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THE CLASS OF FUNCTIONS SPIRALLIKE WITH RESPECT TO A BOUNDARY POINT

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The aim of this paper is to present an analytic characterization of the class of functions δ-spirallike with respect to a boundary point. The method of proof is based on Julia's lemma.

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1. Introduction. In this paper, we study the class $\mathcal{S}_0^*(\delta)$ of δ-spirallike functions with respect to a boundary point. Spirallikeness with respect to a boundary point is a fresh idea being the subject of studies in [1, 2]. Cited papers developed the method based, on the one hand, on the analytic formula for the class $\mathcal{S}_0^*$ of functions starlike with respect to a boundary point proposed and proved partially by Robertson [10], and, on the other hand, on some dynamical system built for $\mathcal{S}_0^*(\delta)$. Lyzzaik [8] completing Robertson's proof solved positively his conjecture. Thereby the full analytic description of functions in $\mathcal{S}_0^*$ was finished. The author [5], by using the Julia lemma, proposed an alternative analytic formula for the class $\mathcal{S}_0^*$ different than Robertson's characterization. The necessary condition for functions to be in $\mathcal{S}_0^*$ was shown and, partially, the sufficient condition. In [7], Lyzzaik and the author complete the proof and in this way the class $\mathcal{S}_0^*$ was equipped with a new analytic characterization.

The use of the Julia lemma has the virtue of looking at the inner property of the class $\mathcal{S}_0^*$ and the other classes defined by the geometric property connected with the boundary point (see, e.g., [6]). In this paper, we apply once again the Julia lemma as a technique to study the class $\mathcal{S}_0^*(\delta)$. Theorem 3.5 demonstrates the basic observation that spirallikeness, as earlier starlikeness with respect to a boundary point, is preserved on each oricycle in the unit disk by every function in $\mathcal{S}_0^*(\delta)$. Theorems 3.6 and 3.8 complete a new analytic characterization of δ-spirallike functions with respect to a boundary point.

2. Preliminaries. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathbb{T} = \partial\mathbb{D}$. For each $k > 0$, consider the oricycle

$$\mathbb{O}_k = \left\{ z \in \mathbb{D} : \frac{|1-z|^2}{1-|z|^2} < k \right\}. \tag{2.1}$$

The oricycle $\mathbb{O}_k$ is a disk in $\mathbb{D}$ whose boundary circle $\partial\mathbb{O}_k$ is tangent to $\mathbb{T}$ at 1.

Let $\Delta = \{z \in \mathbb{D} : |\arg(1-z)| < b,\ |z-1| < \rho\}$, $b \in (0, \pi/2)$, $\rho < 2\cos b$, be a Stolz angle at 1.

Let $\mathcal{A}$ denote the set of all analytic functions in $\mathbb{D}$. The subset of $\mathcal{A}$ of all univalent functions will be denoted by $\mathcal{S}$. The set of all $\omega \in \mathcal{A}$ such that $|\omega(z)| < 1$ for $z \in \mathbb{D}$ will be denoted by $\mathcal{B}$.

An angular limit of $f \in \mathcal{A}$ at $\zeta \in \mathbb{T}$ will be denoted by $f_\angle(\zeta)$. An angular derivative of $f \in \mathcal{A}$ at $\zeta \in \mathbb{T}$ will be denoted by $f'_\angle(\zeta)$.

Let $f \in \mathcal{A}$. Assume that there exists a finite radial limit $\lim_{r\to 1^-} f(r) = \upsilon$ at 1. Denote by

$$Q(z) = \frac{(z-1)f'(z)}{f(z)-\upsilon}, \quad z \in \mathbb{D}, \tag{2.2}$$

the Visser-Ostrowski quotient of f at 1 (see, e.g., [9, page 251]). We say that f satisfies the Visser-Ostrowski condition at 1 if $Q_\angle(1) = 1$ (see, e.g., [9, page 252]).

We recall now the Julia lemma (see [4]; see also [11, pages 68–72]).

Lemma 2.1 (Julia). *Let $\omega \in \mathcal{B}$. Assume that there exists a sequence (z_n) of points in $\mathbb{D}$ such that*

$$\lim_{n\to\infty} z_n = 1, \qquad \lim_{n\to\infty} \omega(z_n) = 1, \tag{2.3}$$

$$\lim_{n\to\infty} \frac{1-|\omega(z_n)|}{1-|z_n|} = \lambda < \infty. \tag{2.4}$$

Then

$$\frac{|1-\omega(z)|^2}{1-|\omega(z)|^2} \le \lambda \frac{|1-z|^2}{1-|z|^2}, \quad z \in \mathbb{D}, \tag{2.5}$$

that is, for every $k > 0$,

$$\omega(\mathbb{O}_k) \subset \mathbb{O}_{\lambda k}. \tag{2.6}$$

Remark 2.2. The constant λ defined in (2.4) is positive (see [11, pages 68–69]).

