □

Lemma 3.2. *Suppose (X^+,Y^+), respectively $(X^-,-Y^-)$, is a comonotone couple with continuous and strictly increasing marginal distributions. Then, for all $u \in (0,1)$, one has the additive relations*

$$F_{S^+}^{-1}(u) = F_X^{-1}(u) + F_Y^{-1}(u), \qquad F_{S^-}^{-1}(u) = F_X^{-1}(u) + F_Y^{-1}(1-u), \tag{3.11}$$

$$\pi_{S^+}\big[F_{S^+}^{-1}(u)\big] = \pi_X\big[F_X^{-1}(u)\big] + \pi_Y\big[F_Y^{-1}(u)\big], \tag{3.12}$$

$$\pi_{S^-}\big[F_{S^-}^{-1}(u)\big] = \pi_X\big[F_X^{-1}(u)\big] + E[Y] - F_Y^{-1}(1-u) - \pi_Y\big[F_Y^{-1}(1-u)\big]. \tag{3.13}$$

Proof. If (X^+,Y^+) is a comonotone couple, it belongs to the copula $C(u,v) = \min(u,v)$. Inserting the expression for the conditional distribution $F_{Y|X=x}(y) = (\partial C/\partial u)[F_X(x),F_Y(y)] = 1_{\{x \le Q_X[F_Y(y)]\}}$ into the formula for the distribution of a sum

$$F_{X+Y}(s) = \int_{-\infty}^{\infty} F_{Y|X=x}(s-x)\,dF_X(x) \tag{3.14}$$

and making the change of variable $F_X(x) = u$, one obtains

$$F_{X+Y}(s) = \int_0^{u_s} du = u_s, \tag{3.15}$$

where u_s solves the equation $F_X^{-1}(u_s) + F_Y^{-1}(u_s) = s$. Therefore, (3.15) is equivalent to $F_{X+Y}^{-1}(u_s) = F_X^{-1}(u_s) + F_Y^{-1}(u_s)$, and since s is arbitrary, the first part of (3.11) is shown. The second part of (3.11) follows similarly using the copula $C(u,v) = \max(u+v-1,0)$. To show (3.13), consider the "spread" function of a random variable X defined by

$$\begin{aligned} T_X(u) := \pi_X\big[F_X^{-1}(u)\big] &= \int_{F_X^{-1}(u)}^{\infty} \big(x - F_X^{-1}(u)\big)\,dF_X(x) \\ &= \int_u^1 \big(F_X^{-1}(t) - F_X^{-1}(u)\big)\,dt. \end{aligned} \tag{3.16}$$

Using (3.11), one immediately obtains from (3.16) that

$$\begin{aligned} T_{S^+}(u) &= \pi_{S^+}\big[F_{S^+}^{-1}(u)\big] \\ &= \int_u^1 \big(F_X^{-1}(t) - F_X^{-1}(u)\big)\,dt + \int_u^1 \big(F_Y^{-1}(t) - F_Y^{-1}(u)\big)\,dt \\ &= \pi_X\big[F_X^{-1}(u)\big] + \pi_Y\big[F_Y^{-1}(u)\big], \end{aligned} \tag{3.17}$$

which shows the first part of (3.13). For the second part of (3.13), one similarly obtains

$$
\begin{aligned}
T_{S^-}(u) &= \pi_{S^-}\left[F_{S^-}^{-1}(u)\right] \\
&= \int_u^1 \left(F_X^{-1}(t)-F_X^{-1}(u)\right)dt + \int_u^1 \left(F_Y^{-1}(1-t)-F_Y^{-1}(1-u)\right)dt \\
&= \pi_X\left[F_X^{-1}(u)\right] + \int_0^1 \left(F_Y^{-1}(z)-F_Y^{-1}(1-u)\right)dz \\
&\quad - \int_{1-u}^1 \left(F_Y^{-1}(z)-F_Y^{-1}(1-u)\right)dz \\
&= \pi_X\left[F_X^{-1}(u)\right] + E[Y] - F_Y^{-1}(1-u) - \pi_Y\left[F_Y^{-1}(1-u)\right].
\end{aligned}
\tag{3.18}
$$

The Lemma is shown. □

Remark 3.3. In case of continuous and strictly increasing margins, the first additive relations in (3.11) and (3.13) extend easily to n-variate sums $S^+ = X_1^+ + \cdots + X_n^+$ of mutually comonotonic random variables:

$$
F_{S^+}^{-1}(u) = \sum_{i=1}^n F_{X_i}^{-1}(u), \qquad \pi_{S^+}\left[F_{S^+}^{-1}(u)\right] = \sum_{i=1}^n \pi_{X_i}\left[F_{X_i}^{-1}(u)\right]. \tag{3.19}
$$

For the quantile, this is already found by Landsberger and Meilijson [35]. Both relations are given by Dhaene et al. [16], Kaas et al. [32], and Hürlimann [26]. Our elementary approach has the advantage to yield the additional result for S^-. These relations are of great importance in economic risk capital evaluations using the value-at-risk and conditional value-at-risk measures. They imply that the maximum CVaR for the aggregate loss $L = L_1 + \cdots + L_n$ of a portfolio $\mathbf{L} = (L_1, \ldots, L_n)$ with fixed marginal losses is attained at the portfolio with mutually comonotone components, and it is equal to the sum of the CVaR of its components (Hürlimann [26, Theorems 2.2 and 2.3]):

$$
\max\left\{\mathrm{CVaR}_\alpha[L]\right\} = \mathrm{CVaR}_\alpha\left[L^+\right] = \sum_{i=1}^n \mathrm{CVaR}_\alpha\left[L_i\right]. \tag{3.20}
$$

