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THE ORLICZ SPACE OF ENTIRE SEQUENCES

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Let Γ denote the space of all entire sequences and $\wedge$ the space of all analytic sequences. This paper is devoted to the study of the general properties of Orlicz space Γ_M of Γ.

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1. Introduction. An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$. If the convexity of Orlicz function M is replaced by $M(x+y) \le M(x)+M(y)$, then this function is called a modulus function, defined and discussed by Ruckle [5] and Maddox [4].

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \le KM(u)$ $(u \ge 0)$. The Δ_2-condition is equivalent to $M(\ell u) \le K.\ \ell M(u)$, for all values of u and for $\ell > 1$.

An Orlicz function M can always be represented in the following integral form: $M(x) = \int_0^x q(t)dt$, where q, known as the kernel of M, is right-differentiable for $t \ge 0$, $q(0) = 0$, $q(t) > 0$ for $t > 0$, q is nondecreasing, and $q(t) \to \infty$ as $t \to \infty$. Lindenstrauss and Tzafriri [3] used the idea of Orlicz function to construct the Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}, \tag{1.1}$$

where $w = \{\text{all complex sequences}\}$.

The space ℓ_M with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\} \tag{1.2}$$

becomes a Banach space which is called an Orlicz sequence space.

2. A complex sequence whose kth term is x_k will be denoted by (x_k) or x. A sequence $x = (x_k)$ is said to be analytic if $\sup_{(k)} |x_k|^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by $\wedge$. A sequence x is called an entire sequence if $\lim_{k\to\infty} |x_k|^{1/k} = 0$. The vector space of all entire sequences will be denoted by Γ.

Definition 2.1. The space consisting of all sequences x in w such that $M(|x_k|^{1/k}/\rho) \to 0$ as $k \to \infty$ for some arbitrarily fixed $\rho > 0$ is denoted by Γ_M, with M being a modulus function. In other words, $\{M(|x_k|^{1/k}/\rho)\}$ is a null sequence. The space Γ_M is a metric space with the metric

$$d(x,y) = \sup_{(k)} M\left(\frac{|x_k - y_k|^{1/k}}{\rho}\right) \tag{2.1}$$

for all $x = \{x_k\}$ and $y = \{y_k\}$ in Γ_M.

Given a sequence $x = \{x_k\}$ whose nth section is the sequence $x^{(n)} = \{x_1, x_2, \dots, x_n, 0, 0, \dots\}$, $\delta^{(n)} = (0,0,\dots,1,0,0,\dots)$, with 1 in the nth place and zeros elsewhere; let $\Phi = \{\text{all finite sequences}\}$.

An FK-space (or a metric space) X is said to have AK property if $(\delta^{(n)})$ is a Schauder basis for X. Or equivalently $x^{(n)} \to x$.

The space is said to have or be an AD space if Φ is dense in X.

We note that AK implies AD by [1].

If X is a sequence space, we give the following definitions:

(i) X' = the continuous dual of X;

(ii) $X^\alpha = \{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty$, for each $x \in X\}$;

(iii) $X^\beta = \{a = (a_k) : \sum_{k=1}^{\infty} a_k x_k$ is convergent, for each $x \in X\}$;

(iv) $X^\gamma = \{a = (a_k) : \sup_{(n)} |\sum_{k=1}^{n} a_k x_k| < \infty$, for each $x \in X\}$;

(v) let X be an FK-space $\supset \Phi$, then $X^f = \{f(\delta^{(n)}) : f \in X'\}$. X^α, X^β, and X^γ are called the α- (or Köthe-Toeplitz-) dual of X, β- (or generalized-Köthe-Toeplitz-) dual of X, and γ-dual of X, respectively.

Note that $X^\alpha \subset X^\beta \subset X^\gamma$. If $X \subset Y$, then $Y^\mu \subset X^\mu$, for $\mu = \alpha$, β, or γ.

Lemma 2.2 (see [6, Theorem 7.2.7]). *Let X be an FK-space $\supset \Phi$. Then*

(i) *$X^\gamma \subset X^f$;*

(ii) *if X has AK, $X^\beta = X^f$;*

(iii) *if X has AD, $X^\beta = X^\gamma$.*

We note that $\Gamma^\alpha = \Gamma^\beta = \Gamma^\gamma = \wedge$.

