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ON THE EXTENDIBILITY OF THE DIOPHANTINE TRIPLE $\{1,5,c\}$

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We study the problem of extendibility of the triples of the form $\{1,5,c\}$. We prove that if $c_k = s_k^2+1$, where (s_k) is a binary recursive sequence, k is a positive integer, and the statement that all solutions of a system of simultaneous Pellian equations $z^2-c_kx^2 = c_k-1$, $5z^2-c_ky^2 = c_k-5$ are given by $(x,y,z) = (0,\pm 2,\pm s_k)$, is valid for $2 \le k \le 31$, then it is valid for all positive integer k.

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1. Introduction. Let n be an integer. A set of positive integers $\{a_1,a_2,\dots,a_m\}$ is said to have the property $D(n)$ if a_ia_j+n is a perfect square for all $1 \le i < j \le m$; such a set is called a Diophantine m-tuple or a P_n set of size m. The problem of construction of such sets was studied by Diophantus (see [4]). A famous conjecture related to this problem is as follows.

Conjecture 1.1. *There does not exist a Diophantine quadruple with the property* $D(-1)$.

For certain triples $\{a,b,c\}$ with $1 \notin \{a,b,c\}$, the validity of this conjecture can be verified by simple use of congruences (see [5]). The case $a = 1$ is more involved and the first important result concerning this conjecture was proved in 1985 by Mohanty and Ramasamy [8]; they proved that the triple $\{1,5,10\}$ cannot be extended. Also, Brown [5] proved the conjecture for the triples $\{n^2+1,\ (n+1)^2+1,\ (2n+1)^2+1\}$, where $n \not\equiv 0 \pmod 4$, for the triples $\{2,\ 2n^2+2n+1,\ 2n^2+6n+5\}$, where $n \equiv 1 \pmod 4$, and proved nonextendibility of triples $\{17,26,68\}$ and $\{1,2,5\}$. In 1998, Kedlaya [7] verified it for the triples $\{1,2,145\}$, $\{1,2,4901\}$, $\{1,5,65\}$, $\{1,5,20737\}$, $\{1,10,17\}$, and $\{1,26,37\}$. Since Dujella [6] has proved the conjecture for all triples of the form $\{1,2,c\}$, the consideration of triples of the form $\{1,5,c\}$ seems to be the natural next step.

In the present paper, we will study the extendibility of all triples of the form $\{1,5,c\}$. In our proof, we will follow the strategy of [6].

2. Preliminaries. Since the triple $\{1,5,c\}$ satisfies the property $D(-1)$, therefore there exist integers s, t such that $c-1 = s^2$ and $5c-1 = t^2$ which imply

$$t^2-5s^2 = 4. \tag{2.1}$$

If this triple can be extended to a Diophantine quadruple, then there are integers d, x, y, z such that

$$d-1 = x^2, \qquad 5d-1 = y^2, \qquad cd-1 = z^2. \tag{2.2}$$

Eliminating d, we get

$$z^2 - cx^2 = c - 1, \qquad z^2 - cy^2 = c - 5; \tag{2.3}$$

it is obvious that if all the solutions of this system are given by $(x,y,z) = (0,\pm 2,\pm\sqrt{c-1})$, then, from (2.2), we get $d = 1$, so the triple $\{1,5,c\}$ cannot be extended.

The Pell equation (2.1) has three classes of solutions and all the solutions are given by

$$\begin{aligned} t'_k + s'_k &= (3+\sqrt{5})(9+4\sqrt{5})^k, \\ t''_k + s''_k &= (-3+\sqrt{5})(9+4\sqrt{5})^k, \\ t'''_k + s'''_k &= 2(9+4\sqrt{5})^k. \end{aligned} \tag{2.4}$$