In contrast to this, the maximum VaR of a portfolio with fixed marginal losses is not attained at the portfolio with mutual components. This assertion is related to Kolmogorov's problem treated by Makarov [38], Rüschendorf [45], Frank et al. [22], Denuit et al. [13], Durrleman et al. [18], Luciano and Marena [37], Cossette et al. [10], and Embrechts et al. [19]. In the comonotonic situation, one has with (3.19) only the additive relation

$$
\mathrm{VaR}_\alpha\left[L^+\right] = \sum_{i=1}^n \mathrm{VaR}_\alpha\left[L_i\right]. \tag{3.21}
$$

Theorem 3.4. *For each $LS_\theta(X,Y)$, $\theta \in [-1,1]$, the distribution and stop-loss transform of the sum $S = X + Y$ are determined as follows. For each $u \in [0,1]$, one has with*

$u_\theta = (1/2)[1-\operatorname{sgn}(\theta)] + \operatorname{sgn}(\theta)u$ *the formulas*

$$
\begin{aligned}
F_S\left[F_X^{-1}(u)+F_Y^{-1}(u_\theta)\right] &= (1-|\theta|)\cdot F_{S^\perp}\left[F_X^{-1}(u)+F_Y^{-1}(u_\theta)\right] + |\theta|\cdot u,\\
\pi_S\left[F_X^{-1}(u)+F_Y^{-1}(u_\theta)\right] &= (1-|\theta|)\cdot \pi_{S^\perp}\left[F_X^{-1}(u)+F_Y^{-1}(u_\theta)\right]\\
&\quad + |\theta|\cdot\Big\{\pi_X\left[F_X^{-1}(u)\right] + \operatorname{sgn}(\theta)\cdot\pi_Y\left[F_Y^{-1}(u_\theta)\right]\\
&\qquad + \frac{1}{2}\left[1-\operatorname{sgn}(\theta)\right]\cdot\left[E[Y]-F_Y^{-1}(u_\theta)\right]\Big\}.
\end{aligned}
\tag{3.22}
$$

Proof. Apply Lemmas 3.1 and 3.2. □

Though not always of simple form, analytical expressions for one of density, distribution, and stop-loss transform of the independent sum $S^\perp = X^\perp + Y^\perp$ from parametric families of margins often exist. A numerical evaluation using computer algebra systems is then easy to implement. For example, this is possible for the often encountered margins from the normal, gamma, and lognormal families of distributions (see Johnson et al. [31] and Hürlimann [25]).

4. A multivariate generalization. We restrict our attention to the construction of n-variate distributions $F(x_1,\dots,x_n)$ whose positive dependent bivariate margins $F_{rs}(x_r,x_s)$ belong to linear Spearman copulas with general Spearman coefficients $\rho_{rs}^S \in [0,1]$. The more complicated case $\rho_{rs}^S \in [-1,1]$ has been illustrated for trivariate distributions by Hürlimann [25, Section 9].

For each $i \in \{1,\dots,n\}$, the n-variate distribution $F^{(i)}(x_1,\dots,x_n)$ belongs to the n-variate copula $C^{(i)}(u_1,\dots,u_n)$ and has bivariate margins $F_{ij}(x_i,x_j)$, $j \neq i$, which belong to the linear Spearman copula

$$
C_{ij}(u_i,u_j) = (1-\theta_{ij})u_iu_j + \theta_{ij}\min(u_i,u_j), \tag{4.1}
$$

where $\theta_{ij} \in [0,1]$, and by symmetry, $\theta_{ji} = \theta_{ij}$. Applying the method of mixtures of independent conditional distributions, one considers the conditional distributions

$$
\begin{aligned}
F_{j|i}(x_j \mid x_i) &= \frac{\partial C_{ij}}{\partial u_i}\left[F_i(x_i),F_j(x_j)\right]\\
&= (1-\theta_{ij})\cdot F_j(x_j) + \theta_{ij}\cdot 1_{\{x_i \le F_i^{-1}[F_j(x_j)]\}}.
\end{aligned}
\tag{4.2}
$$

Denote by $\theta^{(i)} = (\theta_{ij},\ j \neq i)$ the vector of the $(1/2)n(n-1)$ dependence parameters, and let $\Delta^{(i)}$ be the set of the 2^{n-1} vectors $\delta^{(i)} = (\delta_{ij},\ j\neq i)$, where $\delta_{ij} \in \{0,1\}$. Then the n-variate mixture $F^{(i)}$ of independent conditional distributions (4.2), defined in (2.9), belongs to the n-variate copula

$$
\begin{aligned}
C^{(i)}(u_1,\dots,u_n) &= \sum_{\delta^{(i)}\in\Delta^{(i)}}\left[\prod_{j\neq i}(1-\theta_{ij})^{1-\delta_{ij}}\theta_{ij}^{\delta_{ij}}\right]\cdot u_i^{\prod_{j\neq i}(1-\delta_{ij})}\\
&\qquad \cdot\left[\prod_{j\neq i}u_j^{1-\delta_{ij}}\right]\cdot\min_{j\neq i}\left(u_j^{1-\delta_{ij}},u_i^{1-\prod_{j\neq i}(1-\delta_{ij})}\right).
\end{aligned}
\tag{4.3}
$$

