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CLASSES OF UNIFORMLY STARLIKE AND CONVEX FUNCTIONS

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Some classes of uniformly starlike and convex functions are introduced. The geometrical properties of these classes and their behavior under certain integral operators are investigated.

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1. Introduction. Let A denote the class of functions of the form $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ which are analytic in the open unit disk $U=\{z:|z|<1\}$. A function f in A is said to be starlike of order β, $0\le\beta<1$, written as $f\in S^*(\beta)$, if $\operatorname{Re}[(zf'(z))/(f(z))]>\beta$. A function $f\in A$ is said to be convex of order β, or $f\in K(\beta)$, if and only if $zf'\in S^*(\beta)$.

Let $\mathrm{SD}(\alpha,\beta)$ be the family of functions f in A satisfying the inequality

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha\left|\frac{zf'(z)}{f(z)}-1\right|+\beta,\quad z\in U,\ \alpha\ge 0,\ 0\le\beta<1. \tag{1.1}$$

We note that for $\alpha>1$, if $f\in\mathrm{SD}(\alpha,\beta)$, then $zf'(z)/f(z)$ lies in the region $G\equiv G(\alpha,\beta)\equiv\{w:\operatorname{Re}w>\alpha|w-1|+\beta\}$, that is, part of the complex plane which contains $w=1$ and is bounded by the ellipse $(u-(\alpha^2-\beta)/(\alpha^2-1))^2+(\alpha^2/(\alpha^2-1))v^2=\alpha^2(1-\beta)^2/(\alpha^2-1)^2$ with vertices at the points $((\alpha+\beta)/(\alpha+1),0)$, $((\alpha-\beta)/(\alpha-1),0)$, $((\alpha^2-\beta)/(\alpha^2-1),(\beta-1)/\sqrt{\alpha^2-1})$, and $((\alpha^2-\beta)/(\alpha^2-1),(1-\beta)/\sqrt{\alpha^2-1})$. Since $\beta<(\alpha+\beta)/(\alpha+1)<1<(\alpha-\beta)/(\alpha-1)$, we have $G\subset\{w:\operatorname{Re}w>\beta\}$ and so $\mathrm{SD}(\alpha,\beta)\subset S^*(\beta)$. For $\alpha=1$ if $f\in\mathrm{SD}(\alpha,\beta)$, then $zf'(z)/f(z)$ belongs to the region which contains $w=2$ and is bounded by parabola $u=(v^2+1-\beta^2)/2(1-\beta)$.

Using the relation between convex and starlike functions, we define $\mathrm{KD}(\alpha,\beta)$ as the class of functions $f\in A$ if and only if $zf'\in\mathrm{SD}(\alpha,\beta)$. For $\alpha=1$ and $\beta=0$, we obtain the class $\mathrm{KD}(1,0)$ of uniformly convex functions, first defined by Goodman [1]. Rønning [3] investigated the class $\mathrm{KD}(1,\beta)$ of uniformly convex functions of order β. For the class $\mathrm{KD}(\alpha,0)$ of α-uniformly convex function, see [2]. In this note, we study the coefficient bounds and Hadamard product or convolution properties of the classes $\mathrm{SD}(\alpha,\beta)$ and $\mathrm{KD}(\alpha,\beta)$. Using these results, we further show that the classes $\mathrm{SD}(\alpha,\beta)$ and $\mathrm{KD}(\alpha,\beta)$ are closed under certain integral operators.

2. Main results. First we give a sufficient coefficient bound for functions in $\mathrm{SD}(\alpha,\beta)$.

Theorem 2.1. *If* $\sum_{n=2}^{\infty}[n(1+\alpha)-(\alpha+\beta)]|a_n|<1-\beta$, *then* $f\in\mathrm{SD}(\alpha,\beta)$.

Proof. By definition, it is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-(1+\beta)-\alpha\left|\frac{zf'(z)}{f(z)}-1\right|\right|<\left|\frac{zf'(z)}{f(z)}+(1-\beta)-\alpha\left|\frac{zf'(z)}{f(z)}-1\right|\right|. \tag{2.1}$$

For the right-hand side and left-hand side of (2.1) we may, respectively, write

$$\begin{aligned} R &= \left|\frac{zf'(z)}{f(z)}+(1-\beta)-\alpha\left|\frac{zf'(z)}{f(z)}-1\right|\right| \\ &= \frac{1}{|f(z)|}\left|zf'(z)+(1-\beta)f(z)-\alpha e^{i\theta}\left|zf'(z)-f(z)\right|\right| \\ &\geq \frac{1}{|f(z)|}\left[(2-\beta)|z|-\sum_{n=2}^{\infty}(n+1-\beta)|a_n||z|^n-\alpha\sum_{n=2}^{\infty}(n-1)|a_n||z|^n\right] \\ &> \frac{|z|}{|f(z)|}\left[2-\beta-\sum_{n=2}^{\infty}(n+1-\beta+n\alpha-\alpha)|a_n|\right], \end{aligned} \tag{2.2}$$

and similarly

$$L=\left|\frac{zf'(z)}{f(z)}-(1+\beta)-\alpha\left|\frac{zf'(z)}{f(z)}-1\right|\right|<\frac{|z|}{|f(z)|}\left[\beta+\sum_{n=2}^{\infty}(n-1-\beta+n\alpha-\alpha)|a_n|\right]. \tag{2.3}$$

