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BEND SETS, N-SEQUENCES, AND MAPPINGS

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The existence of an N-sequence in a continuum is a common obstruction that implies nonsmoothness, noncontractibility, nonselectibility, and nonexistence of any mean. The aim of the present paper is to investigate if some variants of the concept of an N-sequence also keep these properties. In particular, mapping properties of bend sets are studied.

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All considered spaces are assumed to be metric and all mappings are continuous. The symbol $\mathbb{N}$ stands for the set of all positive integers. Given a space X and its subspaces A and B with $A \subset B$, we denote by $\mathrm{cl}_B(A)$ and $\mathrm{bd}_B(A)$ the closure and the boundary of A with respect to B, respectively.

A *continuum* means a compact connected space. A 1-dimensional continuum is called a *curve*. A continuum is said to be *hereditarily unicoherent* provided that the intersection of every two of its subcontinua is connected. A *dendroid* means an arcwise connected and hereditarily unicoherent continuum. A *ramification point* in a dendroid X means a vertex of a simple triod contained in X. A *fan* denotes a dendroid having exactly one ramification point.

A continuum X is said to be *uniformly arcwise connected* provided that it is arcwise connected and that for each $\varepsilon > 0$ there is a $k \in \mathbb{N}$ such that every arc in X contains k points that cut it into subarcs of diameters less than ε. By [14, Theorem 3.5, page 322] a dendroid is uniformly arcwise connected if and only if it is a (continuous) image of the *Cantor fan* (i.e., the cone over the Cantor middle-thirds set).

Given a continuum X we let $C(X)$ denote the hyperspace of all nonempty subcontinua of X equipped with the *Hausdorff metric* (equivalently, with the Vietoris topology; see, e.g., [20, (0.1), page 1, and (0.12), page 10] or [12, page 9]).

A dendroid X is said to be *smooth at a point* $p \in X$ provided that for each point $a \in X$ and for each sequence of points $\{a_n : n \in \mathbb{N}\}$ in X that converges to a, the sequence of arcs $\{pa_n : n \in \mathbb{N}\}$ converges to the arc pa (in the sense of the Hausdorff metric). A dendroid X is said to be *smooth* provided that there is a point $p \in X$ such that X is smooth at p. The point p is then called an *initial point of* X.

Given spaces X and Y, a mapping $H : X \times [0,1] \to Y$ is called a *homotopy*. Two mappings $f, g : X \to Y$ are said to be *homotopic* provided that there exists a homotopy H such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for each $x \in X$. If every mapping $f : X \to Y$ is homotopic to a constant mapping, then X is said to be *contractible with respect to* Y. A space X is said to be *contractible* provided that there are a homotopy $H : X \times [0,1] \to X$

and a point $p \in X$ such that for each point $x \in X$ we have $H(x,0) = x$ and $H(x,1) = p$. It is known that X is contractible if and only if it is contractible with respect to every space Y, see [15, Section 54, VI, Theorem 2, page 374].

By a *selection* for $C(X)$ we mean a mapping $\sigma : C(X) \to X$ such that $\sigma(A) \in A$ for each $A \in C(X)$. Note that a selection for $C(X)$ is a retraction of $C(X)$ onto X. A continuum X is said to be *selectible* provided that there is a selection for $C(X)$.

A selection $\sigma : C(X) \to X$ is said to be *rigid* provided that if $A, B \in C(X)$ and $\sigma(B) \in A \subset B$, then $\sigma(A) = \sigma(B)$.

A *mean* on a space X is a mapping $\mu : X \times X \to X$ such that $\mu(x,y) = \mu(y,x)$ and $\mu(x,x) = x$ for every $x, y \in X$ (in other words, it is a symmetric, idempotent, continuous binary operation on X). If also $\mu(x,\mu(y,z)) = \mu(\mu(x,y),z)$ for every $x,y,z \in X$, then the mean μ is said to be *associative*.

We start with recalling basic results related to these concepts.

Theorem 1. *The following results are known.*

(1.1) *Each smooth dendroid is uniformly arcwise connected,* [8, Corollary 16, page 318].

(1.2) *Each contractible curve is a uniformly arcwise connected dendroid,* [3, Propositions 1, 4, and 5, page 73] *and* [9, Theorem 3, page 94].

(1.3) *A locally connected curve is contractible if and only if it is a dendrite, see, for example,* [5, (0.3), page 561].