This representation shows that each $C^{(i)}$ is a convex combination of the n different elementary copulas

$$EC^{r}(u_1,\dots,u_n)=\min_{1\le j\le r}(u_j)\cdot\left[\prod_{i=r+1}^{n}u_i\right],\quad r=0,2,3,\dots,n. \tag{4.4}$$

A distribution with copula EC^r is the convolution of a distribution with r comonotone components and a distribution with $n-r$ independent components. This observation is useful for the analytical evaluation of the distribution and stop-loss transform of dependent sums from convex combinations of these elementary copulas (see Theorem 5.1).

To obtain the bivariate copula $C_{rs}^{(i)}(u_r,u_s)$, which belongs to the bivariate margin F_{rs} of $F^{(i)}$, one sets $u_k=1$ for all $k\neq r,s$ in (4.3) to get the bivariate linear Spearman copula

$$C_{rs}^{(i)}(u_r,u_s)=\begin{cases}(1-\theta_{rs})u_ru_s+\theta_{rs}\min(u_r,u_s), & i=r \text{ or } i=s,\\ (1-\theta_{ir}\theta_{is})u_ru_s+\theta_{ir}\theta_{is}\min(u_r,u_s), & i\neq r,s.\end{cases} \tag{4.5}$$

It follows that the Spearman correlation coefficient of $C_{rs}^{(i)}$ is equal to

$$(\rho^S)_{rs}^{(i)}=\begin{cases}\theta_{rs}, & i=r \text{ or } i=s,\\ \theta_{ir}\theta_{is}, & i\neq r,s.\end{cases} \tag{4.6}$$

Therefore, for $r=i$ or $s=i$, the distribution $F^{(i)}$ has the desired linear Spearman bivariate margins F_{rs} with Spearman's rho θ_{rs}. Unfortunately, for the other indices $r,s\neq i$, the bivariate margin F_{rs} has the Spearman correlation coefficient $\theta_{ir}\theta_{is}$, which in general differs from the parameter θ_{rs}. To construct an n-variate distribution F, whose linear Spearman bivariate margins F_{rs} may have more general Spearman's rho $\rho_{rs}^S\in[0,1]$, we consider the convex combination of the copulas $C^{(i)}$, $i\in\{1,\dots,n\}$, defined for all $\theta=(\theta_{ij})$, $\theta_{ij}\neq 1$, by

$$\begin{gathered}C(u_1,\dots,u_n)=\frac{1}{c_n(\theta)}\cdot\sum_{i=1}^{n}\left[\prod_{j\neq i}\frac{1}{1-\theta_{ij}}\right]\cdot C^{(i)}(u_1,\dots,u_n),\\ c_n(\theta)=\sum_{i=1}^{n}\prod_{j\neq i}\frac{1}{1-\theta_{ij}}.\end{gathered} \tag{4.7}$$

If $\theta_{ij}=1$ for all i, j, one sets $C(u_1,\dots,u_n)=\min_{1\le j\le n}(u_j)$, which is the copula of n comonotone random variables. Using (4.6), one sees that the linear Spearman bivariate margins F_{rs} have Spearman's rho determined by

$$\rho_{rs}^S=\frac{1}{c_n(\theta)}\cdot\sum_{i=1}^{n}\left[\prod_{j\neq i}\frac{1}{1-\theta_{ij}}\right]\cdot\left[\theta_{rs}(\varepsilon_i^r+\varepsilon_i^s)+\theta_{ir}\theta_{is}(1-\varepsilon_i^r)(1-\varepsilon_i^s)\right], \tag{4.8}$$

where ε_i^j is a Kronecker symbol such that $\varepsilon_i^j=1$ if $j=i$ and $\varepsilon_i^j=0$ if $j\neq i$. Though it has not been shown that the functions (4.8), which map $\theta=(\theta_{ij})$, $\theta_{ij}\neq 1$, to $\rho^S=(\rho_{ij}^S)$, $\rho_{ij}^S\neq 1$, are one-to-one, the constructed copula (4.7) is sufficiently general and simple to yield tractable positive dependent n-variate distributions with bivariate margins equal or at least close to given linear Spearman bivariate margins. By appropriate choice of the univariate margins, say gamma or lognormal margins, the obtained parametric family of n-variate copulas satisfies the four desirable properties in Section 2.