For $\omega \in \mathcal{B}$ with $\omega_\angle(1) = 1$, let

$$\Lambda = \sup\left\{ \frac{|1-\omega(z)|^2}{1-|\omega(z)|^2} \cdot \frac{1-|z|^2}{|1-z|^2} : z \in \mathbb{D} \right\}. \tag{2.7}$$

The next lemma is a converse of the Julia lemma (see [11, page 72] and [3, pages 42–44]).

Lemma 2.3. *Let $\omega \in \mathcal{B}$. If (2.5) holds for some $\lambda > 0$, then there exists a sequence (z_n) of points in $\mathbb{D}$ satisfying (2.3) and such that*

$$\lim_{n\to\infty} \frac{1-|\omega(z_n)|}{1-|z_n|} = \Lambda \le \lambda. \tag{2.8}$$

Remark 2.4. In fact, in Lemma 2.3, we can find a sequence of real numbers (x_n) in $(0,1)$ satisfying (2.3) and (2.8). Also it can be proved that then $\omega_\angle(1) = 1$ and

$$\lim_{\Delta \ni z \to 1} \frac{1-\omega(z)}{1-z} = \lim_{\Delta \ni z \to 1} \omega'(z) = \omega'_\angle(1) = \Lambda \tag{2.9}$$

for every Stolz angle Δ (see [3, pages 42–44]).

For our need, it will be convenient to define the following classes of functions introduced in [5].

Definition 2.5. Fix $\lambda \in (0,\infty]$. $\omega \in \mathcal{B}$ is said to belong to the class $\mathcal{B}(\lambda)$ if $\omega_\angle(1) = 1$ and $\omega'_\angle(1) = \lambda$.

Let $\mathcal{P}(\lambda)$ denote the class of all functions p of the form

$$p(z) = 4\frac{1-\omega(z)}{1+\omega(z)}, \quad z \in \mathbb{D}, \tag{2.10}$$

where $\omega \in \mathcal{B}(\lambda)$.

Remark 2.6. Note that $p \in \mathcal{P}(\lambda)$ if and only if $p_\angle(1) = 0$ and $p'_\angle(1) = -2\lambda$.

3. Spirallikeness with respect to a boundary point.

3.1. Let for $w \in \mathbb{C}$ and $A \subset \mathbb{C}$, $wA = \{wu : u \in A\}$.

We start with the following definition.

Definition 3.1. Fix $\delta \in (-\pi/2, \pi/2)$ and let $L(\delta) = \{\exp(e^{-i\delta}t) : t \le 0\}$ be the logarithmic spiral joint 0 and 1. Clearly, $L(0)$ is a line segment $(0,1]$. Let $\mathcal{Z}_0^*(\delta)$ denote the class of simply connected domains $\Omega \subset \mathbb{C}$ with $0 \in \partial\Omega$ and such that $wL(\delta) \subset \Omega$ for every $w \in \Omega$. Let $\mathcal{S}_0^*(\delta) \subset \mathcal{S}$ denote the corresponding class of functions mapping $\mathbb{D}$ onto domains in $\mathcal{Z}_0^*(\delta)$.

Domains in $\mathcal{Z}_0^*(\delta)$ and functions in $\mathcal{S}_0^*(\delta)$ will be called δ-spirallike with respect to the boundary point at the origin.

For $\delta = 0$, we get the class $\mathcal{S}_0^*$, that is, $\mathcal{S}_0^* = \mathcal{S}_0^*(0)$. Recall that f belongs to $\mathcal{S}_0^*$ if and only if it is univalent in $\mathbb{D}$ and $f(\mathbb{D})$ is a starlike domain with respect to the boundary point at the origin, that is, the line segment $(0,w]$ is a subset of $f(\mathbb{D})$ for every $w \in f(\mathbb{D})$ (for more about the class $\mathcal{S}_0^*$, see [5, 7, 8, 10]).

Let $f \in \mathcal{S}_0^*(\delta)$ for $\delta \in (-\pi/2, \pi/2)$, and fix $w_1 \in f(\mathbb{D})$. Then $w_1L(\delta) \subset \Omega$ is a curve ending at the origin, so by [9, Proposition 2.1, page 29], the preimage of $w_1L(\delta)$ is a curve in $\mathbb{D}$ ending at some point ζ_0 of $\mathbb{T}$. Applying [9, Corollary 2.17, page 35], we conclude that f has the angular limit zero at ζ_0.