Proposition 2.3. *$\Gamma \subset \Gamma_M$, with the hypothesis that $M(|x_k|^{1/k}/\rho) \le |x_k|^{1/k}$.*

Proof. Let $x \in \Gamma$. Then we have the following implications:

$$|x_k|^{1/k} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{2.2}$$

But $M(|x_k|^{1/k}/\rho) \le |x_k|^{1/k}$, by our assumption, implies that

$$\begin{aligned} M\left(\frac{|x_k|^{1/k}}{\rho}\right) &\longrightarrow 0 \quad \text{as } k \longrightarrow \infty \ (\text{by } (2.2)) \\ &\Longrightarrow x \in \Gamma_M \\ &\Longrightarrow \Gamma \subset \Gamma_M. \end{aligned} \tag{2.3}$$

This completes the proof. □

Proposition 2.4. *Γ_M has AK where M is a modulus function.*

Proof. Let $x = \{x_k\} \in \Gamma_M$, but then $\{M(|x_k|^{1/k}/\rho)\} \in \Gamma$, and hence

$$\sup_{k \geq n+1} M\left(\frac{|x_k|^{1/k}}{\rho}\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.4}$$

By using (2.4), $d(x, x^{[n]}) = \sup_{k \geq n+1} M(|x_k|^{1/k}/\rho) \to 0$ as $n \to \infty$, which implies that $x^{[n]} \to x$ as $n \to \infty$, implying that Γ_M has AK. This completes the proof. □

Proposition 2.5. *Γ_M is solid.*

Proof. Let $|x_k| \leq |y_k|$ and let $y = (y_k) \in \Gamma_M$. $M(|x_k|^{1/k}/\rho) \leq M(|y_k|^{1/k}/\rho)$, because M is nondecreasing. But $M(|y_k|^{1/k}/\rho) \in \Gamma$ because $y \in \Gamma_M$. That is, $M(|y_k|^{1/k}/\rho) \to 0$ as $k \to \infty$ and $M(|x_k|^{1/k}/\rho) \to 0$ as $k \to \infty$. Therefore $x = \{x_k\} \in \Gamma_M$. This completes the proof. □

Proposition 2.6. *Let M be an Orlicz function which satisfies the Δ_2-condition. Then $\Gamma \subset \Gamma_M$.*

Proof. Let

$$x \in \Gamma. \tag{2.5}$$

Then $|x_k|^{1/k} \leq \varepsilon$ for sufficiently large k and every $\varepsilon > 0$. But then by taking $\rho \geq 1/2$,

$$\begin{aligned}
M\left(\frac{|x_k|^{1/k}}{\rho}\right) &\leq M\left(\frac{\varepsilon}{\rho}\right) \quad (\text{because } M \text{ is nondecreasing}) \\
&\leq M(2\varepsilon) \\
\Longrightarrow M\left(\frac{|x_k|^{1/k}}{\rho}\right) &\leq KM(\varepsilon) \quad (\text{by the } \Delta_2\text{-condition, for some } K > 0) \\
&\leq \varepsilon \quad \left(\text{by defining } M(\varepsilon) < \frac{\varepsilon}{k}\right) \\
\Longrightarrow M\left(\frac{|x_k|^{1/k}}{\rho}\right) &\longrightarrow 0 \quad \text{as } k \longrightarrow \infty.
\end{aligned} \tag{2.6}$$

Hence $x \in \Gamma_M$.

From (2.5) and since

$$x \in \Gamma_M, \tag{2.7}$$

we get

$$\Gamma \subset \Gamma_M. \tag{2.8}$$

This completes the proof. □

Proposition 2.7. *If M is a modulus function, then Γ_M is a linear set over the set of complex numbers $\mathbb{C}$.*

Proof. Let $x, y \in \Gamma_M$ and $\alpha, \beta \in \mathbb{C}$. In order to prove the result, we need to find some ρ_3 such that

$$M\left(\frac{|\alpha x_k + \beta y_k|^{1/k}}{\rho_3}\right) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{2.9}$$

Since $x, y \in \Gamma_M$, there exist some positive ρ_1 and ρ_2 such that

$$\begin{aligned} M\left(\frac{|x_k|^{1/k}}{\rho_1}\right) &\longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \\ M\left(\frac{|y_k|^{1/k}}{\rho_2}\right) &\longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \end{aligned} \tag{2.10}$$

Since M is a nondecreasing modulus function, we have

$$\begin{aligned} M\left(\frac{|\alpha x_k + \beta y_k|^{1/k}}{\rho_3}\right) &\le M\left(\frac{|\alpha x_k|^{1/k}}{\rho_3} + \frac{|\beta y_k|^{1/k}}{\rho_3}\right) \\ &\le M\left(\frac{|\alpha|^{1/k}|x_k|^{1/k}}{\rho_3} + \frac{|\beta|^{1/k}|y_k|^{1/k}}{\rho_3}\right) \\ &\le M\left(\frac{|\alpha|\,|x_k|^{1/k}}{\rho_3} + \frac{|\beta|\,|y_k|^{1/k}}{\rho_3}\right). \end{aligned} \tag{2.11}$$