To obtain expressions which can be implemented, insert (4.3) into (4.7) and rearrange terms to get the formula

$$\begin{aligned} C(u_1,\dots,u_n) &= \frac{1}{c_n(\theta)} \\ &\quad \cdot \left\{ n \cdot \left[\prod_{i=1}^{n} u_i \right] \right. \\ &\qquad \left. + \sum_{r=2}^{n} \sum_{i_1 \neq \cdots \neq i_r} \left[\prod_{j=2}^{r} \frac{\theta_{i_1 i_j}}{1-\theta_{i_1 i_j}} \right] \cdot \min_{1 \le j \le r} (u_{i_j}) \cdot \left[\prod_{k \notin \{i_1,\dots,i_r\}} u_k \right] \right\}. \end{aligned} \tag{4.9}$$

In particular, this shows that the constructed n-variate copula is a convex combination of elementary copulas of the type defined in (4.4). Regrouping these terms further, one obtains simpler expressions. For example, if $n = 3$, one has

$$\begin{aligned} C(u_1,u_2,u_3) = c_3(\theta)^{-1} \cdot \left\{ 3u_1u_2u_3 + 2\sum_{i<j} \left(\frac{\theta_{ij}}{1-\theta_{ij}} \right) \min(u_i,u_j) \left[\prod_{k \neq i,j} u_k \right] \right. \\ \left. + \left[\sum_{k \neq j \neq i} \frac{\theta_{ij}\theta_{ik}}{(1-\theta_{ij})(1-\theta_{ik})} \right] \min(u_1,u_2,u_3) \right\}. \end{aligned} \tag{4.10}$$

5. Properties of the multivariate linear Spearman copula. In the present section, some interesting and useful properties of the multivariate Spearman copula will be derived. We begin with the analytical exact evaluation of the distribution and stop-loss transform of dependent sums from an n-variate distribution with copula (4.9). In general, suppose an n-dimensional copula is a convex combination of other copulas, say $C = \sum \lambda_j C^j$. Then the distribution $F_S(s)$ and stop-loss transform $\pi_S(s)$ of dependent sums $S = \sum_{i=1}^{n} X_i^j$ from the multivariate model with copula C are the convex combinations of the distributions $F_{S^j}(s)$ and stop-loss transform $\pi_{S^j}(s)$ of the dependent sums $S^j = \sum_{i=1}^{n} X_i^j$ from the multivariate models with copulas C^j, that is, $F_S(s) = \sum \lambda_j F_{S^j}(s)$ and $\pi_S(s) = \sum \lambda_j \pi_{S^j}(s)$. Since this result applies to the n-variate copula (4.9), it suffices, up to permutations of variables, to discuss the evaluation of the distribution and stop-loss transform of sums from an elementary copula of the type EC^r in (4.4).

A multivariate distribution with copula EC^0 belongs to a random vector $(X_1,\dots,X_n)$ with independent components, while a distribution with copula EC^n belongs to a random vector with comonotone components. For EC^0, the distribution and stop-loss transform of sums are obtained using convolution formulas, while for EC^n, they are obtained through the addition of the same quantities from the individual components as stated in Remark 3.3. For example, the case of gamma marginals has been thoroughly discussed by Hürlimann [26]. There remains the derivation of summation formulas for the other $n-2$ copulas. We restrict the attention to nonnegative random variables with continuous and strictly increasing distributions whose densities exist.

Given random variables X_i, $1 \le i \le n$, with fixed marginal distributions $F_i(x)$, suppose that the distribution of the random vector $(X_1^+,\dots,X_r^+,X_{r+1}^\perp,\dots,X_n^\perp)$ belongs to the copula EC^r, $2 \le r \le n-1$. More precisely, $X_1^+,\dots,X_r^+$ represent the comonotonic version of $X_1,\dots,X_r$, $X_{r+1}^\perp,\dots,X_n^\perp$ represent the independent version of $X_{r+1},\dots,X_n$, and $(X_1^+,\dots,X_r^+)$ is independent from X_i, $r+1 \le i \le n$.

Theorem 5.1. *Suppose* $(X_1^+,\dots,X_r^+,X_{r+1}^\perp,\dots,X_n^\perp)$ *is a random vector whose distribution belongs to the copula* EC^r, $2 \le r \le n-1$. *Assume that the continuous and strictly increasing marginal distributions* $F_i(x)$ *with support* $[0,\infty)$ *have densities* $f_i(x)$, $1 \le i \le n$, *and set* $X = \sum_{i=r+1}^n X_i^\perp$. *Then the distribution and stop-loss transform of the sum* $S = X + \sum_{i=1}^r X_i^+$ *are determined by the formulas*

$$F_S(s) = \int_0^{u_s} u f_X\left[s - \sum_{i=1}^r F_i^{-1}(u)\right] \cdot \left[\sum_{i=1}^r f_i[F_i^{-1}(u)]^{-1}\right] du, \tag{5.1}$$

$$\pi_S(s) = E[S] - s + \int_0^{u_s} u F_X\left[s - \sum_{i=1}^r F_i^{-1}(u)\right] \cdot \left[\sum_{i=1}^r f_i[F_i^{-1}(u)]^{-1}\right] du, \tag{5.2}$$

where u_s *solves the equation*

$$\sum_{i=1}^r F_i^{-1}(u_s) = s. \tag{5.3}$$

Proof. Set $Y = \sum_{i=1}^r X_i^+$ and use Dhaene and Goovaerts [15, Lemma 2] to obtain the formulas $\pi_S(s) = E[S] - s + I(s)$ and $F_S(s) = 1 + (d/ds)\pi_S(s) = I'(s)$, with

$$I(s) = \int_0^s F_{(X,Y)}(x, s-x)\,dx. \tag{5.4}$$

By assumption, X is independent from Y, hence $F_{(X,Y)}(x,w) = F_X(x) \cdot F_Y(w)$. Inserting in (6.4) and making the change of variable $F_Y(t) = u$, one successively obtains