Proposition 3.2. *Every function $f \in \mathcal{S}_0^*(\delta)$, $\delta \in (-\pi/2, \pi/2)$, has the angular limit zero at some point $\zeta_0 \in \mathbb{T}$, that is, $f_\angle(\zeta_0) = 0$.*

In the following considerations, we assume that $\zeta_0 = 1$, that is, we use the boundary normalization $f_\angle(1) = 0$.

3.2. In the proofs of the main theorems of this paper, we will need two lemmas proved in [5].

Lemma 3.3. *Every sequence* (a_n) *of positive numbers with*

$$\lim_{n\to\infty}(a_1a_2\cdots a_n)=0 \tag{3.1}$$

has a convergent subsequence (a_{n_k}) *and*

$$0\le\lim_{k\to\infty}a_{n_k}=a\le 1. \tag{3.2}$$

Lemma 3.4. *Let* $f\in\mathcal{S}$ *have a radial limit* $\lim_{r\to 1^-}f(r)=\upsilon$. *Then there exist* $\lambda\in[0,1]$ *and a sequence* (r_n) *with* $0<r_n<1$ *and* $\lim_{n\to\infty}r_n=1$ *such that*

$$\lim_{n\to\infty}|Q(r_n)|=2\lambda. \tag{3.3}$$

3.3. The theorem below says that every function in $\mathcal{S}_0^*(\delta)$ having a boundary normalization $f_\angle(1)=0$ preserves spirallikeness with respect to a boundary point on each oricycle in $\mathbb{D}$. This information will be used later to find an analytic formula for functions in $\mathcal{S}_0^*(\delta)$.

Theorem 3.5. *Fix* $\delta\in(-\pi/2,\pi/2)$ *and let* $f\in\mathcal{S}$. *Then* $f\in\mathcal{S}_0^*(\delta)$ *and* $f_\angle(1)=0$ *if and only if* $f(\mathbb{O}_k)\in\mathcal{L}_0^*(\delta)$ *for every* $k>0$.

Proof. Assume that $f\in\mathcal{S}_0^*(\delta)$ and $f_\angle(1)=0$. For each $t\le 0$, define

$$\omega_t(z)=f^{-1}\left(\exp\left(e^{-i\delta}t\right)f(z)\right),\quad z\in\mathbb{D}. \tag{3.4}$$

Since $f(\mathbb{D})\in\mathcal{L}_0^*(\delta)$, $\exp(e^{-i\delta}t)f(z)\in f(\mathbb{D})$ for every $t\le 0$, $z\in\mathbb{D}$, and the univalence of f shows that ω_t is well defined for each $t\le 0$.

Now, fix $t<0$ and $w_1\in f(\mathbb{D})$. Hence $w_1L(\delta)\subset f(\mathbb{D})$. For $n\in\mathbb{N}$, let

$$w_n=\exp\left(e^{-i\delta}(n-1)t\right)w_1 \tag{3.5}$$

and $z_n=f^{-1}(w_n)$. Since the sequence (w_n) is placed on the logarithmic spiral $w_1L(\delta)$ and $\lim_{n\to\infty}w_n=0$, $\lim_{n\to\infty}z_n=1$ by Proposition 3.2. Observe that

$$\omega_t(z_n)=f^{-1}\left(\exp\left(e^{-i\delta}t\right)w_n\right)=f^{-1}\left(\exp\left(e^{-i\delta}nt\right)w_1\right)=z_{n+1}. \tag{3.6}$$

Let now

$$a_n=\frac{1-|\omega_t(z_n)|}{1-|z_n|},\quad n\in\mathbb{N}. \tag{3.7}$$

Hence

$$a_n=\frac{1-|z_{n+1}|}{1-|z_n|} \tag{3.8}$$

for all $n \in \mathbb{N}$. Consequently,

$$\begin{aligned}\lim_{n\to\infty}(a_1a_2\cdots a_n) &= \lim_{n\to\infty}\left(\frac{1-|z_2|}{1-|z_1|}\frac{1-|z_3|}{1-|z_2|}\cdots\frac{1-|z_n|}{1-|z_{n-1}|}\frac{1-|z_{n+1}|}{1-|z_n|}\right)\\ &= \lim_{n\to\infty}\frac{1-|z_{n+1}|}{1-|z_1|} = 0.\end{aligned} \tag{3.9}$$

Lemma 3.3 yields a convergent subsequence (a_{n_k}) such that

$$0 \le \lim_{k\to\infty} a_{n_k} = \lambda(t) \le 1, \tag{3.10}$$

which means that

$$\lim_{k\to\infty}\frac{1-|\omega_t(z_{n_k})|}{1-|z_{n_k}|} = \lambda(t) \le 1 \tag{3.11}$$

for each $t < 0$. In view of Remark 2.2, $\lambda(t) > 0$ for every $t < 0$.