(1.4) *Each selectible continuum is a uniformly arcwise connected dendroid,* [21, Lemma 3, page 370] *and* [4, Proposition 2, page 110].

(1.5) *A locally connected continuum is selectible if and only if it is a dendrite,* [21, Corollary, page 371].

(1.6) *A continuum X is a smooth dendroid if and only if there exists a rigid selection for the hyperspace $C(X)$ of its subcontinua,* [25, Theorem 2, page 1043]. *Thus each smooth dendroid is selectible, but not conversely,* [21, Theorem 3, pages 372–374] *and* [4, Propositions 3 and 4, pages 110–111].

(1.7) *If a curve admits an associative mean, then it is a smooth (thus uniformly arcwise connected) dendroid,* [7, Theorem 5.21, page 20]*, so there exists a rigid selection for the hyperspace $C(X)$ of its subcontinua,* by (1.6).

(1.8) *A locally connected curve admits a mean if and only if it is a dendrite, see* [24, page 85] *and compare* [7, Proposition 5.30, page 22].

A dendroid X is said to contain a *zigzag* provided that there exist in X an arc pq, a sequence of arcs p_nq_n, and two sequences of points p'_n and q'_n situated in these arcs in such a manner that $p_n < q'_n < p'_n < q_n$ (where $<$ denotes the natural order on p_nq_n from p_n to q_n) for which the following conditions hold: $pq = \operatorname{Lim} p_nq_n$, $p = \lim p_n = \lim p'_n$, and $q = \lim q_n = \lim q'_n$ (see [11, page 78]). Examples of fans containing a zigzag are pictured in [11, Figures 5 and 6, page 92] and in [9, page 95].

A dendroid X is said to be of *type N* (between points p and q) provided that there exist in X two sequences of arcs $p_np'_n$ and $q_nq'_n$ and points $p''_n \in q_nq'_n \setminus \{q_n, q'_n\}$ and $q''_n \in p_np'_n \setminus \{p_n, p'_n\}$ such that

(a) $pq = \operatorname{Lim} p_n p'_n = \operatorname{Lim} q_n q'_n$,
(b) $p = \lim p_n = \lim p'_n = \lim p''_n$ and $q = \lim q_n = \lim q'_n = \lim q''_n$.
(See [22, page 837].) This concept should not be confused with the one under the same name, where a continuum of type N is defined by means of some conditions imposed on the bonding maps in an expansion of the continuum as the inverse limit of an inverse sequence of arcs, see [2].

It is evident that if a dendroid contains a zigzag, then it is of type N, [23, page 393], but not conversely, even for fans, [5, Example 2.7, page 563].

Theorem 2. *The following results are known.*
(2.1) *If a dendroid is of type N, then it is nonsmooth,* [18, Theorem 2.4, page 81].
(2.2) *If a dendroid is of type N, then it is noncontractible,* [22, Corollary 2.2, page 839] *(compare also* [23, Theorem 2.1, page 392] *and* [11, Theorem 2.1, page 81]*).*
(2.3) *If a dendroid is of type N, then it is nonselectible,* [19, page 548].
(2.4) *If a dendroid is of type N, then it admits no mean,* [13, Theorem 2.2, page 99] *and* [7, Corollary 5.40, page 23]*; compare* [1, Theorem 3.5, page 42].

Besides the results mentioned in Theorems 1 and 2 there are many unsolved problems and open questions concerning various interrelations between the considered notions. The reader is referred to [6, Sections 3–5] to see a current list of problems related to the present paper.

The following concept has been introduced in [10, page 121]. Let a dendroid X be of type N between p and q, with sequences $\{p_n\}$, $\{p'_n\}$, $\{p''_n\}$, $\{q_n\}$, $\{q'_n\}$, $\{q''_n\}$ as in the definition of type N (so satisfying conditions (a) and (b)), and let a mapping $g : X \to Y$ be a surjection from X onto a dendroid Y. The triade (X, g, Y) is said to *have property* $(*)$ provided that
(c) $g(p) \neq g(q)$;
(d) $g(p_n q''_n) \cap g(q''_n p'_n) = \{g(q''_n)\}$ for each $n \in \mathbb{N}$;
(e) $g(q_n p''_n) \cap g(p''_n q'_n) = \{g(p''_n)\}$ for each $n \in \mathbb{N}$.