Hence, if the triple $\{1,5,c\}$ is a Diophantine triple with the property $D(-1)$, then there exists a positive integer k such that the integer c has the following three formulas (see [3]):

$$c = c'_k = \frac{1}{10}\left[(7+3\sqrt{5})^2(161+72\sqrt{5})^k + (7-3\sqrt{5})^2(161-72\sqrt{5})^k + 6\right], \tag{2.5}$$

$$c = c''_k = \frac{1}{10}\left[(7-3\sqrt{5})^2(161+72\sqrt{5})^k + (7+3\sqrt{5})^2(161-72\sqrt{5})^k + 6\right], \tag{2.6}$$

$$c = c'''_k = \frac{1}{5}\left[(161+72\sqrt{5})^k + (161-72\sqrt{5})^k + 3\right]. \tag{2.7}$$

The main result of this paper is in the following theorem, where c_k denotes one of the formulas in (2.5), (2.6), and (2.7).

Theorem 2.1. *Let k be a positive integer and let $c_k = s_k^2 + 1$, where (s_k) is a binary recursive sequence. If the statement that all solutions of a system of simultaneous Pellian equations*

$$z^2 - c_k x^2 = c_k - 1, \qquad 5z^2 - c_k y^2 = c_k - 5 \tag{2.8}$$

are given by $(x,y,z) = (0,\pm 2,\pm s_k)$ is valid for $k \le 31$, then it is valid for all positive integer k.

Remark 2.2. The theorem is true when $k = 0$ [5] and $k = 1$ (see [1, 7, 8]). So we will suppose that $k \ge 2$. For simplicity, we will omit the index k and we will divide the proof of the theorem into many lemmas.

3. A system of Pellian equations.

There are finite sets

$$\begin{aligned} &\left\{z_0^{(i)} + x_0^{(i)}\sqrt{c} : i = 1,2,\ldots,i_0\right\}, \\ &\left\{z_1^{(j)}\sqrt{5} + y_1^{(j)}\sqrt{c} : j = 1,2,\ldots,j_0\right\}, \end{aligned} \tag{3.1}$$

of elements of $Z\lfloor\sqrt{c}\rfloor$ and $Z\lfloor\sqrt{5c}\rfloor$, respectively, such that all solutions of (2.8) are given by

$$z+x\sqrt{c}=\left(z_0^{(i)}+x_0^{(i)}\sqrt{c}\right)(2c-1+2s\sqrt{c})^m,\quad i=1,\dots,m\geq 0, \tag{3.2}$$

$$z\sqrt{5}+y\sqrt{c}=\left(z_1^{(j)}\sqrt{5}+y_1^{(j)}\sqrt{c}\right)(10c_k-1+2t\sqrt{5c})^n,\quad i=1,\dots,n\geq 0, \tag{3.3}$$

respectively (see [6]).

From (3.2), we conclude that $z=v_m^{(i)}$ for some index i and integer m, where

$$v_0^{(i)}=z_0^{(i)},\qquad v_1^{(i)}=(2c-1)z_0^{(i)}+2scx_0^{(i)},\qquad v_{m+2}^{(i)}=(4c-2)v_{m+1}^{(i)}-v_m^{(i)}, \tag{3.4}$$

and from (3.3), we conclude that $z=w_n^{(j)}$ for some index j and integer n, where

$$w_0^{(j)}=z_1^{(j)},\qquad w_1^{(j)}=(10c-1)z_1^{(j)}+2tcy_1^{(j)},\qquad w_{n+2}^{(j)}=(20c-2)w_{n+1}^{(j)}-w_n^{(j)}. \tag{3.5}$$

Thus we reformulated system (2.8) to finitely many Diophantine equations of the form

$$v_m^{(i)}=w_n^{(j)}. \tag{3.6}$$

If we choose representatives $z_0^{(i)}+x_0^{(i)}\sqrt{c}$ and $z_1^{(j)}\sqrt{5}+y_1^{(j)}\sqrt{c}$ such that $|z_0^{(i)}|$ and $|z_1^{(j)}|$ are minimal, then, by [9, Theorem 108a], we have the following estimates:

$$\begin{gathered}0<\left|z_0^{(i)}\right|\leq\sqrt{\frac{1}{2}2c(c-1)}<c,\\ 0<\left|z_0^{(i)}\right|\leq\sqrt{c\cdot(c-5)}<c.\end{gathered} \tag{3.7}$$

4. Application of congruence relations. In the following lemma, we prove that if (2.2) has a nontrivial solution, then the initial terms of sequences $v_m^{(i)}$ and $w_n^{(j)}$ are restricted.