Hence, each ω_t satisfies the assumptions of the Julia lemma, and since $\lambda(t) \le 1$ for every $t < 0$, we derive that $\omega_t(\mathbb{O}_k) \subset \mathbb{O}_{\lambda(t)k} \subset \mathbb{O}_k$ for every $k > 0$. This yields $\exp(e^{-i\delta}t)f(\mathbb{O}_k) \subset f(\mathbb{O}_k)$ for every $t < 0$, and hence $f(\mathbb{O}_k) \in \mathscr{L}_0^*(\delta)$.

Conversely, assume that $f(\mathbb{O}_k) \in \mathscr{L}_0^*(\delta)$ for every $k > 0$. Since $0 \in \bigcap_{k>0}\partial f(\mathbb{O}_k)$ and

$$f(\mathbb{D}) = \bigcup_{k>0} f(\mathbb{O}_k), \tag{3.12}$$

it follows that $0 \in \partial f(\mathbb{D})$ and $f(\mathbb{D}) \in \mathscr{L}_0^*(\Delta)$, so $f \in \mathscr{S}_0^*(\delta)$. We show that $f_\angle(1) = 0$. Fix $k > 0$ and $w_1 \in f(\mathbb{O}_k)$. Then $w_1L(\delta) \subset f(\mathbb{O}_k)$ is a curve ending at $0 \in \partial f(\mathbb{D})$. By [9, Proposition 2.14, page 29], $f^{-1}(w_1L(\delta))$ is a curve in $\mathbb{D}$ ending at some point ζ_0 of $\mathbb{T}$. Since $f^{-1}(w_1L(\delta)) \subset \mathbb{O}_k$ and $\overline{\mathbb{O}_k} \cap \mathbb{T} = \{1\}$, we have $\zeta_0 = 1$. The proof of the theorem is finished. □

Using Theorem 3.5, we characterize functions in $\mathscr{S}_0^*(\delta)$ as follows.

Theorem 3.6. *Fix $\delta \in (-\pi/2, \pi/2)$. If $f \in \mathscr{S}_0^*(\delta)$ and $f_\angle(1) = 0$, then there exist $\lambda \in (0,1]$ and $\omega \in \mathscr{B}(\lambda)$ such that*

$$-e^{i\delta}(1-z)^2\frac{f'(z)}{f(z)} = 4\frac{1-\omega(z)}{1+\omega(z)}, \quad z \in \mathbb{D}. \tag{3.13}$$

Proof. The case $\delta = 0$ reduces to [5, Theorem 3.1]. Therefore we assume that $\delta \in (-\pi/2, \pi/2) \setminus \{0\}$. Let $O_k = \partial\mathbb{O}_k \setminus \{1\}$ and $\Gamma_k = \partial f(\mathbb{O}_k)$ for every $k > 0$. First, we show that

$$\operatorname{Re}\left\{e^{i\delta}(1-z)^2\frac{f'(z)}{f(z)}\right\} < 0, \quad z \in \mathbb{D}. \tag{3.14}$$

We prove that the last inequality is true for all points on O_k for every $k > 0$. Now, fix $k > 0$, $z \in O_k$ and let $w = f(z)$. We parametrize O_k as follows:

$$O_k : z = \gamma_k(\theta) = \frac{1+ke^{i\theta}}{1+k}, \quad \theta \in (0, 2\pi). \tag{3.15}$$

Thus O_k is positively oriented. Denote by $\tau(z)$ the tangent vector to Γ_k at $w = f(z)$, that is, $\tau(z) = \gamma_k'(\theta)f'(\gamma_k(\theta))$, where $z = \gamma_k(\theta)$. Since

$$\begin{aligned}(1-\gamma_k(\theta))^2 &= \frac{k^2}{(1+k)^2}(1-e^{i\theta})^2 = -\frac{4k\sin^2(\theta/2)}{(k+1)i}\left(\frac{k}{k+1}e^{i\theta}i\right)\\ &= \frac{4k\sin^2(\theta/2)}{k+1}\gamma_k'(\theta)i = 2\operatorname{Re}\{1-\gamma_k(\theta)\}\gamma_k'(\theta)i, \quad \theta\in(0,2\pi),\end{aligned} \tag{3.16}$$

we have

$$\tau(z) = \frac{-i(1-\gamma_k(\theta))^2 f'(\gamma_k(\theta))}{2\operatorname{Re}\{1-\gamma_k(\theta)\}} = \frac{-i(1-z)^2 f'(z)}{2\operatorname{Re}\{1-z\}}. \tag{3.17}$$