As an application of the introduced notion of a triade having property $(*)$ to contractibility of dendroids the following result is proved (even in a more general formulation) in [10].

Theorem 3 [10, Theorem, page 121]**.** *Let a surjective mapping $g : X \to Y$ between dendroids X and Y be given such that (X, g, Y) has property $(*)$, and let a mapping $f : X \to Y$ be homotopic to g. Then $\{g(p), g(q)\} \subset f(pq) \subset f(X)$. Consequently, X is noncontractible relative to Y, so Y is noncontractible.*

Below we give further applications of the notion, namely to smoothness, selectibility, and to the concept of a mean. To this aim, recall a concept of a bend set that is due to Maćkowiak [19, page 548].

Let a continuum X and its subcontinuum $A \subset X$ be given. A set $B \subset A$ is said to be a *bend set of A* provided that there are two sequences $\{A_n : n \in \mathbb{N}\}$ and $\{A'_n : n \in \mathbb{N}\}$ of subcontinua of X such that
(f) $A_n \cap A'_n \neq \emptyset$ for each $n \in \mathbb{N}$;
(g) $A = \operatorname{Lim} A_n = \operatorname{Lim} A'_n$;
(h) $B = \operatorname{Lim}(A_n \cap A'_n)$.

A continuum X is said to have the *bend intersection property* provided that for each subcontinuum A of X the intersection of all bend sets of A is nonempty.

The following are applications of the above concept. For the first result quoted below, see [17, Theorem 5, page 124].

Theorem 4. *A dendroid X is not of type N if and only if for each arc $A \subset X$ the intersection of all bend sets of A is nonempty.*

An example of a dendroid X is constructed in [17, Example 7, page 126] such that for each subarc A of X the intersection of all bend sets of A is nonempty, while X does not have the bend intersection property.

Theorem 5. *Let X be a dendroid. Each of the following conditions implies that X has the bend intersection property:*

(5.1) *X is selectible,* [19, Corollary, page 548]*;*
(5.2) *X is smooth,* by (1.6) and (5.1)*;*
(5.3) *X is a contractible fan,* [16, Theorem 2, page 416]*;*
(5.4) *X admits an associative mean,* by (1.7) and (5.2)*.*

The bend intersection property for a dendroid X implies neither (5.1) nor (5.2), see [19, Example 1, page 548], as well as neither part of (5.3), see [7, Example 5.52, page 25]. In connection with (5.3) it is natural to ask whether the assumption that X is a fan is essential in this result (see [17, Question 8, page 126]).

Question 6. Does every contractible dendroid have the bend intersection property?

A similar question arises concerning (5.4). One may ask if the assumption of associativity of the mean is indispensable in this result.

Question 7. Let a dendroid X admit a (nonassociative) mean. Must then the intersection of all bend sets of each subcontinuum of X be nonempty?

Theorem 8. *Let a continuum X contain a subcontinuum $A \subset X$ and two sequences $\{A_n : n \in \mathbb{N}\}$ and $\{A'_n : n \in \mathbb{N}\}$ of subcontinua of X such that conditions (f) and (g) are satisfied, and let $B \subset A$ be a bend set of A. Let $g : X \to Y$ be a surjection. If*

(8.1) *the sequence $\{g(A_n) \cap g(A'_n) : n \in \mathbb{N}\}$ is convergent,*

then $\operatorname{Lim}[g(A_n) \cap g(A'_n)]$ *is a bend set of $g(A)$ that contains $g(B)$.*

If, additionally, $g \mid (A_n \cup A'_n)$ is one-to-one for sufficiently large $n \in \mathbb{N}$, then $g(B) = \operatorname{Lim}[g(A_n) \cap g(A'_n)]$.

Proof. Indeed, there are two sequences $\{g(A_n) : n \in \mathbb{N}\}$ and $\{g(A'_n) : n \in \mathbb{N}\}$ of subcontinua of Y such that (1) $\emptyset \neq g(A_n \cap A'_n) \subset g(A_n) \cap g(A'_n)$, (2) $g(A) = \operatorname{Lim} g(A_n) = \operatorname{Lim} g(A'_n)$ by continuity of g, and (3) $g(B) = g[\operatorname{Lim}(A_n \cap A'_n)] = \operatorname{Lim} g(A_n \cap A'_n) \subset \operatorname{Lim}[g(A_n) \cap g(A'_n)]$.