Lemma 4.1. *Let $k\geq 2$ be the least positive integer (if it exists) for which the statement of Theorem 2.1 is not valid. Let $1\leq i\leq i_0$, $1\leq j\leq j_0$, and let $v_m^{(i)}$ and $w_n^{(j)}$ be the sequences defined in (3.4) and (3.5). If the equation $v_m^{(i)}=w_n^{(j)}$ has a solution, then $|z_0^{(i)}|=|z_1^{(j)}|=s$.*

Proof. From (3.4) and (3.5), it follows easily by induction that

$$\begin{aligned}v_{2m}^{(i)}&\equiv z_0^{(i)}\ (\mathrm{mod}\ 2c),\\ w_{2n}^{(j)}&\equiv z_1^{(j)}\ (\mathrm{mod}\ 2c),\\ v_{2m+1}^{(i)}&\equiv -z_0^{(i)}\ (\mathrm{mod}\ 2c),\\ w_{2n+1}^{(j)}&\equiv -z_1^{(j)}\ (\mathrm{mod}\ 2c).\end{aligned} \tag{4.1}$$

Therefore, if the equation $v_m^{(i)} = w_n^{(j)}$ has a solution in integers m and n, then we must have $|z_0^{(i)}| = |z_1^{(j)}|$. Now, let $d_0 = ((z_0^{(i)})^2+1)/c$; then we have

$$d_0-1=\left(x_0^{(i)}\right)^2, \qquad 5d_0-1=\left(y_1^{(j)}\right)^2, \qquad cd_0-1=\left(z_0^{(i)}\right)^2, \\ d_0 \le \frac{c^2-c+1}{c} < c. \tag{4.2}$$

Assume that $d_0 > 1$. It follows from (4.2) that there exists a positive integer $l < k$ such that $d_0 = c_l$. But now the system

$$z^2 - c_l x^2 = c_l - 1, \qquad 5z^2 - c_l y^2 = c_l - 5 \tag{4.3}$$

has a nontrivial solution $(x,y,z) = (s_k, t_k, z_0^{(i)})$, contradicting the minimality of k. So, $d_0 = 1$ and $|z_0^{(i)}| = s$. □

The following lemma can be proved easily by induction (we will omit the superscripts (i) and (j)).

Lemma 4.2. *Let $\{v_m\}$ and $\{w_n\}$ be the sequences which have the initial terms in Lemma 4.1; then*

$$v_m \equiv (-1)^m(z_0 - 2cm^2 z_0 - 2csmx_0) \pmod{8c^2}, \\ w_n \equiv (-1)^n(z_1 - 10cn^2 z_1 - 2ctny_1) \pmod{8c^2}. \tag{4.4}$$

Remark 4.3. Since we may restrict ourselves to positive solutions of system (2.8), we may assume that $z_0 = z_1 = s$. Notice that $x_0 = 0$ and $y_1 = \pm 2$.

Lemma 4.4. *If $v_m = w_n$, then m and n are both even or odd.*

Proof. Suppose m is odd and n is even and let $m = 2r$ and $n = 2l+1$. Lemma 4.2 and the relation $z_0 = z_1 = s$ imply

$$s \equiv cs(2l+1)^2 + 20cr^2 s \pm 4ctr \pmod{4c^2} \tag{4.5}$$

and we have a contradiction to the fact that c does not divide s.