$$\begin{aligned} I(s) &= \int_0^s F_X(s-t)F_Y(t)\,dt = \int_0^{F_Y(s)} u F_X[s - Q_Y(u)] \cdot Q_Y'(u)\,du \\ &= \int_0^{u_s} u F_X\left[s - \sum_{i=1}^r F_i^{-1}(u)\right] \cdot \left[\sum_{i=1}^r f_i[F_i^{-1}(u)]\right] du, \end{aligned} \tag{5.5}$$

where the last equality follows from the fact that $Y = \sum_{i=1}^r X_i^+$ is a comonotone sum, and the definition of u_s in (5.3). The formula (5.2) is shown. Formula (5.1) follows from

$$\begin{aligned} F_S(s) &= I'(s) = F_X(0) \cdot F_Y(s) + \int_0^s f_X(s-t)F_Y(t)\,dt \\ &= \int_0^s f_X(s-t)F_Y(t)\,dt \end{aligned} \tag{5.6}$$

making the same change of variable $F_Y(t) = u$. □

Next, taking pattern from the recent contributions by Denuit et al. [12, Theorem 3.1] and Hürlimann [26, Remark 2.1], it is important to know if the constructed n-variate "positive dependent" distributions associated to random vectors $(X_1,\dots,X_n)$ are such that the dependent sums $S=\sum_{i=1}^n X_i$ are always bounded in convex order by the corresponding independent sum $S^\perp=\sum_{i=1}^n X_i^\perp$ and the comonotone sum $S^+=\sum_{i=1}^n X_i^+$.

Theorem 5.2. *Suppose $(X_1,\dots,X_n)$ is a random vector whose distribution belongs to the copula (4.9). Then one has the stochastic inequalities $S^\perp \le_{sl}\le S \le_{sl} S^+$.*

Proof. Since the copula (4.9) is a convex combination of elementary copulas of the type (4.4) and the operation of building dependent sums from random vector with such copulas is preserved under stop-loss order, it suffices to show the assertion for the elementary EC_n^r in (4.4) (the lower dimension is added for distinction). One applies induction on n. For $n=2$, the result is trivial because EC_2^0 yields $S^\perp$ and EC_2^2 yields S^+. Assume that the result holds for the dimension n and show it for $n+1$. One has the product representation

$$EC_{n+1}^r(u_1,\dots,u_{n+1})=\begin{cases}EC_n^r(u_1,\dots,u_n)\cdot u_{n+1}, & r\in\{0,2,3,\dots,n\},\\ \min(u_1,\dots,u_n)\cdot u_{n+1}, & r=n+1,\end{cases} \tag{5.7}$$

which shows that X_{n+1} is independent of $(X_1,\dots,X_n)$, hence also of $S_n=\sum_{i=1}^n X_i$. Since the stop-loss order is preserved under convolutions, it follows from the induction assumption $S_n^\perp \le_{sl}\le S_n \le_{sl} S_n^+$ that $S_{n+1}^\perp = S_n^\perp + X_{n+1}^\perp \le_{sl}\le S_n + X_{n+1}^\perp = S_{n+1} \le_{sl} S_{n+1}^+$. □

6. Multivariate conditional value-at-risk and risk allocation. Given a random variable X with survival function $S_X(x)=\Pr(X>x)$, consider the univariate *stop-loss transform* defined by $\pi_X(x)=E[(X-x)_+]=\int_x^\infty S_X(t)dt$. It is related to the *mean excess function* $m_X(x)=E[X-x\mid X>x]$ through $\pi_X(x)=S_X(x)\cdot m_X(x)$. The extension of these notions to a multivariate setting is straightforward.

Let $\mathbf{X}=(X_1,\dots,X_n)$ be a random vector with n-variate survival function $S_X(x)=\Pr(X_1>x_1,\dots,X_n>x_n)$. Then the ith component of the *stop-loss transform vector* $(\pi_X^1(x),\dots,\pi_X^n(x))$ is defined by $\pi_X^i(x)=\int_{x_i}^\infty S_X(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n)dt$, $i=1,\dots,n$. It is related to the *mean excess vector* $(m_X^1(x),\dots,m_X^n(x))$, $m_X^i(x)=E[X_i-x_i\mid X_j>x_j,\ j=1,\dots,n]$, through the relationships $\pi_X^i(x)=S_X(x)\cdot m_X^i(x)$, $i=1,\dots,n$.

In the univariate case, the conditional value-at-risk of X to the confidence level α satisfies the relations

$$\begin{aligned}\mathrm{CVaR}_\alpha[X]&=E[X\mid X>\mathrm{VaR}_\alpha[X]]\\&=\mathrm{VaR}_\alpha[X]+m_X[\mathrm{VaR}_\alpha[X]]\\&=\mathrm{VaR}_\alpha[X]+\frac{1}{1-\alpha}\pi_X[\mathrm{VaR}_\alpha[X]].\end{aligned} \tag{6.1}$$

In the multivariate situation, the *conditional value-at-risk vector* to the confidence level α, denoted by $\mathrm{CVaR}_\alpha[X]=(\mathrm{CVaR}_\alpha^1[X],\dots,\mathrm{CVaR}_\alpha^n[X])$, satisfies similarly the defining relations

$$\begin{aligned}\mathrm{CVaR}_\alpha^i[X] &= E\lfloor X_i \mid X_j > \mathrm{VaR}_\alpha[X_j],\ j=1,\ldots,n\rfloor\\ &= \mathrm{VaR}_\alpha[X_i] + m_X^i[\mathrm{VaR}_\alpha[X]]\\ &= \mathrm{VaR}_\alpha[X_i] + \frac{\pi_X^i[\mathrm{VaR}_\alpha[X]]}{S_X[\mathrm{VaR}_\alpha[X]]},\quad i=1,\ldots,n,\end{aligned}\tag{6.2}$$

where $\mathrm{VaR}_\alpha[X] = (\mathrm{VaR}_\alpha[X_1],\ldots,\mathrm{VaR}_\alpha[X_n])$ defines a *value-at-risk vector*.