Take ρ_3 such that

$$\frac{1}{\rho_3} = \min\left\{\frac{1}{|\alpha|}\frac{1}{\rho_1}, \frac{1}{|\beta|}\frac{1}{\rho_2}\right\}. \tag{2.12}$$

Then

$$\begin{aligned} M\left(\frac{|\alpha x_k + \beta y_k|^{1/k}}{\rho_3}\right) &\le M\left(\frac{|x_k|^{1/k}}{\rho_1} + \frac{|y_k|^{1/k}}{\rho_2}\right) \\ &\le M\left(\frac{|x_k|^{1/k}}{\rho_1}\right) + M\left(\frac{|y_k|^{1/k}}{\rho_2}\right) \\ &\longrightarrow 0 \quad (\text{by } (2.10)). \end{aligned} \tag{2.13}$$

Hence

$$M\left(\frac{|\alpha x_k + \beta y_k|^{1/k}}{\rho_3}\right) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{2.14}$$

So $(\alpha x + \beta y) \in \Gamma_M$. Therefore Γ_M is linear. This completes the proof. □

Definition 2.8. Let $p = (p_k)$ be any sequence of positive real numbers. Then

$$\Gamma_M(p) = \left\{ x = \{x_k\} : \left(M\left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{p_k} \longrightarrow 0 \text{ as } k \longrightarrow \infty \right\}. \tag{2.15}$$

Suppose that p_k is a constant for all k, then $\Gamma_M(p) = \Gamma_M$.

Proposition 2.9. *Let $0 \le p_k \le q_k$ and let $\{q_k/p_k\}$ be bounded. Then $\Gamma_M(q) \subset \Gamma_M(p)$.*

Proof. Let

$$x \in \Gamma_M(q), \tag{2.16}$$

$$\left(M\left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{q_k} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{2.17}$$

Let $t_k = (M(|x_k|^{1/k}/\rho))^{q_k}$ and $\lambda_k = p_k/q_k$. Since $p_k \le q_k$, we have $0 \le \lambda_k \le 1$.
Take $0 < \lambda < \lambda_k$. Define

$$\begin{gathered} u_k = \begin{cases} t_k & (t_k \ge 1) \\ 0 & (t_k < 1), \end{cases} \\ v_k = \begin{cases} 0 & (t_k \ge 1) \\ t_k & (t_k < 1), \end{cases} \\ t_k = u_k + v_k, \qquad t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}. \end{gathered} \tag{2.18}$$

Now it follows that

$$u_k^{\lambda_k} \le u_k \le t_k, \qquad v_k^{\lambda_k} \le v_k^{\lambda}. \tag{2.19}$$

Since$t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$, then $t_k^{\lambda_k} \le t_k + v_k^{\lambda}$.

$$\begin{aligned} \left(\left(M\left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{q_k} \right)^{\lambda_k} &\le \left(M\left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{q_k} \\ \Longrightarrow \left(\left(M\left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{q_k} \right)^{p_k/q_k} &\le \left(M\left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{q_k} \\ \Longrightarrow \left(M\left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{p_k} &\le \left(M\left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{q_k}. \end{aligned} \tag{2.20}$$

But

$$\left(M\left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{q_k} \longrightarrow 0 \quad (\text{by } (2.17)). \tag{2.21}$$

Hence $(M(|x_k|^{1/k}/\rho))^{p_k} \to 0$ as $k \to \infty$. Hence

$$x \in \Gamma_M(p). \tag{2.22}$$

From (2.16) and (2.22), we get

$$\Gamma_M(q) \subset \Gamma_M(p). \tag{2.23}$$

This completes the proof. □

Proposition 2.10. (a) *Let* $0 < \inf p_k \le p_k \le 1$. *Then* $\Gamma_M(p) \subset \Gamma_M$.
(b) *Let* $1 \le p_k \le \sup p_k < \infty$. *Then* $\Gamma_M \subset \Gamma_M(p)$.