Note that $y \in \operatorname{Lim}[g(A_n) \cap g(A'_n)]$ implies that there exists a sequence of points $y_n \in g(A_n) \cap g(A'_n)$ with $y = \lim y_n$, whence it follows that there are two sequences of points $x_n \in A_n$ and $x'_n \in A'_n$ such that $y_n = g(x_n) = g(x'_n)$. By compactness of X and continuity of g we get $y \in g(A)$. Thus $\operatorname{Lim}[g(A_n) \cap g(A'_n)] \subset g(A)$, whence the first part of the conclusion follows.

If $g \mid (A_n \cup A'_n)$ is one-to-one, then (under the same notation) $x_n = x'_n \in A_n \cap A'_n$, and thus $y \in g(B)$ as needed, and the equality for $g(B)$ is shown. The proof is complete. □

Corollary 9. *Let a continuum X contain a subcontinuum $A \subset X$ and two sequences $\{A_n : n \in \mathbb{N}\}$ and $\{A'_n : n \in \mathbb{N}\}$ of subcontinua of X such that conditions (f) and (g) are satisfied, and let $B \subset A$ be a bend set of A. Let a continuum Y be hereditarily unicoherent and let $g : X \to Y$ be a surjection. Then $\operatorname{Ls}[g(A_n) \cap g(A'_n)]$ contains a bend set of $g(A)$ that contains $g(B)$.*

Proof. Put, for shortness, $I_n = g(A_n) \cap g(A'_n)$ and note that since Y is hereditarily unicoherent, the intersections I_n are continua. Since B is a bend set of A, condition (h) is satisfied, whence we have

(9.1) $\emptyset \neq g(B) = g[\operatorname{Lim}(A_n \cap A'_n)] = \operatorname{Lim} g(A_n \cap A'_n) \subset \operatorname{Li}[g(A_n) \cap g(A'_n)]$.

Thus $\operatorname{Li} I_n \neq \emptyset$, whence by [15, Section 47, Theorem 6, page 171] it follows that $\operatorname{Ls} I_n$ is a continuum. Since the hyperspace $C(X)$ is compact, the sequence I_n contains a convergent subsequence I_{n_m}. Putting $C = \operatorname{Lim}_m I_{n_m}$, we get, by (9.1),

$$g(B) \subset \operatorname{Li}\left[g(A_n) \cap g(A'_n)\right] \subset C \subset \operatorname{Ls} I_n \subset g(A). \tag{1}$$

Therefore C is a bend set of $g(A)$. □

As a consequence of Theorems 3, 4, 5, and 8 we get the following.

Corollary 10. *Let a surjective mapping $g : X \to Y$ between dendroids X and Y be given such that (X, g, Y) has property $(*)$. Then the singletons $\{g(p)\}$ and $\{g(q)\}$ are bend sets of $g(pq)$, and therefore Y is nonsmooth, noncontractible, nonselectible, and it admits no associative mean.*

The example below illustrates an application of the concept of a triade having property $(*)$ (see [10, Example, page 123]).

Example 11. There exist dendroids X and Y and a mapping $g : X \to Y$ such that

(11.1) X is of type N,

(11.2) Y is not of type N,

(11.3) the triade (X, g, Y) has property $(*)$.

Consequently, Y is neither smooth, nor contractible, nor selectible, and it admits no associative mean.

Proof. In the Cartesian coordinates in the plane put $K = \{0\} \times [-3/2, 2]$, $L = [0,1] \times \{2\}$ and, for each $n \in \mathbb{N}$, let

$$\begin{aligned} K_n = {} & \left(\left\{\frac{1}{n}\right\} \times \left[-\frac{3}{2}, 2\right]\right) \cup \left(\left[\frac{1}{n} - \frac{1}{2^{3n}}, \frac{1}{n}\right] \times \left\{-\frac{3}{2}\right\}\right) \cup \left(\left\{\frac{1}{n} - \frac{1}{2^{3n}}\right\} \times \left[-\frac{3}{2}, \frac{3}{2}\right]\right) \\ & \cup \left(\left[\frac{1}{n} - \frac{2}{2^{3n}}, \frac{1}{n} - \frac{1}{2^{3n}}\right] \times \left\{\frac{3}{2}\right\}\right) \cup \left(\left\{\frac{1}{n} - \frac{2}{2^{3n}}\right\} \times \left[-\frac{3}{2}, \frac{3}{2}\right]\right). \end{aligned} \tag{2}$$

Define $X = K \cup L \cup \bigcup\{K_n : n \in \mathbb{N}\}$. Thus X is a dendroid and it is of type N between points $p = (0,3/2)$ and $q = (0,-3/2)$.

