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ON THE TOPOLOGY OF D-METRIC SPACES AND GENERATION OF D-METRIC SPACES FROM METRIC SPACES

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An example of a D-metric space is given, in which D-metric convergence does not define a topology and in which a convergent sequence can have infinitely many limits. Certain methods for constructing D-metric spaces from a given metric space are developed and are used in constructing (1) an example of a D-metric space in which D-metric convergence defines a topology which is T_1 but not Hausdorff, and (2) an example of a D-metric space in which D-metric convergence defines a metrizable topology but the D-metric is not continuous even in a single variable.

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1. Introduction. Dhage [2] introduced the notion of D-metric spaces and claimed that D-metric convergence defines a Hausdorff topology and that the D-metric is (sequentially) continuous in all the three variables. Many authors (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]) have taken these claims for granted and used them in proving fixed point theorems in D-metric spaces.

In this paper, we give examples to show that in a D-metric space

(1) D-metric convergence does not always define a topology,
(2) even when D-metric convergence defines a topology, it need not be Hausdorff,
(3) even when D-metric convergence defines a metrizable topology, the D-metric need not be continuous even in a single variable.

In fact, we develop certain methods for constructing D-metric spaces from a given metric space and obtain from them, as by-products, examples illustrating the last two assertions. We also introduce the notions of strong convergence, and very strong convergence in a D-metric space and study in a decisive way the mutual implications among the three notions of convergence, strong convergence, and very strong convergence.

Throughout this paper, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}^+$ the set of all nonnegative real numbers, $\mathbb{N}$ the set of all positive integers, and $(\mathbb{R}^+)^{3*} = \{(t_1,t_2,t_3) \in (\mathbb{R}^+)^3 : t_1 \le t_2+t_3,\ t_2 \le t_3+t_1, t_3 \le t_1+t_2\}$.

Note 1.1. If (X,d) is a metric space, then $(d(x,y),d(y,z),d(z,x)) \in (\mathbb{R}^+)^{3*}$ for all $x,y,z \in X$.

Definition 1.2 [2]. Let X be a nonempty set. A function $\rho : X\times X\times X \to [0,\infty)$ is called a D-metric on X if

(i) $\rho(x,y,z) = 0$ if and only if $x = y = z$ (coincidence),

(ii) $\rho(x,y,z) = \rho(p(x,y,z))$ for all $x,y,z \in X$ and for any permutation $p(x,y,z)$ of x,y,z (symmetry),

(iii) $\rho(x,y,z) \le \rho(x,y,a)+\rho(x,a,z)+\rho(a,y,z)$ for all $x,y,z,a \in X$ (tetrahedral inequality).

If X is a nonempty set and ρ is D-metric on X, then the ordered pair (X,ρ) is called a D-metric space. When the D-metric ρ is understood, we say that X is a D-metric space.

Definition 1.3 [2, 8]. A sequence $\{x_n\}$ in a D-metric space (X,ρ) is said to be convergent (or ρ-convergent) if there exists an element x of X with the following property: given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\rho(x_m,x_n,x) < \varepsilon$ for all $m,n \ge N$. In such a case, $\{x_n\}$ is said to converge to x and x is called a limit of $\{x_n\}$.

Definition 1.4 [2, 8]. A sequence $\{x_n\}$ in a D-metric space (X,ρ) is said to be Cauchy (or ρ-Cauchy) if, given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\rho(x_m,x_n,x_p) < \varepsilon$ for all $m,n,p \ge N$.

Remark 1.5. The definition of ρ-Cauchy sequence as given by Dhage [2] appears to be slightly different from Definition 1.4, but it is actually equivalent to it. It can be shown that in a D-metric space every convergent sequence is Cauchy.

Definition 1.6 [2, 8]. A D-metric space (X,ρ) is said to be complete (or ρ-complete) if every ρ-Cauchy sequence in X is ρ-convergent in X.

Notation 1.7. For a subset E of a D-metric space (X,ρ), E^c denotes $\{x \in X$: there is a sequence in E which converges to x under the D-metric $\rho\}$. For any set X, $P(X)$ denotes the power set of X, that is, the collection of all subsets of X.

We now give an example of a complete D-metric space in which D-metric convergence does not define a topology and in which there are convergent sequences with infinitely many limits.

Example 1.8. Let $X = A \cup B \cup \{0\}$, where $A = \{1/2^n : n \in \mathbb{N}\}$ and $B = \{2^n : n \in \mathbb{N}\}$. Define $\rho : X \times X \times X \to \mathbb{R}^+$ as follows:

(i) $\rho(x,y,z) = 0$ if $x = y = z$,

(ii) $\rho(x,y,z) = \min\{\max\{x,y\},\max\{y,z\},\max\{z,x\}\}$ if $x,y,z \in A \cup \{0\}$, 0 does not occur more than once among x, y, z, and at least two among x, y, z are distinct,

(iii) $\rho(x,y,z) = 1$ if 0 and at least one element of B occur among x, y, z, or 0 occurs exactly twice among x, y, z,

(iv) $\rho(x,y,z) = \min\{x,y,z\}$ if $x,y,z \in A \cup B$ and exactly one element of B occurs exactly once among x, y, z,

(v) $\rho(x,y,z) = \min\{\max\{1/x,1/y\},\max\{1/y,1/z\},\max\{1/z,1/x\}\}$, if $x,y,z \in A \cup B$ and exactly one element of A occurs exactly once among x, y, z,

(vi) $\rho(x,y,z) = |1/x-1/y|+|1/y-1/z|+|1/z-1/x|$ if $x,y,z \in B$.

Then (X,ρ) is a complete D-metric space. But ρ-convergence does not define a topology on X.

Proof. Clearly ρ is symmetric in all the three variables and $\rho(x,y,z) = 0$ if and only if $x = y = z$. We note that $\rho(x,y,z) \le 1$ for all $x,y,z \in X$. Let $x,y,z,u \in X$.

Case (i). $x = y = z$.

Then $\rho(x,y,z) = 0 \le \rho(u,y,z) + \rho(x,u,z) + \rho(x,y,u)$.

Case (ii). $x,y,z \in A \cup \{0\}$, 0 does not occur more than once among x, y, z, and at least two among x, y, z are distinct.

We may assume that $x \ge y \ge z$. If $u \in A \cup \{0\}$, then $\rho(x,y,z) = y \le \rho(u,y,z) + \rho(x,u,z) + \rho(x,y,u)$, since when $u > y$, $\rho(u,y,z) = y$; when $u = y$, $\rho(x,y,z) = \rho(x,u,z)$; and when $u < y$, $\rho(x,y,u) = y$. If $u \in B$, then

$$\begin{aligned}\rho(x,y,z) &= y = \min\{x,y,u\} = \rho(x,y,u)\\ &\le \rho(u,y,z) + \rho(x,u,z) + \rho(x,y,u).\end{aligned} \tag{1.1}$$

Case (iii). 0 occurs exactly twice among x, y, z.

We may assume that $x = y = 0$. Then $z \ne 0$. If $u \in X \setminus \{0\}$, then

$$\begin{aligned}\rho(x,y,z) &= \rho(0,0,z) = 1 = \rho(0,0,u) = \rho(x,y,u)\\ &\le \rho(u,y,z) + \rho(x,u,z) + \rho(x,y,u).\end{aligned} \tag{1.2}$$

If $u = 0$, then

$$\begin{aligned}\rho(x,y,z) &= \rho(0,0,z) = 1 = \rho(u,y,z)\\ &\le \rho(u,y,z) + \rho(x,u,z) + \rho(x,y,u).\end{aligned} \tag{1.3}$$

Case (iv). 0 and at least one element of B occur among x, y, z.

We may assume that $x = 0$ and $y \in B$. Then

$$\begin{aligned}\rho(x,y,z) &= \rho(0,y,z) = 1 = \rho(0,y,u) = \rho(x,y,u)\\ &\le \rho(u,y,z) + \rho(x,u,z) + \rho(x,y,u).\end{aligned} \tag{1.4}$$

Case (v). $x,y,z \in A \cup B$ and exactly one element of B occurs exactly once among x, y, z.

We may assume that $x \in B$. Then $y,z \in A$. We may also assume that $y \ge z$. If $u \in B$, then

$$\begin{aligned}\rho(x,y,z) &= \min\{x,y,z\} = z = \rho(u,y,z)\\ &\le \rho(u,y,z) + \rho(x,u,z) + \rho(x,y,u).\end{aligned} \tag{1.5}$$

If $u \in A \cup \{0\}$, then

$$\rho(x,y,z) = \min\{x,y,z\} = z \le \rho(u,y,z) + \rho(x,u,z) + \rho(x,y,u), \tag{1.6}$$

since when $u < z$, $\rho(u,y,z) = z$; when $u = z$, $\rho(x,y,z) = \rho(x,y,u)$; and when $u > z$, $\rho(u,y,z) = \min\{u,y\} \ge z$.

Case (vi). $x,y,z \in A \cup B$ and exactly one element of A occurs exactly once among x, y, z.

We may assume that $x \in A$. Then $y, z \in B$. We may also assume that $y \geq z$. If $u \in A$, then

$$\begin{aligned}\rho(x,y,z) &= \max\left\{\frac{1}{y},\frac{1}{z}\right\} = \frac{1}{z} = \rho(u,y,z) \\ &\leq \rho(u,y,z)+\rho(x,u,z)+\rho(x,y,u).\end{aligned} \tag{1.7}$$

If $u = 0$, then

$$\begin{aligned}\rho(x,y,z) &= \max\left\{\frac{1}{y},\frac{1}{z}\right\} = \frac{1}{z} \leq \frac{1}{2} < 1 = \rho(u,y,z) \\ &\leq \rho(u,y,z)+\rho(x,u,z)+\rho(x,y,u).\end{aligned} \tag{1.8}$$

If $u \in B$, then

$$\begin{aligned}\rho(x,y,z) &= \max\left\{\frac{1}{y},\frac{1}{z}\right\} = \frac{1}{z} \\ &\leq \max\left\{\frac{1}{u},\frac{1}{z}\right\} = \rho(x,u,z) \\ &\leq \rho(u,y,z)+\rho(x,u,z)+\rho(x,y,u).\end{aligned} \tag{1.9}$$

Case (vii). $x, y, z \in B$.