The same proof holds for the case where m is even and n odd. □

Lemma 4.5. *If $v_m = w_n$, then $n \le m \le n\sqrt{5}$.*

Proof. From relations (3.4) and (3.5), $w_1 > v_1$. Let $w_l > v_l$, where $l > 0$; then

$$w_{l+2} < (20c-2)w_{l+1} - v_l = (20c-2)w_{l+1} - \left[(4c-2)v_{l+1} - v_{l+2}\right], \tag{4.6}$$

hence

$$w_{l+2} - v_{l+2} < (20c-2)w_{l+1} - (4c-2)v_{l+1}. \tag{4.7}$$

But $(20c-2)w_{l+1} - (4c-2)v_{l+1} > 0$, which implies $w_{l+2} < v_{l+2}$. So, if the equation $v_m = w_n$ has a solution and $n \ne 0$, then $v_n < v_m = w_n$. But the sequence v_m is increasing, so $m > n$.

Now, from (3.4), we have

$$v_m = \frac{s}{2}\left[(2c-1+2s\sqrt{c})^m + (2c-1-2s\sqrt{c})^m\right] > \frac{1}{2}(2c-1+2s\sqrt{c})^m, \tag{4.8}$$

and from (3.5), we have

$$\begin{aligned} w_n &= \frac{1}{2\sqrt{5}}(s\sqrt{5}\pm 2\sqrt{c})\left[(10c-1+2t\sqrt{5c})^n + (10c-1-2t\sqrt{5c})^n\right] \\ &< \frac{s\sqrt{5}+2\sqrt{c}+1}{2\sqrt{5}}(10c-1+2t\sqrt{5c})^n < \frac{1}{2}(10c-1+2t\sqrt{5c})^{n+1/2}. \end{aligned} \tag{4.9}$$

Since $k \geq 2$, therefore from (2.5), (2.6), and (2.7), we have $c \geq 3026$. Thus $v_m = w_n$ implies

$$\frac{m}{n+1/2} < \frac{\ln(10c-1+2t\sqrt{5c})}{\ln(2c-1+2s\sqrt{c})} < 1.1712. \tag{4.10}$$

If $n = 0$, then $m = 0$, and if $n \geq 1$, then (4.10) implies

$$m < 1.1712n + 0.5856 < n\sqrt{5}. \tag{4.11}$$

□

Lemma 4.6. *If $v_m = w_n$ and $n \neq 0$, then $n > (1/2)\sqrt[4]{c}$.*

Proof. (1) The case where m and n are both even.
We assume that $n < (1/2)\sqrt[4]{c}$. Using Lemma 4.2 and from $v_m = w_n$, we get

$$2c(2m)^2 s + 2cs(2m)x_0 \equiv 10c(2n)^2 s - 2ct(2n)y_1 \pmod{8c^2}. \tag{4.12}$$

But $x_0 = 0$ and $y_1 = \pm 2$, so

$$8cm^2 s \equiv 40cn^2 s \pm 8ctn \pmod{8c^2}, \tag{4.13}$$

which implies

$$s(5n^2 - m^2) \equiv \pm tn \pmod{c}. \tag{4.14}$$

On the other hand, we have, from Lemma 4.5,

$$|s(5n^2 - m^2)| \leq \sqrt{c}4n^2 < 4\sqrt{c}\left(\frac{1}{2}\sqrt[4]{c}\right)^2 = c. \tag{4.15}$$

Also, since $c > \sqrt{5/4}\sqrt[4]{c^3}$, then

$$tn < \sqrt{5c}n < \sqrt{5}\sqrt{c}\frac{1}{2}\sqrt[4]{c} = \sqrt{\frac{5}{4}}\sqrt[4]{c^3} < c. \tag{4.16}$$

