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SOME EXACT INEQUALITIES OF HARDY-LITTLEWOOD-POLYA TYPE FOR PERIODIC FUNCTIONS

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We investigate the following problem: for a given $A \geq 0$, find the infimum of the set of $B \geq 0$ such that the inequality $\|x^{(k)}\|_2^2 \leq A\|x^{(r)}\|_2^2 + B\|x\|_2^2$, for $k, r \in \mathbb{N} \cup \{0\}$, $0 \leq k < r$, holds for all sufficiently smooth functions.

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1. Introduction. Let $G = \mathbb{R}$ or $G = \mathbb{T} = [0, 2\pi)$. By $L_2(G)$, we will denote the spaces of all measurable functions $x : G \to \mathbb{R}$ such that

$$\|x\|_2 = \|x\|_{L_2(G)} := \left\{ \int_G |x(t)|^2 dt \right\}^{1/2} < \infty. \tag{1.1}$$

Denote by $L_2^r(G)$ $(r \in \mathbb{N})$ the space of all functions x such that $x^{(r-1)}$ are locally absolutely continuous and $x^{(r)} \in L_2(G)$, and set $L_{2,2}^r(G) = L_2(G) \cap L_2^r(G)$ (in the case $G = \mathbb{T}$, we mean that spaces $L_2(G)$ and $L_2^r(G)$ consist of 2π-periodic functions). Note that $L_2^r(G) \subset L_2(G)$ if $G = \mathbb{T}$.

It is well known that the exact inequality of Hardy [3]

$$\|x^{(k)}\|_2^2 \leq \|x\|_2^{2(1-k/r)} \|x^{(r)}\|_2^{2(k/r)}, \quad k \in \mathbb{N},\ 0 < k < r, \tag{1.2}$$

holds for every function $x \in L_{2,2}^r(\mathbb{R})$.

For any $A > 0$ and any $x \in L_{2,2}^r(\mathbb{R})$, from inequality (1.2), we get

$$\|x^{(k)}\|_2^2 \leq \left\{ \left(\frac{k}{Ar} \right)^{k/(r-k)} \|x\|_2^2 \right\}^{(r-k)/r} \left\{ \frac{Ar}{k} \|x^{(r)}\|_2^2 \right\}^{k/r}. \tag{1.3}$$

Using Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad 1 \leq p < \infty,\ a, b > 0, \tag{1.4}$$

with $p = r/(r-k)$ and $p' = r/k$, we get, for any $A > 0$ and any $x \in L_{2,2}^r(\mathbb{R})$, the following inequality:

$$\|x^{(k)}\|_2^2 \leq A\|x^{(r)}\|_2^2 + \frac{r-k}{r} \left(\frac{k}{Ar} \right)^{k/(r-k)} \|x\|_2^2. \tag{1.5}$$

This inequality is the best possible in the next sense: for a given $A > 0$, the infimum of constants B such that the inequality

$$||x^{(k)}||_2^2 \le A||x^{(r)}||_2^2 + B\|x\|_2^2 \tag{1.6}$$

holds for all functions $x \in L_{2,2}^r(\mathbb{R})$ is equal to

$$\frac{r-k}{r}\left(\frac{k}{Ar}\right)^{k/(r-k)}. \tag{1.7}$$

As is well known, inequality (1.2) (and consequently (1.5)) holds true for any function $x \in L_{2,2}^r(\mathbb{T})$. However, the constant (1.7) is not the best possible in general (for a given constant A). Therefore, the main problem which we will study in this paper is the following.

For a given $A \ge 0$, find the infimum of constants B such that inequality (1.6) holds for all functions $x \in L_{2,2}^r(\mathbb{T})$.

We will denote this infimum by $\Psi(\mathbb{T};r,k;A)$. We will investigate also the analogous problem in the presence of some restrictions on the spectrum of functions $x \in L_{2,2}^r(\mathbb{T})$.

Note that Babenko and Rassias [1] investigated the problem on exact inequalities for functions $x \in L_{2,2}^r(\mathbb{T})$. They have found, for a given $A \ge 0$, the infimum of constants B such that the inequality

$$||x^{(k)}||_2^2 \le A\|x\|_2^2 + B||x^{(r)}||_2^2 \tag{1.8}$$

holds for all functions $x \in L_{2,2}^r(\mathbb{T})$.

For more information related to this subject, see, for example, [2, 4, 5, 6].