Let

$$w(t) = f(z)\exp(e^{-i\delta}t), \quad t \le 0, \tag{3.18}$$

be a parametrization of $f(z)L(\delta)$ and let $w'(0) = \lim_{t\to 0^-} w'(t) = e^{-i\delta}f(z)$ be the one-sided tangent vector to the logarithmic spiral $f(z)L(\delta)$ at $f(z)$. By $\varphi(z)$ we denote the directed angle from the vector $iw'(0)$ to $\tau(z)$, that is,

$$\begin{aligned}\varphi(z) &= \arg\{\tau(z)\} - \arg\{iw'(0)\}\\ &= \arg\left\{\frac{-i(1-z)^2f'(z)}{2\operatorname{Re}\{1-z\}}\right\} - \arg\{ie^{-i\delta}f(z)\}\\ &= \arg\left\{-e^{i\delta}\frac{(1-z)^2f'(z)}{f(z)}\right\}.\end{aligned} \tag{3.19}$$

By Theorem 3.5, $f(\mathbb{O}_k) \in \mathcal{X}_0^*(\delta)$ for every $k > 0$. Hence it is easy to see that

$$wL(\delta) \subset \overline{f(\mathbb{O}_k)}, \tag{3.20}$$

where $w = f(z) \in \Gamma_k$. Indeed, let $w_0 \in wL(\delta)$ be arbitrary. Thus $w_0 = wu_0$ for some $u_0 \in L(\delta)$. Since $w \in \Gamma_k$, there exists a sequence (w_n) of points in $f(\mathbb{O}_k)$ convergent to w. The inclusion $w_nL(\delta) \subset f(\mathbb{O}_k)$ yields that w_nu_0 is a point of $f(\mathbb{O}_k)$ for every $n \in \mathbb{N}$. At the end, the convergence of the sequence (w_nu_0) of points of $f(\mathbb{O}_k)$ to w_0 implies that $w_0 \in \overline{f(\mathbb{O}_k)}$. Since w_0 was arbitrary, our claim is proved.

Let l be a line going through $f(z)$ with $w'(0)$ as the directional vector. Then l divides the plane into two closed half-planes H_1 and H_2. One of them, say H_1, contains the origin and the spiral $f(z)L(\delta)$. We assume first that $\delta \in (-\pi/2, 0)$. This means that the spiral $L(\delta)$ has the shape such that it attains 1 from the lower half-plane. Moreover, $f(z)L(\delta)$ parametrized as above turns round the origin in the counterclockwise direction. Hence, $iw'(0)$ lies in H_1. By Theorem 3.5, $f(\mathbb{O}_k) \in \mathcal{X}_0^*(\delta)$. Hence, and from (3.20), it follows that either Γ_k is tangent both to $f(z)L(\delta)$ (one-sided) and to l at $f(z)$, and then $\tau(z)$ lies in l so in H_1, or by [9, Proposition 2.13, page 28], there is a crosscut $C \subset l$ of $f(\mathbb{O}_k)$ with one endpoint at $f(z)$. Thus, by [9, Proposition 2.12, page 27], $f(\mathbb{O}_k)$ has exactly two components, one of them, say G, lies in H_2. Clearly, $\partial G = C \cup \Gamma$, where $\Gamma \subset \Gamma_k$ ends at $f(z)$. Hence Γ is a subset of H_2 and, since it is part of a positively oriented closed analytic curve Γ_k, we deduce finally that the tangent vector $\tau(z)$ to Γ_k at $f(z)$ lies in H_1. In a similar way, we can prove that both vectors $iw'(0)$ and $\tau(z)$ lie together in H_2 as $\delta \in (0,\pi/2)$. This, (3.19), and the fact that $iw'(0)$ is orthogonal to l yield

$$|\varphi(z)| \leq \frac{\pi}{2}. \tag{3.21}$$

As $k > 0$ and $z \in O_k$ was arbitrary, this is true in $\mathbb{D}$.