We may assume that $1/x \geq 1/y \geq 1/z$. If $u = 0$, then

$$\begin{aligned}\rho(x,y,z) &= 2\left(\frac{1}{x}-\frac{1}{z}\right) < 1 = \rho(u,y,z) \\ &\leq \rho(u,y,z)+\rho(x,u,z)+\rho(x,y,u).\end{aligned} \tag{1.10}$$

If $u \in A$, then

$$\begin{aligned}\rho(x,y,z) &= 2\left(\frac{1}{x}-\frac{1}{z}\right) \leq \frac{1}{x}+\frac{1}{x} \\ &= \rho(x,u,z)+\rho(x,y,u) \\ &\leq \rho(u,y,z)+\rho(x,u,z)+\rho(x,y,u).\end{aligned} \tag{1.11}$$

If $u \in B$, then

$$\begin{aligned}\rho(x,y,z) &= \left|\frac{1}{x}-\frac{1}{y}\right|+\left|\frac{1}{y}-\frac{1}{z}\right|+\left|\frac{1}{z}-\frac{1}{x}\right| \\ &\leq \rho(u,y,z)+\rho(x,u,z)+\rho(x,y,u),\end{aligned} \tag{1.12}$$

since $|1/x-1/y| \leq \rho(x,y,u)$, $|1/y-1/z| \leq \rho(u,y,z)$, and $|1/z-1/x| \leq \rho(x,u,z)$. Thus, for all $x,y,z,u \in X$, we have $\rho(x,y,z) \leq \rho(u,y,z)+\rho(x,u,z)+\rho(x,y,u)$. Hence ρ is a D-metric on X.

To show that (X,ρ) is D-complete.

Let $\{x_n\}$ be a Cauchy sequence in X.

Case 1. There exists $N \in \mathbb{N}$ such that $x_n = x_N$ for all $n \geq N$.

In this case, evidently $\{x_n\}$ converges to x_N.

Case 2. Given $N \in \mathbb{N}$, there exist $i, j \in \mathbb{N}$ such that $i > N$, $j > N$, and $x_i \neq x_j$.

Then there exists $N_0 \in \mathbb{N}$ such that $x_i \neq 0$ for each $i \geq N_0$, since $\rho(0,0,x) = 1$, for all $x \in X \backslash \{0\}$, and $\{x_n\}$ is Cauchy.

Subcase (i). There exists $N_1 \in \mathbb{N}$ such that $N_1 \geq N_0$ and $x_i \in A$ for all $i \geq N_1$.

Claim 1.9. *$x_n \to 0$ as $n \to \infty$ in the usual sense.*

Suppose the claim does not hold. Then there exists a positive real number ε such that $x_n \geq \varepsilon$ for infinitely many $n \in \mathbb{N}$. Given $N \in \mathbb{N}$, we can choose $i, j, k \in \mathbb{N}$ such that $k > j > i > \max\{N, N_1\}$, $x_i \geq \varepsilon$, $x_j \geq \varepsilon$, and $x_k \neq x_j$. Then

$$\rho(x_i, x_j, x_k) = \min\{\max\{x_i, x_j\}, \max\{x_j, x_k\}, \max\{x_k, x_i\}\} \geq \varepsilon. \tag{1.13}$$

This is a contradiction since $\{x_n\}$ is Cauchy. Hence the claim.

For $m, n \geq N_1$ and $a \in A \cup \{0\}$, we have

$$\begin{aligned} \rho(a, x_m, x_n) &= \min\{\max\{a, x_m\}, \max\{x_m, x_n\}, \max\{x_n, a\}\} \\ &\leq \max\{x_m, x_n\} \longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty. \end{aligned} \tag{1.14}$$

Hence $\{x_n\}$ converges to a for any $a \in A \cup \{0\}$. It can also be shown that $\{x_n\}$ converges to b for any $b \in B$.

Subcase (ii). There exists $N_2 \in \mathbb{N}$ such that $N_2 \geq N_0$ and $x_i \in B$ for all $i \geq N_2$.

Claim 1.10. *$x_n \to +\infty$ as $n \to +\infty$.*

Suppose the claim does not hold. Then there exists a positive real number M such that $x_n \leq M$ for infinitely many $n \in \mathbb{N}$. Given $N \in \mathbb{N}$, we can find $i, j, k \in \mathbb{N}$ such that $k > j > i > \max\{N, N_2\}$, $x_i \leq M$, $x_j \leq M$, and $x_j \neq x_k$. Then $\rho(x_i, x_j, x_k) \geq |1/x_j - 1/x_k| \geq 1/2x_j \geq 1/2M$. This is a contradiction since $\{x_n\}$ is Cauchy. Hence the claim.

For $m, n \geq N_2$ and $a \in A$, we have

$$\rho(a, x_m, x_n) = \max\left\{\frac{1}{x_m}, \frac{1}{x_n}\right\} \longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty. \tag{1.15}$$

Hence $\{x_n\}$ converges to a for any $a \in A$.

Subcase (iii). Given $N \in \mathbb{N}$, there exist $i, j \in \mathbb{N}$ such that $i > N$, $j > N$, $x_i \in A$, and $x_j \in B$.

Claim 1.11. *Any element of A occurs only finitely many times in the sequence $\{x_n\}$.*

Suppose the claim does not hold. Then there exists $a_0 \in A$ such that $a_0 = x_n$ for infinitely many $n \in \mathbb{N}$. Let $N \in \mathbb{N}$. Then there exist $i, j, k \in \mathbb{N}$ such that $k > j > i > N$, $x_i = x_j = a_0$, and $x_k \in B$. Then $\rho(x_i, x_j, x_k) = \min\{x_i, x_j, x_k\} = a_0$. This is a contradiction since $\{x_n\}$ is Cauchy. Hence the claim.

Claim 1.12. *Any element of B occurs only finitely many times in the sequence $\{x_n\}$.*

Suppose the claim does not hold. Then there exists $b \in B$ such that $b = x_n$ for infinitely many $n \in \mathbb{N}$. Let $N \in \mathbb{N}$. Then there exist $i, j, k \in \mathbb{N}$ such that $k > j > i > N$, $x_i = x_j = b$, and $x_k \in A$. Then $\rho(x_i, x_j, x_k) = 1/b$. This is a contradiction since $\{x_n\}$ is Cauchy. Hence the claim.

Let $c \in A$. Let $\varepsilon > 0$. From Claim 1.11, it follows that there exists $N_3 \in \mathbb{N}$ such that $x_n < \min\{\varepsilon, c\}$ whenever $n \geq N_3$ and $x_n \in A$. From Claim 1.12, it follows that there exists $N_4 \in \mathbb{N}$ such that $x_n > 1/\varepsilon$ whenever $n \geq N_4$ and $x_n \in B$. Let $N_5 = \max\{N_0, N_3, N_4\}$. Let $m, n \in \mathbb{N}$ be such that $m \geq N_5$ and $n \geq N_5$. Then $x_n, x_m \in A \cup B$. If both $x_n, x_m \in A$, then $\rho(c, x_n, x_m) = \max\{x_n, x_m\} < \varepsilon$. If both $x_n, x_m \in B$, then $\rho(c, x_n, x_m) = \max\{1/x_n, 1/x_m\} < \varepsilon$. Suppose that one of x_n, x_m belongs to A and the other belongs to B. We may assume that $x_n \in A$ and $x_m \in B$. Then $\rho(c, x_n, x_m) = \min\{c, x_n, x_m\} = x_n < \varepsilon$. Thus $\rho(c, x_n, x_m) < \varepsilon$ for all $n, m \geq N_5$. Hence $\{x_n\}$ converges to c.

Thus, in any case, $\{x_n\}$ is convergent in X with respect to the D-metric ρ. Hence (X, ρ) is a complete D-metric space.

To show that $(B^c)^c \neq B^c$.

Let $p \in B^c$. Then there exists a sequence $\{x_n\}$ in B such that $\{x_n\}$ converges to p. Hence $\{x_n\}$ is Cauchy. If there exists $N \in \mathbb{N}$ such that $x_k = x_N$ for all $k \geq N$, then $\rho(p, x_N, x_N) = 0$ and hence $p = x_N \in B$.

Suppose that such an N does not exist. Then given $N \in \mathbb{N}$, there exist $i, j \in \mathbb{N}$ such that $i > N$, $j > N$, and $x_i \neq x_j$. As in Subcase (ii) of Case 2 in the proof of the completeness of ρ, it can be shown that $x_n \to +\infty$ as $n \to \infty$ and that $\{x_n\}$ converges to x for any $x \in A$.

For any $x \in B$,

$$\rho(x, x_n, x_m) = \left|\frac{1}{x} - \frac{1}{x_n}\right| + \left|\frac{1}{x_n} - \frac{1}{x_m}\right| + \left|\frac{1}{x_m} - \frac{1}{x}\right| \longrightarrow \frac{2}{x} \quad \text{as } m, n \longrightarrow \infty. \tag{1.16}$$

Hence $\{x_n\}$ does not converge to x for any $x \in B$. We have $\rho(0, x_n, x_m) = 1$ for all $m, n \in \mathbb{N}$. Therefore $\{x_n\}$ does not converge to 0. Hence $p \in A$. Thus $B^c \subseteq B \cup A$. Clearly $B \subseteq B^c$. $\{2^n\}$ converges to x for any x in A. Hence $A \subseteq B^c$. Therefore $A \cup B = B^c$. Since $\{1/2^n\}$ is a sequence in A and it converges to x for any $x \in X$, $(B^c)^c = X$. Since $0 \notin A \cup B$, $(B^c)^c \neq B^c$. Therefore the function $f : P(X) \to P(X)$ defined as $f(E) = E^c$ for all $E \in P(X)$ is not a closure operator. Hence ρ-convergence does not define a topology on X. □

Definition 1.13. Let (X, ρ) be a D-metric space and $\{x_n\}$ a sequence in X. $\{x_n\}$ is said to converge strongly to an element x of X if

(i) $\rho(x, x_m, x_n) \to 0$ as $m, n \to \infty$,

(ii) $\{\rho(y, y, x_n)\}$ converges to $\rho(y, y, x)$ for all y in X.