2. Main results

Theorem 2.1. *Let $k,r \in \mathbb{N}$, $k < r$. Then for any $A \ge 0$ and any $x \in L_{2,2}^r(\mathbb{T})$,*

$$||x^{(k)}||_2^2 \le A||x^{(r)}||_2^2 + (v_0^{2k} - Av_0^{2r})\|x\|_2^2 = A||x^{(r)}||_2^2 + \varphi(A,v_0)\|x\|_2^2 \tag{2.1}$$

holds if v_0 is such that $\eta(v_0+1) \le A \le \eta(v_0)$, where

$$\eta(v) = \frac{v^{2k} - (v-1)^{2k}}{v^{2r} - (v-1)^{2r}}. \tag{2.2}$$

Given A, the constant $\varphi(A,v_0)$ in (2.1) is the best possible; that is,

$$\Psi(\mathbb{T};r,k;A) = (v_0^{2k} - Av_0^{2r}), \tag{2.3}$$

where v_0 is such that $\eta(v_0+1) \le A \le \eta(v_0)$.

Proof. Let

$$\begin{aligned} e_v(t) &:= \frac{1}{2\pi}e^{ivt}, \quad v \in \mathbb{Z},\ t \in \mathbb{R}, \\ c_v(x) &= \int_0^{2\pi} x(t)e_v(t)\,dt \end{aligned} \tag{2.4}$$

be Fourier coefficients of a function x, and let

$$\sum_{v\in\mathbb{Z}} c_v(x)e_v(t) \tag{2.5}$$

be the Fourier series of a function x.

For any $x \in L^r_{2,2}(\mathbb{T})$, $0 < k < r$, and any $A \geq 0$, using Parseval's equality, we get

$$\begin{aligned} \|x^{(k)}\|_2^2 &= \sum_{\substack{v\in\mathbb{Z}\\ v\neq 0}} |c_v(x)|^2 v^{2k} \\ &= A\sum_{\substack{v\in\mathbb{Z}\\ v\neq 0}} |c_v(x)|^2 v^{2r} + \sum_{\substack{v\in\mathbb{Z}\\ v\neq 0}} |c_v(x)|^2 v^{2r}\left[\frac{v^{2k}}{v^{2r}} - A\right] \\ &= A\|x^{(r)}\|_2^2 + \sum_{\substack{v\in\mathbb{Z}\\ v\neq 0}} |c_v(x)|^2 [v^{2k} - Av^{2r}] \\ &\leq A\|x^{(r)}\|_2^2 + \max_{v\in\mathbb{N}}[v^{2k} - Av^{2r}] \sum_{\substack{v\in\mathbb{Z}\\ v\neq 0}} |c_v(x)|^2 \\ &= A\|x^{(r)}\|_2^2 + \max_{v\in\mathbb{N}}[v^{2k} - Av^{2r}]\|x\|_2^2. \end{aligned} \tag{2.6}$$

Set

$$\varphi(A,v) := v^{2k} - Av^{2r}; \tag{2.7}$$

then the last inequality can be written in the form

$$\|x^{(k)}\|_2^2 \leq A\|x^{(r)}\|_2^2 + \max_{v\in\mathbb{N}} \varphi(A,v)\|x\|_2^2. \tag{2.8}$$

Our goal now is to find for a given $A \geq 0$ the value of

$$\max_{v\in\mathbb{N}} \varphi(A,v). \tag{2.9}$$

We consider the difference

$$\begin{aligned} \delta_v &= \varphi(A,v) - \varphi(A,v-1) \\ &= v^{2k} - Av^{2r} - (v-1)^{2k} + A(v-1)^{2r} \\ &= A[(v-1)^{2r} - v^{2r}] - [(v-1)^{2k} - v^{2k}] \\ &= [(v)^{2r} - (v-1)^{2r}]\left[\frac{v^{2k} - (v-1)^{2k}}{v^{2r} - (v-1)^{2r}} - A\right]. \end{aligned} \tag{2.10}$$

Set, for $v \in \mathbb{N}$,

$$\eta(v) := \frac{v^{2k} - (v-1)^{2k}}{v^{2r} - (v-1)^{2r}}; \tag{2.11}$$

then the last equality can be written in the form

$$\delta_v = [(v)^{2r} - (v-1)^{2r}][\eta(v) - A]. \tag{2.12}$$

It is not difficult to see that

$$\operatorname{sgn}\delta_v = \operatorname{sgn}[\eta(v) - A]. \tag{2.13}$$

We now study the function $\eta(v)$.