Now, the required condition (2.1) is satisfied, since

$$R-L>\frac{|z|}{|f(z)|}\left[2(1-\beta)-2\sum_{n=2}^{\infty}[n(1+\alpha)-(\alpha+\beta)]|a_n|\right]>0. \tag{2.4}$$

The following two theorems follow from the above Theorem 2.1 in conjunction with a convolution result of Ruscheweyh and Sheil-Small [5] and the already discussed relation between the classes $\mathrm{SD}(\alpha,\beta)$ and $\mathrm{KD}(\alpha,\beta)$. □

Theorem 2.2. *If $\sum_{n=2}^{\infty} n[n(1+\alpha)-(\alpha+\beta)]|a_n|<1-\beta$, then $f\in\mathrm{KD}(\alpha,\beta)$.*

Theorem 2.3. *The classes $\mathrm{SD}(\alpha,\beta)$ and $\mathrm{KD}(\alpha,\beta)$ are closed under Hadamard product or convolution with convex functions in U.*

From Theorem 2.3 and the fact that

$$F(z)=\frac{1+\lambda}{z^{\lambda}}\int_0^z t^{\lambda-1}f(t)\,dt=f(z)*\sum_{n=1}^{\infty}\frac{1+\lambda}{n+\lambda}z^n, \quad \operatorname{Re}\lambda\geq 0, \tag{2.5}$$

we obtain the following corollary upon noting that $\sum_{n=1}^{\infty}((1+\lambda)/(n+\lambda))z^n$ is convex in U.

Corollary 2.4. *If f is in $\mathrm{SD}(\alpha,\beta)$ or $\mathrm{KD}(\alpha,\beta)$, so is $F(z)$ given by (2.5).*

Similarly, the following corollary is obtained for

$$G(z)=\int_0^z\frac{f(t)-f(\mu t)}{t(1-\mu)}dt=f(z)*\left(z+\sum_{n=2}^{\infty}\frac{1-\mu^n}{n(1-\mu)}z^n\right), \quad |\mu|\leq 1,\ \mu\neq 1. \tag{2.6}$$

Corollary 2.5. *If f is in* $\mathrm{SD}(\alpha,\beta)$ *or* $\mathrm{KD}(\alpha,\beta)$*, so is* $G(z)$ *given by (2.6).*

We observed that if $\alpha > 1$ and if $f \in \mathrm{SD}(\alpha,\beta)$, then $(zf'(z)/f(z))_{z\in U} \subset E$, where E is the region bounded by the ellipse $(u-(\alpha^2-\beta)/(\alpha^2-1))^2 + (\alpha^2/(\alpha^2-1))v^2 = \alpha^2(1-\beta)^2/(\alpha^2-1)^2$ with the parametric form

$$w(t) = \frac{\alpha^2-\beta}{\alpha^2-1} + \frac{\alpha(1-\beta)}{\alpha^2-1}\cos t + \frac{i(1-\beta)}{\sqrt{\alpha^2-1}}\sin t, \quad 0 \le t < 2\pi. \tag{2.7}$$

Thus for $\alpha > 1$ and z in the punctured unit disk $U-\{0\}$, we have $f \in \mathrm{SD}(\alpha,\beta)$ if and only if $zf'(z)/f(z) \ne w(t)$ or $zf'(z) - w(t)f(z) \ne 0$. By Ruscheweyh derivatives (see [4]), we obtain $f \in \mathrm{SD}(\alpha,\beta)$, if and only if $f(z) * [z/(1-z)^2 - w(t)(z/(1-z))] \ne 0$, $z \in U-\{0\}$. Consequently, $f \in \mathrm{SD}(\alpha,\beta)$, $\alpha > 1$, if and only if $f(z) * h(z)/z \ne 0$, $z \in U$ where h is given by the normalized function

$$h(z) = \frac{1}{1-w(t)}\left[\frac{z}{(1-z)^2} - w(t)\frac{z}{1-z}\right] \tag{2.8}$$

and w is given by (2.7). Conversely, if $f(z) * h(z)/z \ne 0$, then $zf'(z)/f(z) \ne w(t)$, $0 \le t < 2\pi$. Hence $(zf'(z)/f(z))_{z\in U}$ lie completely inside E or its compliment E^c. Since $(zf'(z)/f(z))_{z=0} = 1 \in E$, $(zf'(z)/f(z))_{z\in U} \subset E$, which implies that $f \in \mathrm{SD}(\alpha,\beta)$. This proves the following theorem.

Theorem 2.6. *The function f belongs to* $\mathrm{SD}(\alpha,\beta)$*, $\alpha > 1$, if and only if $f(z) * h(z)/z \ne 0$, $z \in U$ where $h(z)$ is given by (2.8).*

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