Consider an equivalence relation $\sim$ on X defined by

$$\begin{aligned}(x_1,y_1)\sim(x_2,y_2) \iff \\ \text{either } \{(x_1,y_1) = (x_2,y_2)\} \text{ or } \{x_1 = x_2 = 0,\ y_1 = -y_2 \in [-1,1]\}.\end{aligned} \tag{3}$$

Then the quotient mapping $g : X \to X/\sim = g(X) = Y$ identifies points $(0,y)$ and $(0,-y)$ for all $y \in [-1,1]$ and it is one-to-one on the rest. Thus the triade (X,g,Y) has property $(*)$. The obtained space Y is a dendroid that contains no N-sequence. Note that $g(pq)$ is a simple triod with the centre $g((0,1)) = g((0,-1))$ and the endpoints $g((0,0))$, $g(p)$, and $g(q)$, and that the singletons $\{g(p)\}$ and $\{g(q)\}$ are bend sets of $g(pq)$. Consequently, Y does not have the bend intersection property. Therefore Y is not contractible by Theorem 3, it is neither selectible nor smooth by (5.1) and (5.2) of Theorem 5, respectively, and it does not admit any associative mean according to (5.4) (or by Corollary 10). □

The following concept generalizes the notion of a triade (X,g,Y) having property $(*)$. A surjective mapping $g : X \to Y$ between dendroids X and Y is said to be *admissible* provided that X is of type N between some points p and q and, if sequences $\{p_n\}$, $\{p'_n\}$, $\{p''_n\}$, $\{q_n\}$, $\{q'_n\}$, $\{q''_n\}$ satisfy the conditions of the definition (conditions (a) and (b), in particular), then

(i) $\operatorname{Ls}[g(p_nq''_n) \cap g(q''_np'_n)] \cap \operatorname{Ls}[g(q_np''_n) \cap g(p''_nq'_n)] = \emptyset$.

Put

(j) $P = \operatorname{Ls}[g(q_np''_n) \cap g(p''_nq'_n)]$ and $Q = \operatorname{Ls}[g(p_nq''_n) \cap g(q''_np'_n)]$.

Observe that the sets P and Q are continua.

The next theorem is related to [10, Theorem 3]. The leading idea of its proof comes from Oversteegen's proof of [22, Theorem 2.1, page 838]; it was used in the proof of [10, Theorem, page 121].

Theorem 12. *Let there be given an admissible mapping $g : X \to Y$ between continua X and Y, and let continua P and Q be defined by (j). Then, for each mapping $f : X \to Y$ homotopic to g, $f(A) \cap P \neq \emptyset \neq f(A) \cap Q$. Consequently, X is not contractible with respect to Y, so Y is noncontractible.*

Proof. Since the mapping $g : X \to Y$ is admissible, the domain continuum X is of type N. So, fix an arc A with endpoints p and q, two sequences of arcs $\{A_n\}$, $\{B_n\}$ (where $n \in \mathbb{N}$) with endpoints p_n, p'_n and q_n, q'_n, respectively, and points $p''_n \in B_n \setminus \{q_n, q'_n\}$ and $q''_n \in A_n \setminus \{p_n, p'_n\}$ such that conditions (a) and (b) are satisfied.

Let $H : X \times [0,1] \to Y$ be a homotopy such that $H(x,0) = g(x)$ and $H(x,1) = f(x)$ for each $x \in X$.

To make notation shorter, put

$$\begin{aligned} T = q_nq'_n \times [0,1], \qquad Z = (\{q_n, q'_n\} \times [0,1]) \cup (q_nq'_n \times \{1\}), \\ S = T \cap H^{-1}(g(q_np''_n) \cap g(p''_nq'_n)), \qquad W = S \cup Z \end{aligned} \tag{4}$$

and note that all these four sets are compact.

For each $n \in \mathbb{N}$, let C_n be the component of the set S that contains the point $(p''_n, 0)$. Note that $H(p''_n, 0) = g(p''_n) \in g(q_n p''_n) \cap g(p''_n q'_n)$, so C_n is well defined.