Suppose now that equality holds in (3.21) for some $z_0 \in \mathbb{D}$. By the maximum principle for harmonic functions, it holds in the whole disk $\mathbb{D}$, which implies that there exists $y \in \mathbb{R} \setminus \{0\}$ so that

$$e^{i\delta}(1-z)^2 \frac{f'(z)}{f(z)} \equiv yi, \quad z \in \mathbb{D}. \tag{3.22}$$

But the solution

$$f(z) = f_0(z) = f(0) \exp\left(\frac{e^{-i\delta} yiz}{1-z}\right), \quad z \in \mathbb{D}, \tag{3.23}$$

of the last equation is not univalent in $\mathbb{D}$. So $f_0 \notin \mathcal{S}_0^*(\delta)$, and hence strict inequality holds in (3.21).

Let $p(z) = -e^{i\delta}(1-z)^2 f'(z)/f(z)$ and let

$$\omega(z) = \frac{4-p(z)}{4+p(z)}, \quad z \in \mathbb{D}. \tag{3.24}$$

Then $\omega(\mathbb{D}) \subset \mathbb{D}$. We now prove that $\omega \in \mathcal{B}(\lambda)$ for some $\lambda \in (0,1]$. Recalling the Visser-Ostrowski quotient, we can write

$$p(z) = e^{i\delta}(1-z)Q(z), \quad z \in \mathbb{D}. \tag{3.25}$$

Since, for every $r \in (0,1)$,

$$|Q(r)| \leq \frac{4}{1+r} \tag{3.26}$$

(see [5, Lemma 2.2, (2.3)]), we have

$$\lim_{r\to 1^-} \{(1-r)Q(r)\} = \lim_{r\to 1^-} \{e^{-i\delta} p(r)\} = 0. \tag{3.27}$$

Hence $\lim_{r\to 1^-} p(r) = 0$ and, in view of (3.24), $\lim_{r\to 1^-} \omega(r) = 1$, so condition (2.3) of the Julia lemma is satisfied. By Lemma 3.4, there exist $\lambda_0 \in [0,1]$ and a sequence (r_n) in $(0,1)$ with $\lim_{n\to\infty} r_n = 1$ such that

$$\lim_{n\to\infty} |Q(r_n)| = 2\lambda_0. \tag{3.28}$$

From (3.24) and (3.27) we have

$$\begin{aligned}\lim_{n\to\infty}\frac{|1-\omega(r_n)|}{1-r_n}&=\lim_{n\to\infty}\left\{\frac{2}{|4+p(r_n)|}\cdot\frac{|p(r_n)|}{1-r_n}\right\}\\&=\lim_{n\to\infty}\left\{\frac{2}{|4+p(r_n)|}|Q(r_n)|\right\}\\&=\lambda_0\in[0,1].\end{aligned}\tag{3.29}$$

But

$$\frac{1-|\omega(r_n)|}{1-r_n}\le\frac{|1-\omega(r_n)|}{1-r_n},\tag{3.30}$$

so we can find a subsequence (r_{n_k}) of (r_n) such that

$$\lim_{k\to\infty}\frac{1-|\omega(r_{n_k})|}{1-r_{n_k}}=\lambda_1\le\lambda_0.\tag{3.31}$$

By Remark 2.2, $\lambda_1\in(0,1]$. Hence ω satisfies the assumptions of the Julia lemma with $\lambda=\lambda_1$. Since then (2.5) holds, by using Lemma 2.3 and Remark 2.4, we see that $\omega\in\mathcal{B}(\Lambda)$, where $\Lambda\le\lambda_1\le 1$ is given by (2.7). This ends the proof of the theorem. □

Corollary 3.7. *If $f\in\mathcal{S}_0^*(\delta)$, $\delta\in(-\pi/2,\pi/2)$, and $f_\angle(1)=0$, then there exists $\lambda\in(0,1]$ such that*

$$\lim_{\Delta\ni z\to 1}Q(z)=2\lambda e^{-i\delta}\tag{3.32}$$

for every Stolz angle Δ.