So, from (4.14), (4.15), and (4.16), we get

$$s(5n^2 - m^2) = \pm tn. \tag{4.17}$$

Also, from (4.14), we have

$$s^2(5n^2-m^2)^2 \equiv t^2n^2 (\operatorname{mod} c). \tag{4.18}$$

But $s^2 \equiv t^2(\operatorname{mod} c)$, so (4.18) becomes

$$(m^2-5n^2)^2 \equiv n^2(\operatorname{mod} c). \tag{4.19}$$

Now, since

$$\begin{gathered}(5n^2-m^2)^2 \le (4n^2)^2 = 16n^4 \le 16\cdot\left(\frac{1}{2}\sqrt[4]{c}\right)^4 = c,\\ n < \frac{1}{2}\sqrt[4]{c} \Longrightarrow n^2 < \frac{1}{4}\sqrt{c} < c,\end{gathered} \tag{4.20}$$

so, from (4.19), (4.20), we get

$$(5n^2-m^2)^2 = n^2. \tag{4.21}$$

Finally, from (4.17) and (4.21), we get $t^2 = s^2$, which is impossible.

(2) The case where m and n are both odd.

We assume that $n < (1/2)\sqrt[4]{c}$. Using Lemma 4.2 and from $v_m = w_n$, where $x_0 = 0$ and $y_1 = \pm 2$, we get

$$s(5n^2-m^2) \equiv \pm 2tn(\operatorname{mod} c). \tag{4.22}$$

As above,

$$|s(5n^2-m^2)| < c, \tag{4.23}$$

and since

$$2tn < 2\sqrt{5c}n < \sqrt{5c}\sqrt[4]{c} < c, \tag{4.24}$$

therefore (4.22), (4.23), and (4.24) imply

$$s(5n^2-m^2) = \pm 2tn. \tag{4.25}$$

Also, from (4.22), we have

$$s^2(5n^2-m^2)^2 \equiv 4t^2n^2(\operatorname{mod} c), \tag{4.26}$$

which implies $(m^2-5n^2)^2 \equiv 4n^2(\operatorname{mod} c)$. But $(5n^2-m^2)^2 < c$ and $4n^2 < c$, so

$$(5n^2-m^2)^2 = 4n^2. \tag{4.27}$$

Finally, from (4.25) and (4.27), we get $t^2 = s^2$, which is impossible. □

5. Linear forms in logarithms

Lemma 5.1. *If $v_m = w_n$, then*

$$0 < n\log(10c-1+2t\sqrt{5c}) - m\log(10c-1+2t\sqrt{5c}) + \log\frac{s\sqrt{5}\pm 2\sqrt{c}}{\sqrt{5c}} < (4c)^{1-n}. \quad (5.1)$$

Proof. We suppose that

$$p = s(2c-1+2s\sqrt{c})^m, \qquad q = \frac{1}{\sqrt{5}}(s\sqrt{5}\pm 2\sqrt{c})(10c-1+2t\sqrt{5c})^n. \quad (5.2)$$

If $v_m = w_n$, then, from (4.8) and (4.9), we get

$$p + s^2 p^{-1} = q + \frac{c-5}{5} q^{-1}. \quad (5.3)$$

It is clear that $p > 1$ and $q > 1$; also

$$p - q = \frac{c-5}{5} q^{-1} - s^2 p^{-1} < (c-1)q^{-1} - (c-1)p^{-1} = (c-1)(p-q)p^{-1}q^{-1}. \quad (5.4)$$

If $p > q$, then from (5.4), we get $pq < c-1$, which is impossible since $q > 1$ and $p > (4s\sqrt{c})s = 4s^2\sqrt{c} = 4(c-1)\sqrt{c} > c > c-1$. Hence $q > p$, and we may assume that $m \geq 1$. Furthermore

$$0 < \log\left(\frac{p}{q}\right)^{-1} = -\log\left(\frac{p}{q}\right) = -\log\left(1 - \frac{q-p}{q}\right). \quad (5.5)$$