Claim 1. $C_n \cap Z \neq \emptyset$.

Suppose, on the contrary, that $C_n \cap Z = \emptyset$. We will show that

(12.1) there is no component J of W such that $J \cap C_n \neq \emptyset \neq J \cap Z$.

Indeed, if there were such J, then C_n would be a proper subcontinuum of J satisfying $C_n \subset J \setminus Z$. Taking an order arc from C_n to J (see [12, Theorem 14.6, page 112]) we would obtain a subcontinuum E of J such that C_n is a proper subset of E and $E \subset J \setminus Z \subset S$. Since C_n is a component of S, we would have $E = C_n$, a contradiction. Thus (12.1) is shown.

Therefore, by [26, Theorem 9.3, page 15] applied to the space W and its disjoint closed subsets C_n and Z, we obtain two disjoint closed subsets F and G of W such that

$$W = F \cup G, \qquad C_n \subset F, \qquad Z \subset G. \tag{5}$$

Let U and V be disjoint open subsets of the 2-cell T such that $F \subset U$ and $G \subset V$. Denote by K the component of $T \setminus U$ containing Z, and let L be the component of $T \setminus K$ containing C_n. Thus L is open as a component of an open set $T \setminus K$ in a locally connected continuum T (see [15, Section 49, II, Theorem 4, page 230]). The set $T \setminus L$ is the union of the continuum K and of all components of $T \setminus K$ different from L. Since each of these components is not separated from K by [15, Section 47, III, Theorem 1, page 172], $T \setminus L$ is connected according to [15, Section 46, II, Theorem 2, page 132]. Therefore, since T is unicoherent, $\mathrm{bd}_T(L) = \mathrm{cl}_T(L) \cap \mathrm{cl}_T(T \setminus L)$ is a continuum. Further, using again local connectedness of T and [15, Section 49, III, Theorem 3, page 238], we have

$$\mathrm{bd}_T(L) \subset \mathrm{bd}_T(T \setminus K) = \mathrm{bd}_T(K) \subset \mathrm{bd}_T(T \setminus U) = \mathrm{bd}_T(U). \tag{6}$$

Notice that $Z \subset K \subset T \setminus L$ and $C_n \subset L$, so each one of the arcs $q_n p''_n \times \{0\}$ and $p''_n q'_n \times \{0\}$ is a connected subset of T that meets both L and $T \setminus L$. Thus there exist points $a \in q_n p''_n$ and $b \in p''_n q'_n$ such that $(a, 0), (b, 0) \in \mathrm{bd}_T(L)$. Then the set $H(\mathrm{bd}_T(L))$ is a subcontinuum of Y that contains the points $g(a)$ and $g(b)$. Since $g(a), g(b) \in g(q_n q'_n)$ and $g(q_n q'_n)$ is an arcwise connected subset of Y, there exists an arc in Y joining $g(a)$ and $g(b)$. By the hereditary unicoherence of Y, such an arc is unique, so we can denote it by $g(a)g(b)$. Using again the hereditary unicoherence of Y we see that the arc $g(a)g(b)$ is contained in both continua $g(q_n p''_n) \cup g(p''_n q'_n)$ and $H(\mathrm{bd}_T(L))$. Since the sets $g(q_n p''_n)$ and $g(p''_n q'_n)$ are closed and each one of them meets $g(a)g(b)$, there exists a point $y \in g(a)g(b) \cap g(q_n p''_n) \cap g(p''_n q'_n)$. So, $y \in H(\mathrm{bd}_T(L))$. Then there is a point $x \in \mathrm{bd}_T(L)$ such that $H(x) = y \in g(q_n p''_n) \cap g(p''_n q'_n)$. Thus

$$x \in S \cap \mathrm{bd}_T(L) \subset S \cap \mathrm{bd}_T(U) \subset S \cap \big(T \setminus (F \cup G)\big) = S \cap (T \setminus W) \subset S \cap (T \setminus S) = \emptyset. \tag{7}$$

This contradiction completes the proof of Claim 1.

Put

$$T' = p_n p'_n \times [0,1], \qquad Z' = (\{p_n, p'_n\} \times [0,1]) \cup (p_n p'_n \times \{1\}),$$
$$S' = T' \cap H^{-1}(g(p_n q''_n) \cap g(q''_n p'_n)), \qquad W' = S' \cup Z', \tag{8}$$

and again note that all these four sets are compact.