We point out the usefulness of this multivariate extension to the evaluation of the economic risk capital of a portfolio of risks $\mathbf{X} = (X_1,\ldots,X_n)$. It is natural to define the economic risk capital of the aggregate risk $S = \sum_{i=1}^n X_i$ as a *multivariate conditional value-at-risk vector* to the confidence level α by setting

$$\mathrm{CVaR}_\alpha[S|X] = E\lfloor S \mid X_j > \mathrm{VaR}_\alpha[X_j],\ j=1,\ldots,n\rfloor.\tag{6.3}$$

Allocating to the risk X_i the ith component of the conditional value-at-risk vector (6.2) defines the *multivariate conditional value-at-risk allocation principle*, which by (6.3) turns out to be an *additive allocation principle*, which satisfies the identity

$$\mathrm{CVaR}_\alpha\left[\sum_{i=1}^n X_i|X\right] = \sum_{i=1}^n \mathrm{CVaR}_\alpha^i[X_i].\tag{6.4}$$

The proposed risk allocation rule yields a simple solution to the difficult risk allocation problem (e.g., Tasche [50] and Denault [11]).

For continuous univariate marginals, the univariate conditional value-at-risk measures $\mathrm{CVaR}_\alpha[\sum_{i=1}^n X_i]$ and $\mathrm{CVaR}_\alpha[X_i]$, $i=1,\ldots,n$, are coherent risk measures in the sense of Artzner et al. [4, 5] (e.g., Acerbi [1] and Acerbi and Tasche [2, 3]). In view of this fact, it seems important to derive connections between these risk measures and their multivariate counterparts. The following main result yields an interesting and surprising result.

Theorem 6.1. *Suppose $\mathbf{X} = (X_1,\ldots,X_n)$ is a random vector with strictly increasing continuous margins whose distribution belongs to the multivariate linear Spearman copula (4.9). Then one has $\mathrm{CVaR}_\alpha^i[X] = \mathrm{CVaR}_\alpha[X_i]$, $i=1,\ldots,n$, that is, each component of the conditional value-at-risk vector coincides with the conditional value-at-risk of the corresponding risk component.*

Proof. In a first step, we show the result for $\mathbf{X}$ with elementary copula (4.4). For simplicity, set $x_{i,\alpha} = \mathrm{VaR}_\alpha[X_i]$, $i=1,\ldots,n$, $x_\alpha = (x_{1,\alpha},\ldots,x_{n,\alpha})$, and $\varepsilon = 1-\alpha$. From the form of the survival copula (4.4), one obtains the survival function

$$S_X(x) = \min_{1\le j\le r}\{S_{X_i}(x_i)\}\left[\prod_{i=r+1}^n S_{X_i}(x_i)\right].\tag{6.5}$$

Using that $S_{X_i}(x_i) = \varepsilon$ and the definition of the stop-loss transform components, one obtains

$$S_X(x_\alpha) = \varepsilon^{n-r+1}, \qquad \pi_X^i(x_\alpha) = \pi_{X_i}(x_{i,\alpha})\varepsilon^{n-r}. \tag{6.6}$$

Inserted in (6.2) one gets, $\mathrm{CVaR}_\alpha^i[X] = x_{i,\alpha} + (1/\varepsilon)\pi_{X_i}(x_{i,\alpha}) = \mathrm{CVaR}_\alpha[X_i]$, which shows the result in this special case. The copula (4.9) is a convex combination of elementary copulas $EC^r(u_{p(1)},\dots,u_{p(n)})$ with appropriate permutations $(p(1),\dots,p(n))$ of $(1,\dots,n)$, for which (6.6) still holds. Since multivariate survival functions and stop-loss transforms are preserved under convex combinations, one sees that $\varepsilon\cdot\pi_X^i(x_\alpha) = \pi_{X_i}(x_{i,\alpha})\cdot S_X(x_\alpha)$ for all $\mathbf{X}$ with copula (4.9), which implies the desired result. □

This result has a remarkable economic implication. In a world of multivariate linear Spearman distributed risks with copula (4.9), the evaluation of the economic risk capital of the aggregate risk using (6.3) automatically yields a "fair" risk allocation in the sense that each risk component becomes allocated to its coherent univariate conditional value-at-risk. Of course, this rule is only applicable in situations where additivity is a desirable property. Whenever a discount for aggregation is prevalent and a subadditive risk allocation is preferred, the economic risk capital is better evaluated using $\mathrm{CVaR}_\alpha[\sum_{i=1}^n X_i]$. In this situation, a nonnegative *diversification effect*, defined by $\mathrm{CVaR}_\alpha[\sum_{i=1}^n X_i] - \sum_{i=1}^n \mathrm{CVaR}_\alpha[X_i]$, which can be evaluated using Theorem 5.1, will occur.