In such a case, x is said to be a strong limit of $\{x_n\}$.

Definition 1.14. Let (X, ρ) be a D-metric space and $\{x_n\}$ a sequence in X. $\{x_n\}$ is said to converge very strongly to an element x of X if

(i) $\rho(x, x_m, x_n) \to 0$ as $m, n \to \infty$,

(ii) $\{\rho(y, z, x_n)\}$ converges to $\rho(y, z, x)$ for any elements y, z of X.

In such a case, x is said to be a very strong limit of $\{x_n\}$.

Remark 1.15. Let $\{x_n\}$ be a sequence in a D-metric space X and $x \in X$. If $\{x_n\}$ converges very strongly to x, then $\{x_n\}$ converges strongly to x. If $\{x_n\}$ converges strongly to x, then it converges to x with respect to ρ. Examples 1.21, 1.39, and 1.40 show that the converse statements are false.

Proposition 1.16. *Let (X,ρ) be a D-metric space. Let $\{x_n\}$ be a sequence in X converging to an element x of X. Then $\{\rho(x,x,x_n)\}$ is convergent.*

Proof. Since $\{x_n\}$ is convergent, it is D-Cauchy. We have $\rho(x,x,x_n) \le \rho(x_m,x,x_n) + \rho(x,x_m,x_n) + \rho(x,x,x_m)$. Hence $\rho(x,x,x_n) - \rho(x,x,x_m) \le 2\rho(x,x_m,x_n)$. Similarly, we have $\rho(x,x,x_m) - \rho(x,x,x_n) \le 2(x,x_n,x_m)$. Hence $|\rho(x,x,x_n) - \rho(x,x,x_m)| \le 2\rho(x,x_n,x_m)$. Since this inequality is true for all $m,n \in \mathbb{N}$ and $\{x_n\}$ converges to x under the D-metric ρ, it follows that $\{\rho(x,x,x_n)\}$ is a Cauchy sequence of real numbers and hence convergent. □

Remark 1.17. Example 1.21 shows that the hypothesis of Proposition 1.16 does not ensure that the limit of $\{\rho(x,x,x_n)\}$ is $\rho(x,x,x)$.

Proposition 1.18. *In a D-metric space, every strongly convergent sequence has a unique strong limit.*

Proof. Let (X,ρ) be a D-metric space and $\{x_n\}$ a strongly convergent sequence in X. Let y, z be strong limits of $\{x_n\}$. Then $\{\rho(y,y,x_n)\}$ converges to both $\rho(y,y,y)$ and $\rho(y,y,z)$. Hence $\rho(y,y,z) = \rho(y,y,y) = 0$. Hence $y = z$. □

Theorem 1.19. *Let (X,d) be a metric space, $x_0 \in X$, and let A be a nonempty subset of $X\setminus\{x_0\}$. Define $\rho : A\times A\times A \to \mathbb{R}^+$ as*

$$\rho(x,y,z) = \begin{cases} 0 & \text{if } x = y = z, \\ \min\{\max\{d(x_0,x),d(x_0,y)\}, & \\ \qquad \max\{d(x_0,y),d(x_0,z)\}, & \\ \qquad \max\{d(x_0,z),d(x_0,x)\}\} & \text{otherwise.} \end{cases} \tag{1.17}$$

Then (A,ρ) is a complete D-metric space and ρ-convergence defines a topology τ on A. If $A \cap \{x \in X : d(x_0,x) < r_0\} = \phi$ for some $r_0 \in (0,\infty)$, then τ is the discrete topology on X; otherwise $\tau = \{\phi\} \cup \{E \subseteq A : \{x \in A : d(x_0,x) < r\} \subseteq E$ for some $r \in (0,\infty)\}$ and, in particular, τ is T_1 but not Hausdorff.

Let $\{x_n\} \subseteq A$. Then $\{x_n\}$ converges to x with respect to ρ for some $x \in A$ and $x_n \ne x$ for all sufficiently large $n \Rightarrow d(x_n,x_0) \to 0$ as $n \to \infty \Rightarrow \{x_n\}$ converges to x with respect to ρ for each x in A. If A has at least two elements, there does not exist a sequence in A which is strongly convergent with respect to ρ.

Proof. Clearly ρ is symmetric in all the three variables and $\rho(x,y,z) = 0$ if and only if $x = y = z$. Let $x,y,z,u \in A$. We may assume that $d(x_0,x) \ge d(x_0,y) \ge d(x_0,z)$. Irrespective of whether $d(x_0,u) < d(x_0,y)$ or $d(x_0,u) \ge d(x_0,y)$, we have

$$\rho(x,y,z) = d(x_0,y) \le \rho(x,y,u). \tag{1.18}$$

Hence $\rho(x,y,z) \leq \rho(u,y,z)+\rho(x,u,z)+\rho(x,y,u)$ for all $x,y,z,u \in A$. Hence ρ is a D-metric on A. Let $\{x_n\}$ be a sequence in A and $x \in A$. If $x_n \neq x$, we have

$$\begin{aligned}\rho(x,x_n,x_n) &= \min\{\max\{d(x_0,x),d(x_0,x_n)\},\\ &\qquad \max\{d(x_0,x_n),d(x_0,x_n)\},\\ &\qquad \max\{d(x_0,x_n),d(x_0,x)\}\}\\ &\geq d(x_0,x_n).\end{aligned} \tag{1.19}$$

That is, $\rho(x,x_n,x_n) \geq d(x_0,x_n)$ if $x_n \neq x$. Hence $d(x_0,x_n) \to 0$ as $n \to \infty$ if $\{x_n\}$ converges to x with respect to ρ for some $x \in A$ and $x_n \neq x$ for all sufficiently large n. We have

$$\begin{aligned}\rho(x_n,x_m,x) &\leq \min\{\max\{d(x_0,x_n),d(x_0,x_m)\},\\ &\qquad \max\{d(x_0,x_m),d(x_0,x)\},\\ &\qquad \max\{d(x_0,x),d(x_0,x_n)\}\}\\ &\leq \max\{d(x_0,x_n),d(x_0,x_m)\} \quad \forall n,m \in \mathbb{N}.\end{aligned} \tag{1.20}$$

Thus $\{x_n\}$ converges to x with respect to ρ for each x in A if $d(x_0,x_n) \to 0$ as $n \to \infty$.

If $x_m \neq x_n$, then we have

$$\begin{aligned}\rho(x_n,x_m,x_n) &= \min\{\max\{d(x_0,x_n),d(x_0,x_m)\},\\ &\qquad \max\{d(x_0,x_m),d(x_0,x_n)\},\\ &\qquad \max\{d(x_0,x_n),d(x_0,x_n)\}\}\\ &\geq d(x_0,x_n).\end{aligned} \tag{1.21}$$

Hence $d(x_0,x_n) \to 0$ as $n \to \infty$ if $\{x_n\}$ is ρ-Cauchy and there does not exist an $N \in \mathbb{N}$ such that $x_n = x_N$ for all $n > N$. If there exists $N \in \mathbb{N}$ such that $x_n = x_N$ for all $n > N$, then, evidently, $\{x_n\}$ converges to x_N with respect to ρ.

If such an N does not exist and $\{x_n\}$ is ρ-Cauchy, then $d(x_0,x_n) \to 0$ as $n \to \infty$, and hence $\{x_n\}$ converges to x with respect to ρ for any x in A. Hence every ρ-Cauchy sequence in A is convergent with respect to ρ. Therefore (A,ρ) is D-complete. If $x_n \neq x$, we have

$$\begin{aligned}\rho(x,x,x_n) &= \min\{\max\{d(x_0,x),d(x_0,x)\},\\ &\qquad \max\{d(x_0,x),d(x_0,x_n)\},\\ &\qquad \max\{d(x_0,x_n),d(x_0,x)\}\}\\ &\geq d(x_0,x).\end{aligned} \tag{1.22}$$

Since $x \in A \subseteq X\backslash\{x_0\}$, $d(x_0,x) > 0$. Hence $\{\rho(x,x,x_n)\}$ does not converge to 0 if $x_n \neq x$ for infinitely many n. Consequently, $\{x_n\}$ is not strongly ρ-convergent if A has at least two elements. Let E be a subset of A. Clearly, $E \subseteq E^c$.

Case 1. $E \cap \{x \in X : d(x,x_0) < r\} = \phi$ for some $r \in (0,\infty)$.

Let $z \in E^c$. Then there exists a sequence $\{x_n\}$ in E such that $\{x_n\}$ converges to z with respect to ρ. Suppose that $x_n \neq z$ for each n. Then $d(x_0,x_n) \to 0$ as $n \to \infty$. Since E $\cap \{x \in X : d(x,x_0) < r\} = \phi$, we have $d(x,x_0) \geq r$ for all $x \in E$. Hence $d(x_n,x_0) \geq r$ for all $n \in \mathbb{N}$. Therefore $\{d(x_n,x_0)\}$ does not converge to 0. Thus we arrived at a contradiction. Consequently, $x_n = z$ for some $n \in \mathbb{N}$. Hence $z \in E$. Therefore $E^c \subseteq E$. Thus $E = E^c$.

Case 2. Case 1 is false.

Then $E \cap \{x \in X : d(x,x_0) < 1/n\} \neq \phi$ for each $n \in \mathbb{N}$. Hence there exists a sequence $\{u_n\}$ in E such that $d(u_n,x_0) < 1/n$ for all $n \in \mathbb{N}$. Therefore $d(u_n,x_0) \to 0$ as $n \to \infty$. Hence $\{u_n\}$ converges to x with respect to ρ for each x in A. Hence $E^c = A$.

Case (I). $A \cap \{x \in X : d(x,x_0) < r_0\} = \phi$ for some $r_0 \in (0,\infty)$.

In this case, for any subset E of A, we have $E^c = E$, and hence $(E^c)^c = E^c = E$. Thus the function f defined on the power set $P(A)$ of A as $f(E) = E^c$ for all $E \in P(A)$ is a closure operator. Therefore ρ-convergence defines a topology τ on A in which every subset of A is closed. Hence $\tau = \{E : E \subseteq A\}$. Consequently, τ is the discrete topology on X.