Proof. (a) Let $x \in \Gamma_M(p)$,

$$\left(M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right)^{p_k} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{2.24}$$

Since $0 < \inf p_k \le p_k \le 1$,

$$\left(M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right) \le \left(M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right)^{p_k}, \tag{2.25}$$

From (2.24) and (2.25) it follows that

$$x \in \Gamma_M. \tag{2.26}$$

Thus

$$\Gamma_M(p) \subset \Gamma_M. \tag{2.27}$$

We have thus proven (a).
(b) Let $p_k \ge 1$ for each k and $\sup p_k < \infty$ and let $x \in \Gamma_M$.

$$M\left(\frac{|x_k|^{1/k}}{\rho}\right) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{2.28}$$

Since $1 \le p_k \le \sup p_k < \infty$, we have

$$\begin{gathered}\left(M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right)^{p_k} \le \left(M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right), \\ \left(M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right)^{p_k} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty \text{ (by using (2.28))}.\end{gathered} \tag{2.29}$$

Therefore $x \in \Gamma_M(p)$. This completes the proof. □

Proposition 2.11. *Let $0 < p_k \le q_k < \infty$ for each k. Then $\Gamma_M(p) \subseteq \Gamma_M(q)$.*

Proof. Let $x \in \Gamma_M(p)$

$$\left(M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right)^{p_k} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{2.30}$$

This implies that $(M(|x_k|^{1/k}/\rho)) \le 1$ for sufficiently large k. Since M is nondecreasing, we get

$$\begin{gathered}\left(M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right)^{q_k} \le \left(M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right)^{p_k} \\ \Longrightarrow \left(M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right)^{q_k} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty \text{ (by using (2.30))}.\end{gathered} \tag{2.31}$$

Since $x \in \Gamma_M(q)$, hence $\Gamma_M(p) \subseteq \Gamma_M(q)$. This completes the proof. □

Proposition 2.12. *$\Gamma_M(p)$ is r-convex for all r, where $0 \le r \le \inf p_k$. Moreover, if $p_k = p \le 1$ for all k, then they are p-convex.*

Proof. We will prove the theorem for $\Gamma_M(p)$. Let $x \in \Gamma_M(p)$ and $r \in (0, \lim_{n\to\infty} \inf p_n)$. Then, there exists k_0 such that $r \le p_k$ for all $k > k_0$.

Now, define

$$g^*(x) = \inf\left\{\rho : M\left(\frac{|x_k - y_k|^{1/k}}{\rho}\right)^r + M\left(\frac{|x_k - y_k|^{1/k}}{\rho}\right)^{p_n}\right\}. \tag{2.32}$$

Since $r \le p_k \le 1$ for all $k > k_0$, g^* is subadditive. Further, for $0 \le |\lambda| \le 1$,

$$|\lambda|^{p_k} \le |\lambda|^r \quad \forall k > k_0. \tag{2.33}$$

Therefore, for each λ, we have

$$g^*(\lambda x) \le |\lambda|^r \cdot g^*(x). \tag{2.34}$$

Now, for $0 < \delta < 1$,

$$U = \{x : g^*(x) \le \delta\}, \tag{2.35}$$

which is an absolutely r-convex set, for

$$|\lambda|^r + |\mu|^r \le 1, \qquad x, y \in U. \tag{2.36}$$

Now

$$\begin{aligned} g^*(\lambda x+\mu y) &\le g^*(\lambda x)+g^*(\mu y) \\ &\le |\lambda|^r g^*(x)+|\mu|^r g^*(y) \\ &\le |\lambda|^r\delta+|\mu|^r\delta \quad \text{(using (2.34) and (2.35))} \\ &\le (|\lambda|^r+|\mu|^r)\delta \\ &\le 1\cdot\delta \quad \text{(by using (2.36))} \\ &\le \delta. \end{aligned} \tag{2.37}$$

If $p_k = p \le 1$ for all k, then for $0 < r < 1$, $U = \{x : g^*(x) \le \delta\}$ is an absolutely p-convex set. This can be obtained by a similar analysis and therefore we omit the details. This completes the proof. □

Proposition 2.13. $(\Gamma_M)^\beta = \wedge$.

Proof

Step 1. $\Gamma \subset \Gamma_M$ by Proposition 2.3; this implies that $(\Gamma_M)^\beta \subset \Gamma^\beta = \wedge$. Therefore,

$$(\Gamma_M)^\beta \subset \wedge. \tag{2.38}$$

Step 2. Let $y \in \wedge$. Then $|y_k| < M^k$ for all k and for some constant $M > 0$.

Let $x \in \Gamma_M$. Then $M(|x_k|^{1/k}/\rho) \to 0$ as $k \to \infty$. Hence $M(|x_k|^{1/k}/\rho) < \varepsilon$ for given $\varepsilon > 0$ for sufficiently large k.