As a topic for future research, it is interesting to study the largest class of distributions or copulas for which the property of Theorem 6.1 holds. As the following example shows, this class is bigger than the convex hull of all permuted elementary copulas (4.4).

Example 6.2 (a bivariate distribution with the property of Theorem 6.1). Consider a random couple (X,Y) with survival function $S_{(X,Y)}(x,y) = (1/2)[e^{-x-y} + e^{-\max(x,y)}]$, $x,y \geq 0$. One has $\pi_{(X,Y)}^1(x,y) = S_{(X,Y)}(x,y) + (1/2)(y-x)_+e^{-y}$, $S_X(x) = \pi_X(x) = (1/2)e^{-x}$, and $S_Y(y) = \pi_Y(y) = (1/2)e^{-y}$. An elementary calculation shows that

$$\begin{aligned}\mathrm{CVaR}_\alpha^1[(X,Y)] &= \mathrm{CVaR}_\alpha[X] = \mathrm{CVaR}_\alpha^2[(X,Y)]\\ &= \mathrm{CVaR}_\alpha[Y] = 1 - \ln\{2(1-\alpha)\}.\end{aligned} \tag{6.7}$$

For a more comprehensive understanding, it will be necessary to discuss in the future the impact of the dependence structure of various multivariate distributions on the calculation of the multivariate conditional value-at-risk vector, which in general differs from the vector of the univariate conditional value-at-risk measures. We illustrate this issue at some simple parametric families of bivariate distributions with exponential margins.

Let (X,Y) be a random couple with exponential survival margins $S_X(x) = \exp(-x/\mu_X)$ and $S_Y(y) = \exp(-y/\mu_Y)$. For comparison, we use the Marshall-Olkin [40] bivariate exponential, which exhibits positive dependence, the negatively dependent exponential Gumbel [24], and the bivariate linear Spearman, which allows both positive and negative

TABLE 6.1. Conditional value-at-risk comparisons.

	Univariate CVaR		Bivariate CVaR	
$\rho = 0.5$	Marshall-Olkin	Linear Spearman	Marshall-Olkin	Linear Spearman
X	4.80445	4.80445	4.80445	4.80445
Y	4.20389	4.20389	4.47718	4.20389
S	7.56412	8.03684	9.28162	9.00835
$\rho = -0.3$	Gumbel	Linear Spearman	Gumbel	Linear Spearman
X	4.80445	4.80445	4.18356	4.80445
Y	4.20389	4.20389	3.66058	4.20389
S	5.37364	5.82982	7.84414	9.00835

dependence. The conditional value-at-risk measures of the margins are $\mathrm{CVaR}_\alpha[X] = (1-\ln\varepsilon)\mu_X$, $\mathrm{CVaR}_\alpha[Y] = (1-\ln\varepsilon)\mu_Y$, with $\varepsilon = 1-\alpha$. Formulas for the other risk measures are derived in a straightforward way. For the dependent sum $S = X+Y$, we use the formula $\mathrm{CVaR}_\alpha[S] = F_S^{-1}(\alpha) + (1/\varepsilon)\pi_S[F_S^{-1}(\alpha)]$, which requires expressions for $F_S(s)$ and $\pi_S(s)$. Pearson's rho is denoted by ρ. A typical numerical example is summarized in Table 6.1, where $\mu_X = 0.85715$ and $\mu_Y = 0.75$.

MARSHALL-OLKIN BIVARIATE EXPONENTIAL [40]. The survival distribution of this model and the required formulas to evaluate Table 6.1 are summarized by the following formulas:

$$
\begin{aligned}
S(x,y) &= \exp\left(-\alpha x - \beta y - \gamma \max(x,y)\right), \quad \alpha,\beta > 0,\ \gamma \ge 0,\\
\mu_X &= \frac{1}{\alpha+\gamma}, \qquad \mu_Y = \frac{1}{\beta+\gamma}, \qquad \rho = \frac{\gamma}{\alpha+\beta+\gamma} \ge 0,\\
\mathrm{CVaR}_\alpha^1[(X,Y)] &= \frac{1}{\alpha}\left[1-\frac{\gamma}{\alpha+\gamma}\varepsilon^{\alpha(\mu_Y-\mu_X)_+}\right] - \ln\varepsilon\cdot\mu_X,\\
\mathrm{CVaR}_\alpha^2[(X,Y)] &= \frac{1}{\beta}\left[1-\frac{\gamma}{\beta+\gamma}\varepsilon^{\alpha(\mu_X-\mu_Y)_+}\right] - \ln\varepsilon\cdot\mu_Y,\\
F_S(x) &= 1-\frac{1}{2}(\alpha+\beta+\gamma)\left(\frac{1}{\alpha+\gamma-\beta}+\frac{1}{\beta+\gamma-\alpha}\right)e^{-(1/2)(\alpha+\beta+\gamma)x}\\
&\quad + \frac{\beta}{\alpha+\gamma-\beta}e^{-(\alpha+\gamma)x} + \frac{\alpha}{\beta+\gamma-\alpha}e^{-(\beta+\gamma)x},\\
\pi_S(x) &= \left(\frac{1}{\alpha+\gamma-\beta}+\frac{1}{\beta+\gamma-\alpha}\right)e^{-(1/2)(\alpha+\beta+\gamma)x}\\
&\quad - \left(\frac{1}{\alpha+\gamma-\beta}-\frac{1}{\alpha+\gamma}\right)e^{-(\alpha+\gamma)x} - \left(\frac{1}{\beta+\gamma-\alpha}-\frac{1}{\beta+\gamma}\right)e^{-(\beta+\gamma)x}.
\end{aligned}
\tag{6.8}
$$