Since $-\log(1-x) < x + x^2$, therefore, from (5.5), we get

$$0 < \log\left(\frac{q}{p}\right) < \frac{q-p}{q} + \left(\frac{q-p}{q}\right)^2. \quad (5.6)$$

But from (5.3), we deduce that $p > q - (c-1)p^{-1} > q - (c-1)$, so

$$p^{-1} < (q-(c-1))^{-1}; \quad (5.7)$$

hence, from (5.3) and (5.7), we get

$$q - p < (c-1)(q-(c-1))^{-1} - \frac{c-5}{5}q^{-1} < \frac{4cq + c^2 + 5}{q(5q - 5c + 5)} < \frac{4cq + c^2 + 5}{q}. \quad (5.8)$$

But $q > (s\sqrt{5} - 2\sqrt{c})(4c)$ implies

$$\frac{c^2}{q} < \frac{c^2}{(s\sqrt{5}-2\sqrt{c})(4c)} < \frac{c^2}{4c} = \frac{c}{4}, \quad (5.9)$$

so (5.8) becomes

$$\frac{q-p}{q} < \left[4c + \frac{c^2}{q} + \frac{5}{q}\right]q^{-1} < \left[4c + \frac{c}{4} + 5\right]q^{-1} = \left(\frac{17}{4}c + 5\right)q^{-1}. \quad (5.10)$$

From (5.6) and (5.10), we get

$$0<\log\frac{q}{p}<\left(\frac{17}{4}c+5\right)q^{-1}+\left(\frac{17}{4}c+5\right)^2q^{-2}. \tag{5.11}$$

Now, we will estimate $((17/4)c+5)q^{-1}$. From (2.5), (2.6), and (2.7), we have $c>20$, so

$$\left(\frac{17}{4}c+5\right)q^{-1}<\left(\frac{17}{4}c+5\right)c^{-1}=\frac{17}{4}+\frac{5}{c}<\frac{17}{4}+\frac{1}{4}=\frac{9}{2}. \tag{5.12}$$

Thus (5.11) becomes

$$\begin{aligned}0&<\log\frac{q}{p}<\left(\frac{17}{4}c+5\right)q^{-1}+\left(\frac{17}{4}c+5\right)^2q^{-2}\\&=\left(\frac{17}{4}c+5\right)q^{-1}\left[1+\left(\frac{17}{4}c+5\right)q^{-1}\right]<\frac{11}{2}\left(\frac{17}{4}c+5\right)q^{-1}\\&=\frac{11}{2}\left(\frac{17}{4}c+5\right)\frac{\sqrt{5}}{s\sqrt{5}\pm2\sqrt{c}}\left(10c-1+2t\sqrt{5c}\right)^{-n}\\&<11\sqrt{5}\left(\frac{17}{8}c+\frac{5}{2}\right)\left(10c-1+2t\sqrt{5c}\right)^{-n}\\&<11(\sqrt{5})\left(3c+\frac{5}{2}\right)\left(4\sqrt{5c-1}\sqrt{5c}\right)^{-n}\\&<11(\sqrt{5})\left(3c+\frac{5}{2}\right)(4c)^{-n}\\&<4c(4c)^{-n}.\end{aligned} \tag{5.13}$$

But

$$\log\frac{q}{p}=n\log\left(10c-1+2t\sqrt{5c}\right)-m\log\left(10c-1+2t\sqrt{5c}\right)+\log\frac{s\sqrt{5}\pm2\sqrt{c}}{\sqrt{5c}}. \tag{5.14}$$

So, (5.13) and (5.14) complete the proof of the lemma. □

Now, to prove the theorem, we apply the following theorem.