Proof. Since

$$\begin{aligned}Q(z)&=e^{-i\delta}\frac{4}{1+\omega(z)}\frac{1-\omega(z)}{1-z},\quad z\in\mathbb{D},\\\lim_{\Delta\ni z\to 1}&\frac{1-\omega(z)}{1-z}=\Lambda\in(0,1]\end{aligned}\tag{3.33}$$

for every Stolz angle Δ, the assertion follows at once with $\lambda=\Lambda$. □

Theorem 3.8. *Fix $\delta\in(-\pi/2,\pi/2)$. Let $f\in\mathcal{A}$ with $f_\angle(1)=0$. If there exist $\lambda\in(0,\cos\delta]$ and a function $\omega\in\mathcal{B}(\lambda)$ such that (3.13) holds, then $f\in\mathcal{S}_0^*(\delta)$.*

Proof. First we show that f is univalent in $\mathbb{D}$. It is immediate from (3.13) that f is locally univalent in $\mathbb{D}$. Let $\omega\in\mathcal{B}(\lambda)$, where $\lambda\in(0,\cos\delta]$, and let g be the solution of the differential equation

$$-(1-z)^2\frac{g'(z)}{g(z)}=4\frac{1-\omega(z)}{1+\omega(z)},\quad z\in\mathbb{D},\tag{3.34}$$

with the boundary condition $g_\angle(1)=0$. As was proved in [7, Theorem 3], g belongs to the class $\mathcal{S}_0^*$, so it is univalent, and $g(\mathbb{D})$ being a simply connected domain lies in a wedge of angle $2\lambda\pi$. Hence there exists a single-valued analytic branch of $\log g$ in $\mathbb{D}$, and

$$g^{e^{-i\delta}}(z) = \exp\{e^{-i\delta}\log g(z)\}, \quad z \in \mathbb{D}, \tag{3.35}$$

is well defined. But, in view of (3.13) and (3.34), we have

$$\frac{g'}{g} = e^{i\delta}\frac{f'}{f}, \tag{3.36}$$

so

$$f = g^{e^{-i\delta}}. \tag{3.37}$$

Since $\lambda \in (0, \cos\delta]$, from the above, the univalence of f in $\mathbb{D}$ follows.

Now, we prove that $f(\mathbb{D}) \in \mathscr{L}_0^*(\delta)$. This is clear, looking at the relation (3.37) between classes $\mathscr{S}_0^*$ and $\mathscr{S}_0^*(\delta)$, which yields the geometric relation between starlikeness and spirallikeness of domains in the plane. To be self-contained, we prove it without using geometric properties of functions in $\mathscr{S}_0^*$. We assume that $\delta \neq 0$, since this case reduces to [7, Theorem 3].

Let $O_k = \partial\mathbb{O}_k \setminus \{1\}$ and $\Gamma_k = \partial f(\mathbb{O}_k)$ for every $k > 0$. Suppose, on the contrary, that $f(\mathbb{D}) \notin \mathscr{L}_0^*(\delta)$. By Theorem 3.5, there exists $k > 0$ such that $f(\mathbb{O}_k) \notin \mathscr{L}_0^*(\delta)$. Hence $w_0L(\delta) \not\subset f(\mathbb{O}_k)$ for some $w_0 \in f(\mathbb{O}_k)$. Thus there exists $w_1 \in (w_0L(\delta) \setminus \{w_0\}) \cap \Gamma_k$ such that the subarc of $w_0L(\delta)$ joining w_1 and w_0 without w_1 is contained in $f(\mathbb{O}_k)$. Since $w_1 \in \Gamma_k$, $w_1 = f(z_1)$ for some $z_1 \in O_k$. Let

$$v(t) = w_0 \exp\{e^{-i\delta}t\}, \quad t \le 0, \tag{3.38}$$

be a parametrization of $w_0L(\delta)$. Clearly,

$$w_1 = v(t_1) = w_0 \exp\{e^{-i\delta}t_1\} \tag{3.39}$$

for some $t_1 < 0$. Let

$$w(t) = w_1 \exp\{e^{-i\delta}s\}, \quad s \le 0, \tag{3.40}$$

be a parametrization of $w_1L(\delta)$. From (3.38), (3.39), and (3.40), we have

$$w(t) = w_0 \exp\{e^{-i\delta}t_1\}\exp\{e^{-i\delta}s\} = w_0 \exp\{e^{-i\delta}(t_1+s)\}, \tag{3.41}$$

which means that $w_1L(\delta)$ is a subset of $w_0L(\delta)$. Moreover,

$$v'(t_1) = e^{-i\delta}w_0\exp\{e^{-i\delta}t_1\} = e^{-i\delta}f(z_1) = w'(0). \tag{3.42}$$