Case (II). $A \cap \{x \in X : d(x,x_0) < r\} \neq \phi$ for any $r \in (0,\infty)$.

In this case, for a subset E of X, we have $E^c = E$ or A according to whether Case 1 or Case 2 holds. Hence $(E^c)^c = E^c$ for all $E \in P(A)$. Therefore the function f defined on $P(A)$ as $f(E) = E^c$ for all $E \in P(A)$ is a closure operator. Thus ρ-convergence defines a topology τ on A with respect to which a subset B of A is closed if and only if $B = E^c$ for some $E \in P(A)$. Hence

$$\begin{aligned}\tau &= \{A \backslash E^c : E \in P(A)\} \\ &= \{\phi\} \cup \{E \in P(A) : \{x \in A : d(x_0,x) < r\} \subseteq E \text{ for some } r \in (0,\infty)\}.\end{aligned} \tag{1.23}$$

If U_1, U_2 are nonempty open sets in τ, then $U_1 \cap U_2 \neq \phi$ since there exist $r_1, r_2 \in (0,\infty)$ such that

$$\{x \in A : d(x_0,x) < r_i\} \subseteq U_i, \quad i = 1,2. \tag{1.24}$$

Hence τ is not Hausdorff. Let p, q be distinct elements of A. Since $x_0 \notin A$, $d(p,x_0)$ and $d(q,x_0)$ are positive real numbers. Let $0 < r < \min\{d(p,x_0), d(q,x_0)\}$. Let $V_0 = \{x \in A : d(x,x_0) < r\}$. Then $V_0 \cup \{p\}$ is a τ-open subset of A containing p but not q, and $V_0 \cup \{q\}$ is a τ-open subset of A containing q but not p. Hence the topology τ is T_1. □

Example 1.20. Let $X = \mathbb{R}$ with the usual metric, $x_0 = 0$, and $A = [1,2]$. Then the function ρ defined in Theorem 1.19 on $A \times A \times A$ reduces to the following:

$$\rho(x,y,z) = \begin{cases} 0 & \text{if } x = y = z, \\ \min\{\max\{x,y\}, \max\{y,z\}, \max\{z,x\}\} & \text{otherwise.} \end{cases} \tag{1.25}$$

From Theorem 1.19 it follows that (A,ρ) is a complete D-metric space and that ρ-convergence defines a topology τ on A, which is the discrete topology on A.

Example 1.21. Let $X = \mathbb{R}$ with the usual metric, $x_0 = 0$, and $A = \{1/n : n \in \mathbb{N}\}$. Then the function ρ defined in Theorem 1.19 on $A\times A\times A$ has the same form as that given in Example 1.20. From Theorem 1.19 it follows that (A,ρ) is a complete D-metric space, any sequence in A which converges to zero in the usual sense converges to x with respect to ρ for each x in A, and that ρ-convergence defines a topology τ on A with respect to which nonempty subset E of A is open if and only if E contains $\{1/n : n \in \mathbb{N}$ and $n \geq N\}$ for some $N \in \mathbb{N}$. Further, τ is T_1 but not Hausdorff. Let $x_n = 1/n$ for all $n \in \mathbb{N}$ and $x_0 = 1/2$. Then $\{x_n\}$ converges to $1/2$ under the D-metric ρ. We have $\rho(x_0,x_0,x_n) = \rho(1/2,1/2,1/n) = 1/2$ for all $n \in \mathbb{N}\setminus\{2\}$. Hence $\{\rho(x_0,x_0,x_n)\}$ does not converge to $0 = \rho(x_0,x_0,x_0)$. We note that $\{1/n\}$ does not converge strongly even though it converges to every element of X.

Theorem 1.22. *Let (X,d) be a metric space, $x_0 \in X$, and let $\{x_n\}$ be a convergent sequence in $X\setminus\{x_0\}$ with limit x_0, A a proper subset of $X\setminus\{x_0\}$ containing $\{x_n\}$, and B a subset of $X\setminus\{x_0\}$ which contains A properly. Define $\rho : B\times B\times B \to \mathbb{R}^+$ as*

$$\rho(x,y,z) = \begin{cases} 0 & \text{if } x = y = z, \\ \min\{\max\{d(x_0,x),d(x_0,y)\}, \\ \qquad \max\{d(x_0,y),d(x_0,z)\}, \\ \qquad \max\{d(x_0,z),d(x_0,x)\}\} & \text{otherwise.} \end{cases} \tag{1.26}$$

Let ρ_0 denote the restriction of ρ to $A\times A\times A$. Then (B,ρ) and (A,ρ_0) are complete D-metric spaces, $A \subseteq B$, but $\{x \in B$: there is a sequence $\{y_n\}$ in A which converges to x with respect to $\rho\} = B \neq A$.

Proof. The proof follows from Theorem 1.19. □

Remark 1.23. If (X,d) is a metric space, $Y \subseteq X$, d_0 is the restriction of d to $Y\times Y$, and (Y,d_0) is complete, then $\{x \in X$: there is a sequence $\{y_n\}$ in Y which converges to $x\} = Y$. Theorem 1.22 shows that an analogous result does not hold in D-metric spaces.

Theorem 1.24. *Suppose that $\Phi : (\mathbb{R}^+)^{3^*} \to \mathbb{R}^+$ is*

(i) *symmetric in all the three variables,*
(ii) *$\Phi(t_1,t_2,t_3) = 0$ if and only if $t_1 = t_2 = t_3 = 0$,*
(iii) *$\Phi(t_1,t_2,t_3) \leq \Phi(t_1,t_2',t_3') + \Phi(t_1',t_2,t_3'') + \Phi(t_1'',t_2'',t_3)$ whenever (t_1,t_2,t_3), (t_1,t_2',t_3'), (t_1',t_2,t_3''), $(t_1'',t_2'',t_3) \in (\mathbb{R}^+)^{3^*}$ and $t_i \leq t_i' + t_i''$ for all $i = 1,2,3$.*

Let d be a metric on X and let $\rho : X\times X\times X \to \mathbb{R}^+$ be defined as

$$\rho(x,y,z) = \Phi(d(x,y),d(y,z),d(z,x)). \tag{1.27}$$

Then ρ is a D-metric on X. If Φ is continuous at $(0,0,0)$, then

(1) *any d-Cauchy sequence in X is ρ-Cauchy,*
(2) *$\{x_n\} \subseteq X$, $x \in X$, and $d(x_n,x) \to 0$ as $n \to \infty \Rightarrow \{x_n\}$ converges to x with respect to the D-metric ρ.*

Suppose that Φ has the following property:

(iv) *given $\varepsilon > 0$, there exists $\delta > 0$ such that $t < \varepsilon$ whenever $t \in \mathbb{R}^+$ and $\Phi(0,t,t) < \delta$.*

Then

(1) *any ρ-Cauchy sequence is d-Cauchy,*

(2) $\{x_n\}$ $(\subseteq X)$ *converges to $x \in X$ with respect to $\rho \Rightarrow d(x_n,x) \to 0$ as $n \to \infty$.*

If Φ is continuous at $(0,0,0)$, $\{\Phi(0,t_n,t_n)\}$ converges to $\Phi(0,t,t)$ whenever $t \in \mathbb{R}^+$, and $\{t_n\}$ is a sequence in $\mathbb{R}^+$ converging to t, then $\{x_n\} \subseteq X$, $x \in X$, and $d(x_n,x) \to 0$ as $n \to \infty \Rightarrow \{x_n\}$ converges strongly to x with respect to the D-metric ρ.

If Φ is continuous at $(0,0,0)$ and is continuous in any two variables, then $\{x_n\} \subseteq X$, $x \in X$, and $d(x_n,x) \to 0$ as $n \to \infty \Rightarrow \{x_n\}$ converges very strongly to x with respect to the D-metric ρ.

Proof. We prove that ρ is a D-metric on X. Since Φ is symmetric in all the three variables, so is ρ. From property (ii) of Φ, it follows that $\rho(x,y,z) = 0$ if and only if $x = y = z$.

Let $x,y,z,u \in X$. From property (iii) of Φ, we have

$$\begin{aligned}\rho(x,y,z) &= \Phi(d(x,y),d(y,z),d(z,x)) \\ &\le \Phi(d(x,y),d(y,u),d(u,x)) \\ &\quad + \Phi(d(u,y),d(y,z),d(z,u)) \\ &\quad + \Phi(d(x,u),d(u,z),d(z,x))\end{aligned} \tag{1.28}$$

since $d(x,y) \le d(u,y)+d(x,u)$, $d(y,z) \le d(y,u)+d(u,z)$, and $d(z,x) \le d(u,x)+d(z,u)$. Hence $\rho(x,y,z) \le \rho(x,y,u)+\rho(u,y,z)+\rho(x,u,z)$ for all $x,y,z,u \in X$. Hence ρ is a D-metric on X.

Suppose that Φ is continuous at $(0,0,0)$.

(1) Let $\{x_n\}$ be a d-Cauchy sequence in X. Then $d(x_n,x_m) \to 0$ as $n,m \to \infty$. We have

$$\begin{aligned}\rho(x_n,x_m,x_k) &= \Phi(d(x_n,x_m),d(x_m,x_k),d(x_k,x_n)) \\ &\longrightarrow \Phi(0,0,0) = 0 \quad \text{as } n,m,k \longrightarrow \infty \quad (\text{since } \Phi \text{ is continuous at } (0,0,0)).\end{aligned} \tag{1.29}$$

Hence $\{x_n\}$ is ρ-Cauchy in X.

(2) Let $\{x_n\} \subseteq X$ and let $x \in X$ be such that $d(x_n,x) \to 0$ as $n \to \infty$. We have

$$\begin{aligned}\rho(x,x_n,x_m) &= \Phi(d(x,x_n),d(x_n,x_m),d(x_m,x)) \\ &\longrightarrow \Phi(0,0,0) = 0 \quad \text{as } n,m \longrightarrow \infty\end{aligned} \tag{1.30}$$

(since every d-convergent sequence is d-Cauchy and Φ is continuous at $(0,0,0)$).