BIVARIATE EXPONENTIAL GUMBEL [24]. The survival distribution of this model and the required formulas to evaluate Table 6.1 are summarized by the following formulas:

$$
\begin{aligned}
S(x,y) &= \exp(-\alpha x - \beta y - \theta x y), \quad \alpha,\beta > 0,\ 0 \le \theta \le \alpha\beta,\\
\mu_X &= \frac{1}{\alpha}, \qquad \mu_Y = \frac{1}{\beta}, \qquad \rho = \alpha\beta\int_0^\infty \frac{e^{-\alpha t}\,dt}{\beta+\theta t} - 1 \le 0,
\end{aligned}
$$

$$\begin{aligned}
\mathrm{CVaR}_\alpha^1[(X,Y)] &= \left[\frac{1}{1-\theta\mu_X\mu_Y\ln\varepsilon} - \ln\varepsilon\right]\mu_X, \\
\mathrm{CVaR}_\alpha^2[(X,Y)] &= \left[\frac{1}{1-\theta\mu_X\mu_Y\ln\varepsilon} - \ln\varepsilon\right]\mu_Y
\end{aligned}$$

$$\begin{aligned}
F_S(x) &= 1 - e^{-\alpha x} - e^{-\beta x} \\
&\quad + \left\{g'(x) - \left[\beta + \frac{\theta}{4}\left(x + \frac{\alpha-\beta}{\theta}\right)\right]g(x)\right\}\exp\left\{-\beta x - \frac{\theta}{4}\left(x + \frac{\alpha-\beta}{\theta}\right)^2\right\}, \\
\pi_S(x) &= \mu_X e^{-\alpha x} + \mu_Y e^{-\beta x} + g(x)\exp\left\{-\beta x - \frac{\theta}{4}\left(x + \frac{\alpha-\beta}{\theta}\right)^2\right\}, \\
g(x) &= \sum_{k=0}^{\infty}\frac{\theta^k}{k!}\frac{1}{(2k+1)2^{2k+1}}\left[\left(x - \frac{\alpha-\beta}{\theta}\right)^{2k+1} - (-1)^{2k+1}\left(x + \frac{\alpha-\beta}{\theta}\right)^{2k+1}\right].
\end{aligned} \tag{6.9}$$

Linear Spearman bivariate exponential. The survival distribution of this model and the required formulas to evaluate Table 6.1 are summarized by the following formulas:

$$\begin{gathered}
S(x,y) = \begin{cases} (1-\theta)e^{-\alpha x-\beta y} + \theta\min\left(e^{-\alpha x}, e^{-\beta y}\right), & \theta\in[0,1], \\ (1+\theta)e^{-\alpha x-\beta y} - \theta\max\left(e^{-\alpha x} + e^{-\beta y} - 1, 0\right), & \theta\in[-1,0], \end{cases} \\
\mu_X = \frac{1}{\alpha}, \qquad \mu_Y = \frac{1}{\beta}, \\
\rho = \theta\left\{1 + \frac{1}{2}[1-\mathrm{sgn}(\theta)]\frac{1}{\mu_Y^2}\int_0^\infty y\ln(1-e^{-\beta y})e^{-\beta y}dy\right\}, \\
\mathrm{CVaR}_\alpha^1[(X,Y)] = \mathrm{CVaR}_\alpha[X], \qquad \mathrm{CVaR}_\alpha^2[(X,Y)] = \mathrm{CVaR}_\alpha[Y] \quad (\text{Theorem 6.1}).
\end{gathered} \tag{6.10}$$

The quantities $F_S(x)$ and $\pi_S(x)$ are calculated using Theorem 3.4.

Some comments and recommendations are in order. By positive dependence, the multivariate conditional value-at-risk for the Marshall-Olkin is greater than for the linear Spearman, while by negative dependence, it is smaller for the Gumbel than for the linear Spearman. If the risk measure should be invariant with respect to the dependence structure, or if the diversification effect should vanish (additive risk allocation), we recommend the use of the multivariate conditional value-at-risk for the linear Spearman. A discrimination of the risk measure with respect to the dependence structure is obtained by using either $\mathrm{CVaR}[S]$ or $\mathrm{CVaR}[S|X]$ for copulas different from the linear Spearman one. A maximal diversification effect is obtained by using $\mathrm{CVaR}[S]$ with the Marshall-Olkin by positive dependence and with the Gumbel by negative dependence. A more stable diversification effect is obtained with the linear Spearman. Whether these observations generalize to other bivariate distributions and extend to the multivariate situation is left to future investigations.

Finally, we want to mention that the approach of the present paper has a lot of alternative significant applications including very recent ones like Cherubini and Luciano [6, 7].

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