For each $n \in \mathbb{N}$, let D_n be the component of the set S' that contains the point $(q''_n, 0)$. Note that $H(q''_n, 0) = g(q''_n) \in g(p_n q''_n) \cap g(q''_n p'_n)$, so D_n is well defined.

By the symmetry of assumptions (or in a similar way as for Claim 1) we obtain the following.

Claim 2. $D_n \cap Z' \neq \emptyset$.

For each $n \in \mathbb{N}$, fix points $c_n \in C_n \cap Z$ and $d_n \in D_n \cap Z'$. For $k \in \mathbb{N}$, take subsequences $\{C_{n_k}\}$, $\{D_{n_k}\}$, $\{c_{n_k}\}$, and $\{d_{n_k}\}$ of the sequences $\{C_n\}$, $\{D_n\}$, $\{c_n\}$, and $\{d_n\}$, correspondingly, which converge to the respective limits C, D, c, and d. Then $(p,0) \in C$, $(q,0) \in D$, and

(12.2) $c \in C \cap [(\{q\} \times [0,1]) \cup (A \times \{1\})]$, $d \in D \cap [(\{p\} \times [0,1]) \cup (A \times \{1\})]$.

Since C_n is a component of the set S, it follows that $C_n \subset T$ and $H(C_n) \subset g(q_n p''_n) \cap g(p''_n q'_n)$. Thus $C \subset A \times [0,1]$ and $H(C) \subset P$. Similarly, $D \subset A \times [0,1]$ and $H(D) \subset Q$.

Now we are ready to prove the theorem. Suppose that the conclusion of the theorem is false. Without loss of generality we may assume that $f(A) \cap Q = \emptyset$ (the case when $f(A) \cap P = \emptyset$ is similar).

By (12.2) we have two possibilities.

If $d \in A \times \{1\}$, then $d = (a,1)$ for some $a \in A$. Thus $f(a) = H(a,1) = H(d) \in Q$. So $f(a) \in f(A) \cap Q \neq \emptyset$. This is a contradiction that shows that $d \notin A \times \{1\}$.

Therefore, by (12.2), $d \in \{p\} \times [0,1]$. Then D is a subcontinuum of the disk $A \times [0,1]$ that contains the point $(q,0)$ and intersects $\{p\} \times [0,1]$. Since C is a subcontinuum of the disk $A \times [0,1]$ that contains the point $(p,0)$ and intersects $(\{q\} \times [0,1]) \cup (A \times \{1\})$ according to (12.2), it follows that $C \cap D \neq \emptyset$. Thus there is a point $e \in C \cap D$. Then $H(e) \in H(C \cap D) \subset H(C) \cap H(D) \subset P \cap Q$. This contradicts (i) and finishes the proof. □

The next result is a consequence of Theorem 12. It extends Theorem 3.

Theorem 13. *Let an admissible mapping $g : X \to Y$ between dendroids X and Y be given. Then X is not contractible with respect to Y and, consequently, Y is not contractible.*

Theorem 14. *Let an admissible mapping $g : X \to Y$ between dendroids X and Y be given, and let points p and q and sequences $\{p_n\}$, $\{p'_n\}$, $\{p''_n\}$, $\{q_n\}$, $\{q'_n\}$, $\{q''_n\}$ be as in the definition of type N. Then $\mathrm{Ls}[g(p_n q''_n) \cap g(q''_n p'_n)]$ and $\mathrm{Ls}[g(q_n p''_n) \cap g(p''_n q'_n)]$ contain (disjoint) bend sets of $g(pq)$.*

Proof. To see that $\mathrm{Ls}[g(p_n q''_n) \cap g(q''_n p'_n)]$ contains a bend set of $g(pq)$ (the argument for $\mathrm{Ls}[g(q_n p''_n) \cap g(p''_n q'_n)]$ is the same) it is enough to apply Corollary 9 with $A_n = p_n q''_n$ and $A'_n = q''_n p'_n$. Now condition (i) guarantees that the two bend sets of $g(pq)$ are disjoint. □

Theorems 13 and 14 imply, according to parts (5.1), (5.2), and (5.4) of Theorem 5, the following corollary.

Corollary 15. *Let an admissible mapping $g : X \to Y$ between dendroids X and Y be given. Then Y is neither smooth, nor contractible, nor selectible, and it admits no associative mean.*

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