Hence $\rho(x,x_n,x_m) \to 0$ as $n,m \to \infty$. Hence $\{x_n\}$ converges to x with respect to ρ. Suppose that Φ has property (iv).

(1) Let $\{x_n\}$ be a ρ-Cauchy sequence in X. Let ε be a positive real number. Then there exists $\delta > 0$ such that $t < \varepsilon$ whenever $t \in \mathbb{R}^+$ and $\Phi(0,t,t) < \delta$. Since $\rho(x_n,x_m,x_n) \to 0$ as $n,m \to \infty$, there exists $N \in \mathbb{N}$ such that $\rho(x_n,x_m,x_n) < \delta$ for all $n,m \ge N$. That is, $\Phi(d(x_n,x_m),d(x_m,x_n),d(x_n,x_n)) < \delta$ for all $n,m \ge N$. In other words, $\Phi(0,d(x_n,x_m),d(x_n,x_m)) < \delta$ for all $n,m \ge N$ (since Φ is symmetric). Hence $d(x_n,x_m) < \varepsilon$ for all $n,m \ge N$. Therefore $\{x_n\}$ is d-Cauchy.

(2) Let $\{x_n\} \subseteq X$ converge to $x \in X$ with respect to the D-metric ρ. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $t < \varepsilon$ whenever $t \in \mathbb{R}^+$ and $\Phi(0,t,t) < \delta$. Since $\rho(x_n,x_n,x) \to 0$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that $\rho(x_n,x_n,x) < \delta$ for all $n \geq N$. That is, $\Phi(d(x_n,x_n),d(x_n,x),d(x,x_n)) < \delta$ for all n $\geq N$. That is $\Phi(0,d(x_n,x),d(x_n,x)) < \delta$ for all $n \geq N$. Hence $d(x_n,x) < \varepsilon$ for all $n \geq N$. Therefore $\{x_n\}$ converges to x with respect to the metric d.

Suppose that Φ is continuous at $(0,0,0)$ and $\{\Phi(0,t_n,t_n)\}$ converges to $\Phi(0,t,t)$ whenever $t \in \mathbb{R}^+$ and $\{t_n\}$ is a sequence in $\mathbb{R}^+$ converging to t. Let $\{x_n\} \subseteq X$ and let $x \in X$ be such that $d(x_n,x) \to 0$ as $n \to \infty$. Let $y \in X$. Then $d(x_n,y) \to d(x,y)$ as $n \to \infty$. Hence $\{\Phi(0,d(x_n,y),d(x_n,y))\} \to \Phi(0,d(x,y),d(x,y))$ as $n \to \infty$. That is, $\rho(y,y,x_n) \to \rho(y,y,x)$ as $n \to \infty$. Since Φ is continuous at $(0,0,0)$, from what we have already proved, it follows that $\{x_n\}$ converges to x with respect to ρ. Hence $\{x_n\}$ converges strongly to x with respect to the D-metric ρ. Suppose that Φ is continuous at $(0,0,0)$ and is continuous in any two variables. Let $\{x_n\} \subseteq X$ and let $x \in X$ be such that $d(x_n,x) \to 0$ as $n \to \infty$. Let $y,z \in X$. Then $\{d(z,x_n)\}$ and $\{d(x_n,y)\}$ converge to $d(z,x)$ and $d(x,y)$, respectively. Since Φ is continuous in any two variables, it follows that $\{\Phi(d(y,z),d(z,x_n),d(x_n,y))\}$ converges to $\Phi(d(y,z),d(z,x),d(x,y))$, that is, $\{\rho(y,z,x_n)\}$ converges to $\rho(y,z,x)$. Since Φ is continuous at $(0,0,0)$, $\{x_n\}$ converges to x with respect to ρ. Hence $\{x_n\}$ converges very strongly to x with respect to ρ. □

Corollary 1.25. *Let (X,d) be a metric space and let $\Phi : (\mathbb{R}^+)^{3^*} \to \mathbb{R}^+$ be continuous at $(0,0,0)$, and have properties (i), (ii), (iii), and (iv) specified in Theorem 1.24. Let ρ be defined as in Theorem 1.24. Then ρ is a D-metric on X, a sequence in X is D-Cauchy if and only if it is ρ-Cauchy, and a sequence $\{x_n\}$ in X converges with respect to d to an element x of X if and only if $\{x_n\}$ converges to x with respect to ρ. In particular, ρ-convergence defines a topology on X which coincides with the metric topology induced by the metric d on X, and X is complete with respect to the metric d if and only if it is complete with respect to the D-metric ρ. Further, the following statements are true.*

(1) If $\{\Phi(0,t_n,t_n)\}$ converges to $\Phi(0,t,t)$ whenever $t \in \mathbb{R}^+$ and $\{t_n\}$ is a sequence in $\mathbb{R}^+$ converging to t, $\{x_n\} \subseteq X$, and $x \in X$, then $\{x_n\}$ converges to x with respect to ρ if and only if $\{x_n\}$ converges strongly to x with respect to ρ.

(2) If Φ is continuous in any two variables, $\{x_n\} \subseteq X$, and $x \in X$, then $\{x_n\}$ converges to x with respect to ρ if and only if $\{x_n\}$ converges very strongly to x with respect to ρ.

(3) If Φ is continuous on $(\mathbb{R}^+)^{3^*}$, then ρ is sequentially continuous in all the three variables, that is, $\{\rho(u_n,v_n,w_n)\}$ converges to $\rho(u,v,w)$ whenever $u,v,w \in X$ and $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ are sequences in X converging to u, v, and w, respectively with respect to ρ.

Note 1.26. Corollary 1.25 is useful in generating a number of D-metrics from a given metric on a set.

We now prove a number of propositions which show that the class of functions $\Phi : (\mathbb{R}^+)^{3^*} \to \mathbb{R}^+$, which are continuous at $(0,t,t)$ for all $t \in \mathbb{R}^+$ and which satisfy properties (i), (ii), (iii), and (iv) specified in Theorem 1.24, is very rich.

Lemma 1.27. *Let $p \in [1,\infty)$. Then $(a+b)^p \geq a^p + b^p$ for all $a,b \in \mathbb{R}^+$.*

Proof. Define $f : \mathbb{R}^+ \to \mathbb{R}$ as $f(t) = (1+t)^p - 1 - t^p$ for all $t \in \mathbb{R}^+$. Then $f'(t) = p(1+t)^{p-1} - pt^{(p-1)} = p[(1+t)^{p-1} - t^{(p-1)}]$. Since $1+t \geq t$ for all $t \in \mathbb{R}^+$ and $p-1 \geq 0$, we have $(1+t)^{p-1} \geq t^{(p-1)}$ for all $t \in \mathbb{R}^+$.

Hence $f'(t) \geq 0$ for all $t \in \mathbb{R}^+$. Therefore f is monotonically increasing on $\mathbb{R}^+$. We have $f(0) = 0$. Hence $f(t) \geq f(0)$ for all $t \in \mathbb{R}^+$. Therefore

$$(1+t)^p \geq 1 + t^p \quad \forall t \in \mathbb{R}^+. \tag{1.31}$$

Let $a,b \in \mathbb{R}^+$. We may assume that $a \geq b$. If $a = 0$, then $b = 0$ and $(a+b)^p = 0 = a^p + b^p$. Suppose that $a > 0$. Then, from what we have already proved above, we have

$$\left(1 + \frac{b}{a}\right)^p \geq 1 + \left(\frac{b}{a}\right)^p, \tag{1.32}$$

that is,

$$\left(\frac{a+b}{a}\right)^p \geq 1 + \frac{b^p}{a^p}. \tag{1.33}$$

Hence $(a+b)^p \geq a^p + b^p$. □

Corollary 1.28. *Let $p \in [1,\infty)$. Then $(a+b+c)^p \geq a^p + b^p + c^p$ for all $a,b,c \in \mathbb{R}^+$.*

Proof. Let $a,b,c \in \mathbb{R}^+$. Then, from Lemma 1.27, we have

$$(a+b+c)^p = \left[(a+b)+c\right]^p \geq (a+b)^p + c^p \geq a^p + b^p + c^p. \tag{1.34}$$

□

Proposition 1.29. *Suppose that $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ is monotonically increasing and $\Psi(t) = 0$ if and only if $t = 0$. Let $p \in [1,\infty)$. Define $\Phi : (\mathbb{R}^+)^{3^*} \to \mathbb{R}^+$ as*

$$\Phi(t_1,t_2,t_3) = \left[\left[\Psi(t_1)\right]^p + \left[\Psi(t_2)\right]^p + \left[\Psi(t_3)\right]^p\right]^{1/p} \quad \forall (t_1,t_2,t_3) \in \left(\mathbb{R}^+\right)^{3^*}. \tag{1.35}$$

Then Φ has properties (i), (ii), (iii), and (iv) specified in Theorem 1.24. If Ψ is continuous at 0, then Φ is continuous at $(0,0,0)$, and if Ψ is continuous on $\mathbb{R}^+$, then Φ is continuous on $(\mathbb{R}^+)^{3^*}$.