Note that $\eta(1) = 1$, $\eta(v) \to 0$ as $v \to \infty$ (since $k < r$), and, for $v \ge 1$,

$$\eta(v) > \eta(v+1). \tag{2.14}$$

Indeed, using Cauchy's theorem,

$$\eta(v) = \frac{k}{r}\frac{\theta_v^{2k}}{\theta_v^{2r}}, \quad v - 1 < \theta_v < v. \tag{2.15}$$

Thus, inequality (2.14) is equivalent to the inequality

$$\frac{k}{r}\frac{\theta_v^{2k}}{\theta_v^{2r}} > \frac{k}{r}\frac{\theta_{v+1}^{2k}}{\theta_{v+1}^{2r}} \tag{2.16}$$

or

$$\left(\frac{\theta_v}{\theta_{v+1}}\right)^{2r-2k} < 1. \tag{2.17}$$

The last inequality is true since $\theta_v < \theta_{v+1}$ and $2r - 2k > 0$.

If, for a given $A \ge 0$, the value v_0 is such that $\eta(v_0 + 1) \le A \le \eta(v_0)$, then for $v \le v_0$, taking into account equality (2.13), we obtain that $\delta_v \ge 0$, and consequently,

$$\varphi(A,1) \le \varphi(A,2) \le \cdots \le \varphi(A,v_0). \tag{2.18}$$

In the case $v > v_0$, we get $\delta_v \le 0$ and then

$$\varphi(A,v_0) \ge \varphi(A,v_0+1) \ge \cdots. \tag{2.19}$$

Therefore,

$$\max_{v\in\mathbb{N}} \varphi(A,v) = \max_{v\in\mathbb{N}} [v^{2k} - Av^{2r}] = \varphi(A,v_0) \tag{2.20}$$

if $\eta(v_0+1) \le A \le \eta(v_0)$. Thus inequality (2.1) is proved.

We now show the evidence of equality (2.3). Let $x(t) = \cos v_0 t$. Then the inequality becomes an equality since

$$\|x^{(k)}\|_2^2 = \pi v_0^{2k}, \qquad \|x\|_2^2 = \pi, \qquad \|x^{(r)}\|_2^2 = \pi v_0^{2r}. \tag{2.21}$$

□

The function $\Psi(\mathbb{T}; r, k; A)$ defined by (2.3) is continuous, linear on any interval $[\eta(v+1), \eta(v)]$, and for any $v \ge 1$,

$$\Psi(\mathbb{T}; r, k; \eta(v+1)) = \frac{v^{2k}(v+1)^{2r} - v^{2r}(v+1)^{2k}}{(v+1)^{2r} - v^{2r}}. \tag{2.22}$$

Claim that

$$v_0^{2k} - A v_0^{2r} < \frac{r-k}{r}\left(\frac{k}{Ar}\right)^{k/(r-k)}. \tag{2.23}$$

To do this, we will consider the function

$$f(A) = \frac{r-k}{r}\left(\frac{k}{Ar}\right)^{k/(r-k)} - v^{2k} + A v^{2r}. \tag{2.24}$$

Differentiating the function f, we get

$$f'(A) = v^{2r} - \left(\frac{k}{r}\right)^{r/(r-k)}\left(\frac{1}{A}\right)^{r/(r-k)} \tag{2.25}$$

and the condition $f'(A) = 0$ implies

$$A_0 = \frac{k}{r} v^{2k-2r}. \tag{2.26}$$

Now we have $f(A_0) = 0$ and our statement is proved.

Let Π_{2n+1} be the set of trigonometric polynomials of order less than or equal to n. Then in view of the Bernstein-type inequality, we have, for any $\tau \in \Pi_{2n+1}$ and any $k \in \mathbb{N}$,

$$\|\tau^{(k)}\|_2^2 \le n^{2k}\|\tau\|_2^2. \tag{2.27}$$

Therefore, for $x = \tau$, inequality (1.6) holds with $A = 0$ and $B = n^{2k}$. Let now $A > 0$. By repeating (with obvious modifications) the proof of Theorem 2.1, we obtain that for any $k, r \in \mathbb{N}$, $k < r$, and any $\tau \in \Pi_{2n+1}$, the following holds:

$$\|\tau^{(k)}\|_2^2 \le A\|\tau^{(r)}\|_2^2 + B\|\tau\|_2^2 = A\|\tau^{(r)}\|_2^2 + \max_{\substack{v\in\mathbb{N}\\ v\le n}} \varphi(A,v)\|\tau\|_2^2. \tag{2.28}$$

We now compute the value

$$\max_{\substack{v\in\mathbb{N}\\ v\le n}} \varphi(A,v). \tag{2.29}$$

Let $\eta(v_0+1) \le A \le \eta(v_0)$, where $v_0 \le n$. Then

$$\max_{\substack{v\in\mathbb{N}\\ v\le n}} \varphi(A,v) = \varphi(A,v_0) = \max_{v\in\mathbb{N}} \varphi(A,v). \tag{2.30}$$

If $\eta(v_0+1) \le A \le \eta(v_0)$, where $v_0 \ge n+1$, we get, taking into account the relations

$$\varphi(A,1) \le \varphi(A,2) \le \cdots \le \varphi(A,n) \le \cdots \le \varphi(A,v_0), \tag{2.31}$$

that

$$\max_{\substack{v\in\mathbb{N}\\ v\le n}} \varphi(A,v) = \varphi(A,n) = n^{2k} - A n^{2r} \tag{2.32}$$

if $A \le \eta(n)$. Therefore, we have proved the following theorem.