Take $\varepsilon = 1/2M$ so that $M(|x_k|/\rho) < 1/(2M)^k$.

But then $M(|x_k y_k|/\rho) \le 1/2^k$ so that $\sum_{k=1}^\infty M(|x_k y_k|/\rho)$ converges. Therefore $\sum_{k=1}^\infty M(x_k y_k/\rho)$ converges. Hence $\sum_{k=1}^\infty x_k y_k$ converges so that $y \in (\Gamma_M)^\beta$. Thus

$$\wedge \subset (\Gamma_M)^\beta. \tag{2.39}$$

Step 3. From (2.38) and (2.39), we obtain

$$(\Gamma_M)^\beta = \wedge. \tag{2.40}$$

This completes the proof. □

Proposition 2.14. $(\Gamma_M)^\mu = \wedge$ *for* $\mu = \alpha, \beta, \gamma, f$.

Proof

Step 1. Γ_M has AK by Proposition 2.4. Hence, by Lemma 2.2(i), we get $(\Gamma_M)^\beta = (\Gamma_M)^f$. But $(\Gamma_M)^\beta = \wedge$. Hence

$$(\Gamma_M)^f = \wedge. \tag{2.41}$$

Step 2. Since AK implies AD, hence by Lemma 2.2(iii) we get $(\Gamma_M)^\beta = (\Gamma_M)^\gamma$. Therefore

$$(\Gamma_M)^\gamma = \wedge. \tag{2.42}$$

Step 3. Γ_M is normal by Proposition 2.5. Hence, by [2, Proposition 2.7], we get

$$(\Gamma_M)^{\alpha} = (\Gamma_M)^{\gamma} = \wedge. \tag{2.43}$$

From (2.41), (2.42), and (2.43), we have

$$(\Gamma_M)^{\alpha} = (\Gamma_M)^{\beta} = (\Gamma_M)^{\gamma} = (\Gamma_M)^{f} = \wedge. \tag{2.44}$$

□

Proposition 2.15. *The dual space of Γ_M is $\wedge$. In other words, $\Gamma_M^* = \wedge$.*

Proof. We recall that δ^k has 1 in the kth place and zeros elsewhere, with

$$\begin{aligned} x = \delta^k, \quad \left\{M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right\} &= \left\{\frac{M(0)^1}{\rho}, \frac{M(0)^{1/2}}{\rho}, \dots, \frac{M(1)^{1/k}}{\rho}, \frac{M(0)^{1/(k+1)}}{\rho}, \dots\right\} \\ &= \left\{0, 0, \dots, \frac{M(1)^{1/k}}{\rho}, 0, \dots\right\} \end{aligned} \tag{2.45}$$

which is a null sequence. Hence $\delta^k \in \Gamma_M$. $f(x) = \sum_{k=1}^{\infty} x_k y_k$ with $x \in \Gamma_M$ and $f \in \Gamma_M^*$, where Γ_M^* is the dual space of Γ_M. Take $x = \delta^k \in \Gamma_M$. Then

$$|y_k| \le \|f\| d(\delta^k, 0) < \infty \quad \forall k. \tag{2.46}$$

Thus (y_k) is a bounded sequence and hence an analytic sequence. In other words, $y \in \wedge$. Therefore $\Gamma_M^* = \wedge$. This completes the proof. □

Lemma 2.16 [6, Theorem 8.6.1]. *$Y \supset X \Leftrightarrow Y^f \subset X^f$, where X is an AD-space and Y an FK-space.*

Proposition 2.17. *Let Y be any FK-space $\supset \Phi$. Then $Y \supset \Gamma_M$ if and only if the sequence $\delta^{(k)}$ is weakly analytic.*

Proof. The following implications establish the result: since Γ_M has AD and by Lemma 2.16,

$$\begin{aligned} Y \supset \Gamma_M &\Longleftrightarrow Y^f \subset (\Gamma_M)^f \\ &\Longleftrightarrow Y^f \subset \wedge \quad \left(\text{since } (\Gamma_M)^f = \wedge\right) \\ &\Longleftrightarrow \text{for each } f \in Y', \text{ the topological dual of } Y \cdot f(\delta^{(k)}) \in \wedge \\ &\Longleftrightarrow f(\delta^{(k)}) \text{ is analytic} \\ &\Longleftrightarrow \delta^{(k)} \text{ is weakly analytic,} \end{aligned} \tag{2.47}$$

this completes the proof. □

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