Proof. Clearly Φ is symmetric in all the three variables:

$$\begin{aligned}
\Phi(t_1,t_2,t_3) = 0 &\Longleftrightarrow \left[\left[\Psi(t_1)\right]^p + \left[\Psi(t_2)\right]^p + \left[\Psi(t_3)\right]^p\right]^{1/p} = 0 \\
&\Longleftrightarrow \left[\Psi(t_i)\right]^p = 0 \quad \forall i \\
&\Longleftrightarrow \Psi(t_i) = 0 \quad \forall i \\
&\Longleftrightarrow t_i = 0 \quad \forall i.
\end{aligned} \tag{1.36}$$

Let $t_1, t_1', t_1'', t_2, t_2', t_2'', t_3, t_3', t_3'' \in \mathbb{R}^+$. Let

$$\begin{aligned} a &= [[\Psi(t_1)]^p + [\Psi(t_2')]^p + [\Psi(t_3')]^p]^{1/p}, \\ b &= [[\Psi(t_1')]^p + [\Psi(t_2)]^p + [\Psi(t_3'')]^p]^{1/p}, \\ c &= [[\Psi(t_1'')]^p + [\Psi(t_2'')]^p + [\Psi(t_3)]^p]^{1/p}. \end{aligned} \tag{1.37}$$

We have

$$\begin{aligned} (a+b+c)^p &\geq a^p + b^p + c^p \\ &= [[\Psi(t_1)] + [\Psi(t_2')] + [\Psi(t_3')]] \\ &\quad + [[\Psi(t_1')]^p + [\Psi(t_2)]^p + [\Psi(t_3'')]^p] \\ &\quad + [[\Psi(t_1'')]^p + [\Psi(t_2'')]^p + [\Psi(t_3)]^p] \\ &\geq [\Psi(t_1)]^p + [\Psi(t_2)]^p + [\Psi(t_3)]^p. \end{aligned} \tag{1.38}$$

Hence $a+b+c \geq [[\Psi(t_1)]^p + [\Psi(t_2)]^p + [\Psi(t_3)]^p]^{1/p}$. Therefore Φ has property (iii). We have

$$\begin{aligned} \Phi(0,t,t) &= [[\Psi(0)]^p + [\Psi(t)]^p + [\Psi(t)]^p]^{1/p} \\ &= [2[\Psi(t)]^p]^{1/p} \\ &= 2^{1/p}\Psi(t). \end{aligned} \tag{1.39}$$

Let ε be a positive real number. Choose $\delta = 2^{1/p}\Psi(\varepsilon)$. Then $\delta > 0$ since $\Psi(t) = 0$ implies $t = 0$.

$$\begin{aligned} \Phi(0,t,t) < \delta &\Longrightarrow 2^{1/p}\Psi(t) < 2^{1/p}\Psi(\varepsilon) \\ &\Longrightarrow \Psi(t) < \Psi(\varepsilon) \\ &\Longrightarrow t < \varepsilon \quad (\text{since } \Psi \text{ is monotonically increasing}). \end{aligned} \tag{1.40}$$

Hence Φ has property (iv). □

Corollary 1.30. *Let $p \in [1,\infty)$. Then the function $\Phi : (\mathbb{R}^+)^{3*} \to \mathbb{R}^+$ defined as $\Phi(t_1,t_2,t_3) = [t_1^p + t_2^p + t_3^p]^{1/p}$ for all $(t_1,t_2,t_3) \in (\mathbb{R}^+)^{3*}$ is continuous on $(\mathbb{R}^+)^{3*}$ and has properties (i), (ii), (iii), and (iv) specified in Theorem 1.24.*

Proof. The proof follows from Proposition 1.29 by taking $\Psi(t) = t$ for all $t \in \mathbb{R}^+$. □

Proposition 1.31. *Suppose that $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ is monotonically increasing and $\Psi(t) = 0$ if and only if $t = 0$. Define $\Phi : (\mathbb{R}^+)^{3*} \to \mathbb{R}^+$ as*

$$\Phi(t_1,t_2,t_3) = \max\{\Psi(t_1),\Psi(t_2),\Psi(t_3)\} \quad \forall (t_1,t_2,t_3) \in (\mathbb{R}^+)^{3*}. \tag{1.41}$$

Then Φ has properties (i), (ii), (iii), and (iv) specified in Theorem 1.24. If Ψ is continuous at 0, then Φ is continuous at $(0,0,0)$, and if Ψ is continuous on $\mathbb{R}^+$, then Φ is continuous on $(\mathbb{R}^+)^{3*}$.

Proof. Clearly Φ is symmetric in all the three variables.

$$\begin{aligned}\Phi(t_1,t_2,t_3)=0 &\Longleftrightarrow \max\{\Psi(t_1),\Psi(t_2),\Psi(t_3)\}=0\\ &\Longleftrightarrow \Psi(t_i)=0 \quad \forall i\\ &\Longleftrightarrow t_i=0 \quad \forall i.\end{aligned} \tag{1.42}$$

Hence Φ has property (ii).

Let $t_1,t_1',t_1'',t_2,t_2',t_2'',t_3,t_3',t_3''\in\mathbb{R}^+$. Then

$$\begin{aligned}\max\{\Psi(t_1),\Psi(t_2),\Psi(t_3)\}\le{}&\max\{\Psi(t_1),\Psi(t_2'),\Psi(t_3')\}\\ &+\max\{\Psi(t_1'),\Psi(t_2),\Psi(t_3'')\}\\ &+\max\{\Psi(t_1''),\Psi(t_2''),\Psi(t_3)\}.\end{aligned} \tag{1.43}$$

Hence Φ has property (iii).

Let ε be a positive real number. Choose $\delta=\Psi(\varepsilon)$. Then $\delta>0$ since $\Psi(t)=0$ implies $t=0$.

$$\begin{aligned}\Phi(0,t,t)<\delta &\Longrightarrow \max\{\Psi(0),\Psi(t),\Psi(t)\}<\Psi(\varepsilon)\\ &\Longrightarrow \Psi(t)<\Psi(\varepsilon)\\ &\Longrightarrow t<\varepsilon \quad (\text{since } \Psi \text{ is monotonically increasing}).\end{aligned} \tag{1.44}$$

Hence Φ has property (iv). □

Corollary 1.32. *The function $\Phi:(\mathbb{R}^+)^{3*}\to\mathbb{R}^+$ defined as $\Phi(t_1,t_2,t_3)=\max\{t_1,t_2,t_3\}$ for all $(t_1,t_2,t_3)\in(\mathbb{R}^+)^{3*}$ is continuous on $(\mathbb{R}^+)^{3*}$ and has properties (i), (ii), (iii), and (iv) specified in Theorem 1.24.*

Proof. The proof follows from Proposition 1.31 by taking $\Psi(t)=t$ for all $t\in\mathbb{R}^+$. □

Proposition 1.33. *Suppose that $\Psi:\mathbb{R}^+\to\mathbb{R}^+$ is monotonically increasing, $\Psi(s+t)\le\Psi(s)+\Psi(t)$ for all $s,t\in\mathbb{R}^+$, and $\Psi(t)=0$ if and only if $t=0$. Define $\Phi:(\mathbb{R}^+)^{3*}\to\mathbb{R}^+$ as*

$$\begin{aligned}\Phi(t_1,t_2,t_3)=\min\{&\max\{\Psi(t_1),\Psi(t_2)\},\max\{\Psi(t_2),\Psi(t_3)\},\\ &\max\{\Psi(t_3),\Psi(t_1)\}\}\quad \forall(t_1,t_2,t_3)\in(\mathbb{R}^+)^{3*}.\end{aligned} \tag{1.45}$$

Then Φ *has properties (i), (ii), (iii), and (iv) specified in Theorem 1.24. If* Ψ *is continuous at* 0*, then* Φ *is continuous at* $(0,0,0)$*, and if* Ψ *is continuous on* $\mathbb{R}^+$*, then* Φ *is continuous on* $(\mathbb{R}^+)^{3*}$*.*

Proof. Clearly Φ is symmetric in all the three variables.

$$\begin{aligned}\Phi(t_1,t_2,t_3)=0 &\Longleftrightarrow \min\{\max\{\Psi(t_1),\Psi(t_2)\},\max\{\Psi(t_2),\Psi(t_3)\},\\ &\qquad \max\{\Psi(t_3),\Psi(t_1)\}\}=0\\ &\Longleftrightarrow \max\{\Psi(t_1),\Psi(t_2)\}=0\\ &\qquad \text{or}\quad \max\{\Psi(t_2),\Psi(t_3)\}=0\\ &\qquad \text{or}\quad \max\{\Psi(t_3),\Psi(t_1)\}=0\\ &\Longleftrightarrow t_1=t_2=0\quad\text{or}\quad t_2=t_3=0\quad\text{or}\quad t_3=t_1=0\\ &\Longleftrightarrow t_1=t_2=t_3=0\quad \left(\text{since } (t_1,t_2,t_3)\in(\mathbb{R}^+)^{3*}\right).\end{aligned}\tag{1.46}$$

Hence Φ has property (ii).

Let $t_1,t_1',t_1'',t_2,t_2',t_2'',t_3,t_3',t_3''\in\mathbb{R}^+$ be such that $t_i\le t_i'+t_i''$ for all $i=1,2,3$. We have

$$\begin{aligned}&\max\{\Psi(t_1),\Psi(t_2')\}+\max\{\Psi(t_1''),\Psi(t_2'')\}\\ &\quad\ge \max\{\Psi(t_1),\Psi(t_2'+t_2'')\}\quad \left(\text{since } \Psi(s+t)\le\Psi(s)+\Psi(t)\right)\\ &\quad\ge \max\{\Psi(t_1),\Psi(t_2)\}\quad \left(\text{since } \Psi \text{ is monotonically increasing}\right.\\ &\qquad\qquad \left.\text{and } t_2\le t_2'+t_2''\right).\end{aligned}\tag{1.47}$$

Clearly, we have

$$\begin{aligned}&\max\{\Psi(t_1),\Psi(t_2')\}+\max\{\Psi(t_2''),\Psi(t_3)\}\ge\max\{\Psi(t_1),\Psi(t_3)\},\\ &\max\{\Psi(t_1),\Psi(t_2')\}+\max\{\Psi(t_3),\Psi(t_1'')\}\ge\max\{\Psi(t_1),\Psi(t_3)\}.\end{aligned}\tag{1.48}$$

Hence

$$\begin{aligned}&\max\{\Psi(t_1),\Psi(t_2')\}+\min\{\max\{\Psi(t_1''),\Psi(t_2'')\},\max\{\Psi(t_2''),\Psi(t_3)\},\\ &\qquad\qquad \max\{\Psi(t_3),\Psi(t_1'')\}\}\\ &\quad\ge\min\{\max\{\Psi(t_1),\Psi(t_2)\},\max\{\Psi(t_1),\Psi(t_3)\}\}.\end{aligned}\tag{1.49}$$