Theorem 2.2. *For any $k,n,r \in \mathbb{N}$, $k < r$, any $\tau \in \Pi_{2n+1}$, and any $A \geq 0$,*

$$\|\tau^{(k)}\|_2^2 \leq A\|\tau^{(r)}\|_2^2 + B\|\tau\|_2^2, \tag{2.33}$$

where

$$B = \varphi(A, v_0) \tag{2.34}$$

if $\eta(v_0+1) \leq A \leq \eta(v_0)$, $v_0 \leq n$, and

$$B = \varphi(A, n) \tag{2.35}$$

if $A \leq \eta(n)$. Inequality (2.33) is the best possible for any $A \geq 0$.

Consider the set of functions $x \in L_{2,2}^r(\mathbb{T})$ such that $c_v(x) = 0$ for $|v| \leq n-1$ (we will denote this set of functions by $L_{2,2}^r(\mathbb{T};n)$). The following inequality is well known for functions $x \in L_{2,2}^r(\mathbb{T};n)$:

$$\|x\|_2^2 \leq \frac{1}{n^{2r}}\|x^{(r)}\|_2^2. \tag{2.36}$$

Thus, for any $k < r$,

$$\|x^{(k)}\|_2^2 \leq \frac{1}{n^{2r-2k}}\|x^{(r)}\|_2^2. \tag{2.37}$$

Then inequality (1.6) for functions $x \in L_{2,2}^r(\mathbb{T};n)$ holds with $B = 0$ and $A \geq 1/n^{2r-2k}$.

By repeating (with obvious modifications) the proof of Theorem 2.1, we obtain that for any $k,r \in \mathbb{N}$, $k < r$, any $x \in L_{2,2}^r(\mathbb{T};n)$, and any $0 \leq A \leq 1/n^{2r-2k}$,

$$\|x^{(k)}\|_2^2 \leq A\|x^{(r)}\|_2^2 + \max_{\substack{v \in \mathbb{N} \\ v \geq n}} \varphi(A,v)\|x\|_2^2. \tag{2.38}$$

We need to find the value of

$$\max_{\substack{v \in \mathbb{N} \\ v \geq n}} \varphi(A,v). \tag{2.39}$$

Note that

$$\eta(n) = \frac{n^{2k} - (n-1)^{2k}}{n^{2r} - (n-1)^{2r}} \leq \frac{n^{2k}}{n^{2r}}. \tag{2.40}$$

To show this, assume that

$$\eta(n) > \frac{n^{2k}}{n^{2r}}, \tag{2.41}$$

then we get

$$\left(\frac{n}{n-1}\right)^{2r} < \left(\frac{n}{n-1}\right)^{2k} \tag{2.42}$$

which is impossible since $n/(n-1) > 1$ and $k < r$.

First let $\eta(v_0+1) \le A \le \eta(v_0)$ where $v_0 \le n$. Then

$$\varphi(A,n) \ge \varphi(A,n+1) \ge \cdots \tag{2.43}$$

and therefore

$$\max_{\substack{v\in\mathbb{N}\\ v\ge n}} \varphi(A,v) = \varphi(A,n) \tag{2.44}$$

if $\eta(v_0+1) \le A \le n^{2k-2r}$.

Let now $\eta(v_0+1) \le A \le \eta(v_0)$ where $v_0 \ge n+1$. In this case, we get

$$\max_{\substack{v\in\mathbb{N}\\ v\ge n}} \varphi(A,v) = \max_{v\in\mathbb{N}} \varphi(A,v) = \varphi(A,v_0). \tag{2.45}$$

Thus we have proved the following theorem.

Theorem 2.3. *For any $k,n,r \in \mathbb{N}$, $k < r$, any $x \in L^r_{2,2}(\mathbb{T};n)$, and any $0 \le A \le n^{2k-2r}$, inequality (1.6) holds where $B = \varphi(A,n)$ if $\eta(v_0+1) \le A \le n^{2k-2r}$, $v_0 \le n$, and $B = \varphi(A,v_0)$ if $\eta(v_0+1) \le A \le \eta(v_0)$, $v_0 \ge n+1$. Inequality (1.6) is the best possible for any $0 \le A \le n^{2k-2r}$.*

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