Therefore the tangent line l to $w_0L(\delta)$ at w_1 has the directional vector $v'(t_1) = w'(0)$ and is the boundary of two closed half-planes denoted by H_1 and H_2. One of them, say H_1, contains the origin. Let $\delta \in (-\pi/2, 0)$. As we remarked in the proof of Theorem 3.6, the spiral $L(\delta)$ has the shape such that it attains 1 from the lower half-plane. Moreover, $w_1L(\delta)$ parametrized as above turns round the origin in the counterclockwise direction. Hence, $iw'(0)$ lies in H_1. Observe that either Γ_k is tangent as well to $w_1L(\delta)$ (one-sided) as to l at w_1 and then $\tau(z_1)$ lies in l, or, by [9, Proposition 2.13, page 28], there is a crosscut $C \subset l$ of $f(\mathbb{O}_k)$ with one endpoint at w_1. Thus, by [9, Proposition 2.12, page 27], $f(\mathbb{O}_k)$ has exactly two components, one of them, say G, lies in H_2. Moreover, $\partial G = C \cup \Gamma$, where $\Gamma \subset \Gamma_k$ ends at w_1. Hence Γ is a subset of H_2 and, since it is part of a positively oriented closed analytic curve Γ_k, we deduce finally that the tangent vector $\tau(z_1)$ to Γ_k at $f(z_1)$ lies in H_2. Since $iw'(0)$ is orthogonal to l and lies in H_1, we deduce that

$$|\varphi(z_1)| \geq \frac{\pi}{2}, \tag{3.43}$$

where $\varphi(z_1)$ denotes the directed angle defined by (3.19), with z_1 instead of z. This contradicts (3.13). Similarly, we get a contradiction assuming that $\delta \in (0,\pi/2)$. □

Remark 3.9. In [1], the authors found necessary and sufficient conditions for functions to be in $\mathcal{S}_0^*(\delta)$ (Theorem 2.1). The analytic formula (2.1) in [1] generalizes the Robertson inequality for starlike functions with respect to a boundary point. In fact, the authors of [1] proved that each spirallike function with respect to a boundary point is a complex power of a corresponding function which is starlike with respect to a boundary point. Formula (3.13) presents an alternative analytic description of the class $\mathcal{S}_0^*(\delta)$. In case $\delta = 0$ ($\mu = 2\pi$ in [1, equation (2.1)]), these two analytic formulas for $\mathcal{S}_0^*(0)$ characterizing starlike functions with respect to a boundary point are equivalent. Looking at [1, Theorem 2.1(III) and Theorems 3.2 and 3.3], we can expect that formulas (2.1) in [1] and (3.13) of the present paper are equivalent, which, in fact, means that in Theorem 3.6 the assumptions $\lambda \in (0,1]$ should be replaced by $\lambda \in (0,\cos\delta]$. This is an open problem.

3.4. Now, we present some examples of functions. In all of the examples below, $\delta \in (-\pi/2,\pi/2)$ and $p(z) = -e^{i\delta}(1-z)^2 f'(z)/f(z)$. It is convenient to express formula (3.13) in terms of the class $\mathcal{P}(\lambda)$. Therefore, in the examples below, we apply Remark 2.6 which says that $p \in \mathcal{P}(\lambda)$ if and only if $p_\angle(1) = 0$ and $p'_\angle(1) = -2\lambda$. In every case, we use Theorem 3.8 reformulated by using the class $\mathcal{P}(\lambda)$.

Example 3.10. (1) $f(z) = ((1-z)/(1+z))^{\beta e^{-i\delta}}$, $\beta > 0$, $z \in \mathbb{D}$.

Then $p(z) = 2\beta(1-z)/(1+z)$. Hence $\operatorname{Re} p(z) > 0$, $z \in \mathbb{D}$, $p(1) = 0$, and $p'(1) = -\beta$. Consequently, $f \in \mathcal{S}_0^*(\delta)$ for $\beta \in (0,2]$. For every $\beta > 2$, $f \notin \mathcal{S}_0^*(\delta)$.

(2) $f(z) = (1-z)^{\beta e^{-i\delta}}$, $\beta > 0$, $z \in \mathbb{D}$.

Then $p(z) = \beta(1-z)$. Hence $\operatorname{Re} p(z) > 0$, $z \in \mathbb{D}$, $p(1) = 0$, and $p'(1) = -\beta$. Consequently, $f \in \mathcal{S}_0^*(\delta)$ for $\beta \in (0,2]$. For every $\beta > 2$, $f \notin \mathcal{S}_0^*(\delta)$.

(3) $f(z) = (1-z)^{2e^{-i\delta}} e^{e^{-i\delta} z}$, $z \in \mathbb{D}$.

Then $p(z) = -z^2 + 1$. Hence $\operatorname{Re} p(z) > 0$, $z \in \mathbb{D}$, $p(1) = 0$, and $p'(1) = -2$. Consequently, $f \in \mathcal{S}_0^*(\delta)$.

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