Similarly,

$$\begin{aligned}
&\max\{\Psi(t_3'),\Psi(t_1)\}+\min\{\max\{\Psi(t_1'),\Psi(t_2)\},\max\{\Psi(t_2),\Psi(t_3'')\},\max\{\Psi(t_3''),\Psi(t_1')\}\}\\
&\qquad\geq\min\{\max\{\Psi(t_1),\Psi(t_2)\},\max\{\Psi(t_1),\Psi(t_3)\}\},\\
&\max\{\Psi(t_1'),\Psi(t_2)\}+\min\{\max\{\Psi(t_1''),\Psi(t_2'')\},\max\{\Psi(t_2''),\Psi(t_3)\},\max\{\Psi(t_3)\},\Psi(t_1'')\}\}\\
&\qquad\geq\min\{\max\{\Psi(t_1),\Psi(t_2)\},\max\{\Psi(t_2),\Psi(t_3)\}\},\\
&\max\{\Psi(t_2),\Psi(t_3'')\}+\min\{\max\{\Psi(t_1),\Psi(t_2')\},\max\{\Psi(t_2'),\Psi(t_3')\},\max\{\Psi(t_3'),\Psi(t_1)\}\}\\
&\qquad\geq\min\{\max\{\Psi(t_1),\Psi(t_2)\},\max\{\Psi(t_2),\Psi(t_3)\}\},\\
&\max\{\Psi(t_2''),\Psi(t_3)\}+\min\{\max\{\Psi(t_1),\Psi(t_2')\},\max\{\Psi(t_2'),\Psi(t_3')\},\max\{\Psi(t_3'),\Psi(t_1)\}\}\\
&\qquad\geq\min\{\max\{\Psi(t_1),\Psi(t_3)\},\max\{\Psi(t_2),\Psi(t_3)\}\},\\
&\max\{\Psi(t_3),\Psi(t_1'')\}+\min\{\max\{\Psi(t_1'),\Psi(t_2)\},\max\{\Psi(t_2),\Psi(t_3'')\},\max\{\Psi(t_3''),\Psi(t_1')\}\}\\
&\qquad\geq\min\{\max\{\Psi(t_1),\Psi(t_3)\},\max\{\Psi(t_2),\Psi(t_3)\}\}.
\end{aligned}\tag{1.50}$$

We have

$$\begin{aligned}
&\max\{\Psi(t_2'),\Psi(t_3')\}+\max\{\Psi(t_3''),\Psi(t_1')\}+\max\{\Psi(t_1''),\Psi(t_2'')\}\\
&\quad\geq\max\{\max\{\Psi(t_3''),\Psi(t_1')\}+\max\{\Psi(t_1''),\Psi(t_2'')\},\\
&\qquad\qquad\max\{\Psi(t_2'),\Psi(t_3')\}+\max\{\Psi(t_1''),\Psi(t_2'')\},\\
&\qquad\qquad\max\{\Psi(t_2'),\Psi(t_3')\}+\max\{\Psi(t_3''),\Psi(t_1')\}\}\\
&\quad\geq\max\{\Psi(t_1')+\Psi(t_1''),\Psi(t_2')+\Psi(t_2''),\Psi(t_3')+\Psi(t_3'')\}\\
&\quad\geq\max\{\Psi(t_1'+t_1''),\Psi(t_2'+t_2''),\Psi(t_3'+t_3'')\}\quad(\Theta\Psi(s+t)\leq\Psi(s)+\Psi(t)\ \forall s,t\in\mathbb{R}^+)\\
&\quad\geq\max\{\Psi(t_1),\Psi(t_2),\Psi(t_3)\}\quad(\text{since }\Psi\text{ is monotonically increasing}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{and }t_i\leq t_i'+t_i''\ \forall i=1,2,3).
\end{aligned}\tag{1.51}$$

Hence

$$\begin{aligned}
&\min\{\max\{\Psi(t_1),\Psi(t_2)\},\max\{\Psi(t_2),\Psi(t_3)\},\max\{\Psi(t_3),\Psi(t_1)\}\}\\
&\quad\leq\min\{\max\{\Psi(t_1),\Psi(t_2')\},\max\{\Psi(t_2'),\Psi(t_3')\},\max\{\Psi(t_3'),\Psi(t_1)\}\}\\
&\qquad+\min\{\max\{\Psi(t_1'),\Psi(t_2)\},\max\{\Psi(t_2),\Psi(t_3'')\},\max\{\Psi(t_3''),\Psi(t_1')\}\}\\
&\qquad+\min\{\max\{\Psi(t_1''),\Psi(t_2'')\},\max\{\Psi(t_2''),\Psi(t_3)\},\max\{\Psi(t_3),\Psi(t_1'')\}\}.
\end{aligned}\tag{1.52}$$

Therefore $\Phi(t_1,t_2,t_3)\leq\Phi(t_1,t_2',t_3')+\Phi(t_1',t_2,t_3'')+\Phi(t_1'',t_2'',t_3)$ whenever (t_1,t_2,t_3), (t_1,t_2',t_3'), (t_1',t_2,t_3''), $(t_1'',t_2'',t_3)\in(\mathbb{R}^+)^{3*}$ and $t_i\leq t_i'+t_i''$ for all $i=1,2,3$. Thus Φ has property (iii).

Let ε be a positive real number. Choose $\delta = \Psi(\varepsilon)$. Then $\delta > 0$ since $\Psi(t) = 0$ implies $t = 0$.

$$\begin{aligned}\Phi(0,t,t) < \delta &\Longrightarrow \min\{\max\{\Psi(0),\Psi(t)\},\max\{\Psi(t),\Psi(t)\},\\ &\qquad \max\{\Psi(t),\Psi(0)\}\} < \Psi(\varepsilon)\\ &\Longrightarrow \min\{\Psi(t),\Psi(t),\Psi(t)\} < \Psi(\varepsilon)\\ &\Longrightarrow \Psi(t) < \Psi(\varepsilon)\\ &\Longrightarrow t < \varepsilon \quad (\text{since } \Psi \text{ is monotonically increasing}).\end{aligned} \tag{1.53}$$

Hence Φ has property (iv). □

Corollary 1.34. *The function* $\Phi : (\mathbb{R}^+)^{3^*} \to \mathbb{R}^+$ *defined as* $\Phi(t_1,t_2,t_3) = \min\{\max\{t_1, t_2\},\max\{t_2,t_3\},\max\{t_3,t_1\}\}$ *for all* $(t_1,t_2,t_3) \in (\mathbb{R}^+)^{3^*}$ *is continuous on* $(\mathbb{R}^+)^{3^*}$ *and has properties (i), (ii), (iii), and (iv) specified in Theorem 1.24.*

Proof. The proof follows from Proposition 1.33 by taking $\Psi(t) = t$ for all $t \in \mathbb{R}^+$. □

Proposition 1.35. *The function* $\Phi : (\mathbb{R}^+)^{3^*} \to \mathbb{R}^+$ *defined as*

$$\Phi(t_1,t_2,t_3) = \begin{cases} t_1+t_2+t_3 & \text{if } \min\{|t_1-t_2|,|t_2-t_3|,|t_3-t_1|\} \le 1,\\ 1+t_1+t_2+t_3 & \text{otherwise,}\end{cases} \tag{1.54}$$

has properties (i), (ii), (iii), and (iv) specified in Theorem 1.24 and is continuous at $(t_1,t_2, t_3) \in (\mathbb{R}^+)^{3^*}$ *if* $\min\{|t_1-t_2|,|t_2-t_3|,|t_3-t_1|\} \ne 1$.

Proof. Clearly Φ has properties (i) and (ii).

Let $(t_1,t_2,t_3),(t_1,t_2',t_3'),(t_1',t_2,t_3''),(t_1'',t_2'',t_3) \in (\mathbb{R}^+)^{3^*}$ be such that $t_i \le t_i'+t_i''$ for all $i = 1,2,3$.

Case (i). $\min\{|t_1-t_2|,|t_2-t_3|,|t_3-t_1|\} \le 1$.
Then

$$\begin{aligned}\Phi(t_1,t_2,t_3) &= t_1+t_2+t_3\\ &\le (t_1+t_2'+t_3')+(t_1'+t_2+t_3'')+(t_1''+t_2''+t_3)\\ &\le \Phi(t_1,t_2',t_3')+\Phi(t_1',t_2,t_3'')+\Phi(t_1'',t_2'',t_3).\end{aligned} \tag{1.55}$$

Case (ii). $\min\{|t_1-t_2|,|t_2-t_3|,|t_3-t_1|\} > 1$.
Then $\max\{t_1,t_2,t_3\} > 1$, hence

$$\begin{aligned}\Phi(t_1,t_2,t_3) &= 1+t_1+t_2+t_3\\ &< 2(t_1+t_2+t_3)\\ &\le (t_1+t_2'+t_3')+(t_1'+t_2+t_3'')+(t_1''+t_2''+t_3)\\ &\qquad (\text{since } t_i \le t_i'+t_i'' \ \ \forall i = 1,2,3).\\ &\le \Phi(t_1,t_2',t_3')+\Phi(t_1',t_2,t_3'')+\Phi(t_1'',t_2'',t_3).\end{aligned} \tag{1.56}$$

Thus, in either case, $\Phi(t_1,t_2,t_3) \le \Phi(t_1,t_2',t_3')+\Phi(t_1',t_2,t_3'')+\Phi(t_1'',t_2'',t_3)$. Hence Φ has property (iii).

Let ε be a positive real number. Let $\delta = \varepsilon$. We have

$$\begin{aligned}\Phi(0,t,t) < \delta &\Longrightarrow 0+t+t < \varepsilon \\ &\Longrightarrow 2t < \varepsilon \\ &\Longrightarrow t < \varepsilon.\end{aligned} \tag{1.57}$$

Hence Φ has property (iv). □

Proposition 1.36. *The function* $\Phi : (\mathbb{R}^+)^{3*} \to \mathbb{R}^+$ *defined as*

$$\Phi(t_1,t_2,t_3) = \begin{cases} t_1+t_2+t_3 & \textit{if } \min\{t_1,t_2,t_3\} \le 1, \\ 1+t_1+t_2+t_3 & \textit{otherwise}, \end{cases} \tag{1.58}$$

has properties (i), (ii), (iii), and (iv) specified in Theorem 1.24 and is continuous at $(t_1,t_2,t_3) \in (\mathbb{R}^+)^{3*}$ *if* $\min\{t_1,t_2,t_3\} \ne 1$.