Theorem 5.2 [2]. *For a linear form $\Omega\neq0$ in logarithms of l algebraic numbers $\alpha_1,\dots,\alpha_l$ with rational coefficients $b_1,\dots,b_l$,*

$$\log|\Omega|\geq-18(l+1)!\,l^{l+1}(32d)^{l+2}h'(\alpha_1)\cdots h'(\alpha_l)\log(2ld)\log B, \tag{5.15}$$

where $B=\max(|b_1|,\dots,|b_l|)$ and where d is the degree of the number field generated by $\alpha_1,\dots,\alpha_l$.

Here

$$h'(\alpha)=\frac{1}{r}\max\left(h(\alpha),|\log\alpha|,1\right) \tag{5.16}$$

and $h(\alpha)$ denotes the standard logarithmic Weil height of α.

6. Proof of Theorem 2.1. (1) The case where m and n are both even.

We consider the equation $v_{2m} = w_{2n}$ with $n \neq 0$. We apply the above theorem and we have $l = 3$, $d = 4$, $B = 2m$, where

$$\begin{aligned}\alpha_1 &= 10c - 1 + 2t\sqrt{5c},\\ \alpha_2 &= 2c - 1 + 2s\sqrt{c},\\ \alpha_3 &= \frac{s\sqrt{5}+2\sqrt{c}}{\sqrt{5}s}.\end{aligned} \tag{6.1}$$

The equations satisfied by α_1, α_2, α_3 are

$$\begin{gathered}\alpha_1^2 - (20c-2)\alpha_1 + 1 = 0,\\ \alpha_2^2 - (4c-2)\alpha_2 + 1 = 0,\\ (5c-5)\alpha_3^2 - (10c-10)\alpha_3 + c - 5 = 0 \Longleftrightarrow \alpha_3^2 - 2\alpha_3 + \frac{c-5}{5c-5} = 0.\end{gathered} \tag{6.2}$$

Hence

$$\begin{aligned}h'(\alpha_1) &= \frac{1}{2}\log\alpha_1 < \frac{1}{2}\log 20c,\\ h'(\alpha_2) &= \frac{1}{2}\log\alpha_2 < \frac{1}{2}\log 4c,\\ h'(\alpha_3) &= \frac{1}{2}\log\frac{s\sqrt{5}+2\sqrt{c}}{\sqrt{5}s} < \frac{1}{2}\log(1+2c).\end{aligned} \tag{6.3}$$

From Lemma 5.1, where n is even, we have

$$\log\Omega < (4c)^{1-2n} = -(2n-1)\log 4c. \tag{6.4}$$

So, from Theorem 5.2, we get

$$(2n-1)\log 4c \le 18\times 4!\times 3^4(32\times 4)^5\times\frac{1}{2}\log(20c)\times\frac{1}{2}\log(4c)\times\frac{1}{2}\log(2c+1)\log 24\times\log 2m. \tag{6.5}$$

Now, using Lemmas 4.5 and 4.6, we get

$$(2n-1) \le 2.07431\times 10^{14}\times\log 8000n^4\times\log\left(800n^4+1\right)\times\left(\log 2\sqrt{5n}\right), \tag{6.6}$$

which implies that

$$n < 2\times 10^{19}, \tag{6.7}$$

and finally,

$$c < 256(10^{76}). \tag{6.8}$$

To find k in the first class, substitute in (2.5); hence

$$k\log(161+72\sqrt{5}) < \log 256+77\log 10-\log(7-3\sqrt{5}), \tag{6.9}$$

which implies $k \le 31$. Similarly, we find that in the other two classes, $k \le 31$.

(2) The case where m and n are both odd.

In this case, using Lemma 5.1, where n is odd, relation (6.4) becomes

$$\log\Omega < (4c)^{-2n} = -(2n-1)\log 4c. \tag{6.10}$$

Hence (6.6) becomes

$$2n \le 2.07431\times 10^{14}\times\log 8000n^4\times\log(800n^4+1)\times(\log 2\sqrt{5n}), \tag{6.11}$$

which implies that $n < 2\times 10^{19}$, and finally $c < 256(10^{76})$, hence $k \le 31$.

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