Proof. The proof is similar to that of Proposition 1.35. □

Proposition 1.37. *Let* $\Phi : (\mathbb{R}^+)^{3*} \to \mathbb{R}^+$ *be defined as*

$$\Phi(t_1,t_2,t_3) = \min\{t_1+t_2, t_2+t_3, t_3+t_1\} \quad \forall (t_1,t_2,t_3) \in (\mathbb{R}^+)^{3*}. \tag{1.59}$$

Then Φ *is continuous on* $(\mathbb{R}^+)^{3*}$ *and has properties (i), (ii), (iii), and (iv) specified in Theorem 1.24.*

Proof. Let $t_1,t_1',t_1'',t_2,t_2',t_2'',t_3,t_3',t_3'' \in \mathbb{R}^+$ be such that $t_i \le t_i' + t_i''$ for all $i = 1,2,3$. We have

$$\begin{gathered}(t_1+t_2') + (t_1''+t_2'') \ge t_1+t_2'+t_2'' \ge t_1+t_2, \qquad (t_1+t_2') + (t_2''+t_3) \ge t_1+t_3, \\ (t_1+t_2') + (t_3+t_1'') \ge t_1+t_3.\end{gathered} \tag{1.60}$$

Hence $(t_1+t_2') + \min\{(t_1''+t_2''),(t_2''+t_3),(t_3+t_1'')\} \ge \min\{t_1+t_2, t_1+t_3\}$. Similarly, we have

$$\begin{aligned}(t_3'+t_1) + \min\{(t_1'+t_2),(t_2+t_3''),(t_3''+t_1')\} &\ge \min\{t_1+t_2,t_1+t_3\}, \\ (t_1'+t_2) + \min\{(t_1''+t_2''),(t_2''+t_3),(t_3+t_1'')\} &\ge \min\{t_1+t_2,t_2+t_3\}, \\ (t_2+t_3'') + \min\{(t_1+t_2'),(t_2'+t_3'),(t_3'+t_1)\} &\ge \min\{t_1+t_2,t_2+t_3\}, \\ (t_2''+t_3) + \min\{(t_1+t_2'),(t_2'+t_3'),(t_3'+t_1)\} &\ge \min\{t_1+t_2,t_2+t_3\}, \\ (t_3+t_1'') + \min\{(t_1'+t_2),(t_2+t_3''),(t_3''+t_1')\} &\ge \min\{t_2+t_3,t_3+t_1\}.\end{aligned} \tag{1.61}$$

We have

$$(t_2'+t_3') + (t_3''+t_1') + (t_1''+t_2'') = (t_1'+t_1'') + (t_2'+t_2'') + (t_3'+t_3'') \ge t_1+t_2+t_3. \tag{1.62}$$

Hence

$$\begin{aligned}\min\{t_1+t_2,t_2+t_3,t_3+t_1\} \le{}& \min\{(t_1+t_2'),(t_2'+t_3'),(t_3'+t_1)\} \\ &+ \min\{(t_1'+t_2),(t_2+t_3''),(t_3''+t_1')\} \\ &+ \min\{(t_1''+t_2''),(t_2''+t_3),(t_3+t_1'')\}.\end{aligned} \tag{1.63}$$

Hence $\Phi(t_1,t_2,t_3) \le \Phi(t_1,t_2',t_3') + \Phi(t_1',t_2,t_3'') + \Phi(t_1'',t_2'',t_3)$ whenever (t_1,t_2,t_3), $(t_1,t_2',t_3'),(t_1',t_2,t_3''),(t_1'',t_2'',t_3) \in (\mathbb{R}^+)^{3*}$ and $t_i \le t_i' + t_i''$ for all $i = 1,2,3$. Hence Φ has property (iii) specified in Theorem 1.24. □

Remark 1.38. Let (X,ρ) be a D-metric space and $\{x_n\}$ a sequence in X. If $\{x_n\}$ converges to an element, say $x \in X$, then $\{\rho(y,z,x_n)\}$ need not converge to $\rho(y,z,x)$ for all $y,z \in X$. The following examples show that it is so, even when every convergent sequence is strongly convergent and has a unique limit and ρ-convergence defines a topology on X which is a metric topology. While in the first example we show the existence of a convergent sequence $\{x_n\}$ with limit, say x, and elements y, z of X such that $\{\rho(y,z,x_n)\}$ is convergent but not to $\rho(y,z,x)$, in the second example we show the existence of a convergent sequence $\{x_n\}$ with limit, say, x, and elements y, z of X such that $\{\rho(y,z,x_n)\}$ is not convergent.

Example 1.39. Define a function $\Phi : (\mathbb{R}^+)^{3*} \to \mathbb{R}^+$ as

$$\Phi(t_1,t_2,t_3) = \begin{cases} t_1+t_2+t_3 & \text{if } \min\{|t_1-t_2|,|t_2-t_3|,|t_3-t_1|\} \le 1, \\ 1+t_1+t_2+t_3 & \text{otherwise.} \end{cases} \tag{1.64}$$

From Proposition 1.35, we know that Φ has properties (i), (ii), (iii), and (iv) specified in Theorem 1.24, and that Φ is continuous at (t_1,t_2,t_3) if $\min\{|t_1-t_2|,|t_2-t_3|,|t_3-t_1|\} \ne 1$. Define $\rho : \mathbb{R}^3 \to \mathbb{R}^+$ as $\rho(x,y,z) = \Phi(|x-y|,|y-z|,|z-x|)$ for all $x,y,z \in \mathbb{R}$. Then, from Corollary 1.25, it follows that ρ is a D-metric on $\mathbb{R}$, $(\mathbb{R},\rho)$ is ρ-complete, and ρ-convergence defines a topology on $\mathbb{R}$ which is nothing but the usual topology on $\mathbb{R}$. Further, if $\{u_n\} \subseteq \mathbb{R}$ and $u \in \mathbb{R}$, then $|u_n-u| \to 0$ as $n \to \infty$ if and only if $\{u_n\}$ converges to u with respect to ρ if and only if $\{u_n\}$ converges to u strongly with respect to ρ. Hence every ρ-convergent sequence has a unique limit. Let $x_n = 1+1/n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a sequence in $\mathbb{R}$. Clearly $\{x_n\}$ converges to 1 in the usual sense. Hence $\{x_n\}$ converges to 1 with respect to ρ. Let $x = 1$, $y = 3$, and $z = 6$. Then we have

$$\begin{aligned} \rho(y,z,x) &= \rho(3,6,1) = \Phi(|3-6|,|6-1|,|1-3|) = \Phi(3,5,2) \\ &= 10 \quad (\text{since } \min\{|3-5|,|5-2|,|2-3|\} = \min\{2,3,1\} = 1), \\ \rho(y,z,x_n) &= \rho\left(3,6,1+\frac{1}{n}\right) = \Phi\left(|3-6|,\left|6-1-\frac{1}{n}\right|,\left|1+\frac{1}{n}-3\right|\right) \\ &= \Phi\left(3,5-\frac{1}{n},2-\frac{1}{n}\right) \\ &= 11-\frac{2}{n} \quad \forall n \ge 2 \left(\text{since } \min\left\{\left|3-5+\frac{1}{n}\right|,|5-2|,\right.\right. \\ &\qquad\qquad \left.\left.\left|\left(2-\frac{1}{n}\right)-3\right|\right\} = \min\left\{3,1+\frac{1}{n},2-\frac{1}{n}\right\} > 1\right) \\ &\longrightarrow 11 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{1.65}$$

Hence $\{\rho(y,z,x_n)\}$ does not converge to $\rho(y,z,x)$.

Example 1.40. Define a function $\Phi : (\mathbb{R}^+)^{3^*} \to \mathbb{R}^+$ as

$$\Phi(t_1,t_2,t_3) = \begin{cases} t_1+t_2+t_3 & \text{if } \min\{t_1,t_2,t_3\} \le 1, \\ 1+t_1+t_2+t_3 & \text{otherwise.} \end{cases} \tag{1.66}$$

From Proposition 1.36 we know that Φ has properties (i), (ii), (iii), and (iv) specified in Theorem 1.24, and that Φ is continuous at $(t_1,t_2,t_3) \in (\mathbb{R}^+)^{3^*}$ if and only if $\min\{t_1,t_2,t_3\} \ne 1$. Define $\rho : \mathbb{R}^3 \to \mathbb{R}^+$ as $\rho(x,y,z) = \Phi(|x-y|,|y-z|,|z-x|)$ for all $x,y,z \in \mathbb{R}$. Then, from Corollary 1.25, it follows that ρ is a D-metric on $\mathbb{R}$, $(\mathbb{R},\rho)$ is D-complete, and ρ-convergence defines a topology on $\mathbb{R}$ which is nothing but the usual topology on $\mathbb{R}$. Further, if $\{u_n\} \subseteq \mathbb{R}$ and $u \in \mathbb{R}$, then $|u_n - u| \to 0$ as $n \to \infty$ if and only if $\{u_n\}$ converges to u with respect to ρ if and only if $\{u_n\}$ converges to u strongly with respect to ρ. Hence every ρ-convergent sequence has a unique limit.

For any $y \in \mathbb{R}$,

$$\begin{aligned} \rho\left(y,y+2,y+3-\frac{1}{n}\right) &= \Phi\left(2,1-\frac{1}{n},3-\frac{1}{n}\right) \\ &= 6-\frac{2}{n} \longrightarrow 6 \quad \text{as } n \longrightarrow \infty, \\ \rho\left(y,y+2,y+3+\frac{1}{n}\right) &= \Phi\left(2,1+\frac{1}{n},3+\frac{1}{n}\right) \\ &= 1+6+\frac{2}{n} \\ &\longrightarrow 7 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{1.67}$$

The sequences $\{y+3-1/n\}$ and $\{y+3+1/n\}$ both converge to $y+3$. Let

$$x_n = \begin{cases} y+3-\dfrac{1}{n} & \text{if } n \text{ is odd,} \\ y+3+\dfrac{1}{n} & \text{if } n \text{ is even.} \end{cases} \tag{1.68}$$

Then $\{x_n\}$ converges to $y+3$, but $\{\rho(y,y+2,x_n)\}$ does not converge.

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