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ON THE EXPANSION THEOREM DESCRIBED BY $H(A,B)$ SPACES

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The aim of this paper is to construct a generalized Fourier analysis for certain Hermitian operators. When A, B are entire functions, then $H(A,B)$ will be the associated reproducing kernel Hilbert spaces of $\mathbb{C}_{n\times n}$-valued functions. In that case, we will construct the expansion theorem for $H(A,B)$ in a comprehensive manner. The spectral functions for the reproducing kernel Hilbert spaces will also be constructed.

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1. Introduction. Let H be the Hilbert space over $\mathbb{C}$ with inner product $[\cdot,\cdot]$ defined by $[f,g]=\int_a^b \overline{g}(t)f(t)dt$, $f,g\in H$, where $^-$ denotes the complex conjugate. Let H^2 be the Hilbert space of all ordered pairs $\{f,g\}$, where $f,g\in H$ with the inner product $\langle\cdot,\cdot\rangle$ defined by $\langle\{f,g\},\{h,k\}\rangle=[f,h]+[g,k]$, $\{f,g\},\{h,k\}\in H^2$. T is called a closed linear relation in H^2 if T is a (closed) linear manifold in H^2. Such T is often considered as a graph of (closed) linear (multivalued) operator. To any matrix-valued Nevanlinna function, there is associated in a natural way a reproducing kernel Hilbert space. This Hilbert space provides a model for a simple symmetric not necessarily densely defined operator (see [5]).

We define the $\mathbb{C}_B^n$ to be the range $R(B)$ in $\mathbb{C}_{n\times 1}$ endowed with the inner product $[Bc,Bd]=d^*Bc$, where B is an entire function and $c,d\in\mathbb{C}$. We note that $\mathbb{C}_B^n$ is equal to the space $L(B\ell)$. As usual, $L^2(d\sigma)$ is the Hilbert space of all $n\times 1$ vector functions f defined on $\mathbb{R}$ such that $\|f\|_\Sigma^2=\int_{\mathbb{R}} f(t)^*d\Sigma(t)f(t)<\infty$ (see [5]). The theory of Hilbert space of entire functions is a detailed description of eigenfunction expansions associated with formally selfadjoint differential operators, then one can construct the expansion theorem for the reproducing kernels associated with the Hilbert space. In Section 2, we collect some basic and essential observations (see [5, 6]), while in Section 3, we give the main idea of de Branges space. In Section 4, we define the $H(A,B)$ spaces when A and B are entire functions. Finally, in Section 5, we give a description of the spectral functions for the case we study.

2. Preliminaries. Let H be a Hilbert space, then $L[H]$ is a set of bounded linear operators from H to H. Furthermore, U is unitary if $U^{-1}=U^*\Rightarrow U$ unitary operator, $D(U)=R(U)=H$, V is isometric if $V^{-1}\subset V^*\to V$ operator, S is symmetric if $S\subset S^*$, A is selfadjoint if $A=A^*$, T is closed $\rho(T)=\{\ell\in\mathbb{C}\mid (T-\ell)^{-1}\in L(H)\}$ resolvent set (closed), $\sigma(T)=\mathbb{C}\setminus\rho(T)$ spectrum (open), and $R_T(\ell)=(T-\ell)^{-1}$, $\ell\in\rho(T)$.

When $A = A^*$, put

$$\begin{gathered} H_0 = H \ominus A(0), \\ A_\infty = \{0\} \times A(0) = \{\{f, g\} \in A \mid f = 0\}, \\ A_0 = A \ominus A_\infty. \end{gathered} \tag{2.1}$$

Then

$$A = A_0 \oplus A_\infty, \tag{2.2}$$

A_0 is a selfadjoint operator (hence densely defined) in H_0:

$$\begin{gathered} \begin{array}{|c|}\hline H_0 \\ \hline A(0) \\ \hline \end{array} \xrightarrow{A_0} \begin{array}{|c|}\hline H_0 \\ \hline A(0) \\ \hline \end{array} \\ 0 \xrightarrow{A_\infty} \text{all of } A(0). \end{gathered} \tag{2.3}$$

Hence

$$\begin{aligned} (A-\ell)^{-1} &= (A_0-\ell)^{-1} \oplus 0, \quad \text{where 0 is a zero operator on } A(0) \\ &= (A_0-\ell)^{-1} P_0, \quad P_0 = \text{orth. proj. of } H \text{ onto } H_0, \\ \rho(A) &= \rho(A_0) \supset \mathbb{C}_0. \end{aligned} \tag{2.4}$$

Let (E_t^0) be the orthogonal spectral family for A_0 on H_0. Put

$$E_t = E_t^0 \oplus 0 = E_t^0 P_0, \tag{2.5}$$

E_t is called the orthogonal spectral family for A in H.

For a 2×2 matrix $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, MT $\equiv \{\{\alpha f + \beta g, \gamma f + \delta g\} \mid \{f, g\} \in T\}$. The matrix M is called a multiplier if $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Then

$$T - \lambda = \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}, \qquad \alpha T = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} T, \qquad T^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} T. \tag{2.6}$$

Observe $E_\infty = P_0$,

$$\begin{gathered} R_A(\ell) = \int \frac{dE_t}{t-\ell} = \int \frac{dE_t^0 P_0}{t-\ell} - R_{A_0}(\ell) P_0, \\ R_A(\ell)^* = R_A(\overline{\ell}), \\ R_A(\ell) - R_A(\lambda) = (\ell - \lambda) R_A(\ell) R_A(\lambda), \\ -\ell R_A(\ell) \longrightarrow P_0, \quad \ell \longrightarrow \infty \text{ along imaginary axis.} \end{gathered} \tag{2.7}$$

Now, consider $S \subset S^*$. Put $M_\ell = M_\ell(S^*) = \{\{f, g\} \in S^* \mid g = \ell f\}$. Note that $D(M_\ell) = R(M_\ell) = \gamma(S^* - \ell) = R(S - \ell)^\perp$. See [1, 2, 4, 25, 26, 27, 28, 29].

Now, the aim of this part is to associate to certain Hermitian operators a generalized Fourier analysis. Let A and B be entire functions, $\mathbb{C}_{n\times n}$-valued such that the

$$K_N(\lambda, u) = \frac{A(\lambda) B^*(\omega) + B(\lambda) A^*(\omega)}{\pi i (\lambda - \omega^*)} \tag{2.8}$$

is positive such that, on $\mathbb{R}$,

$$\begin{gathered} A(\lambda)B^*(\lambda)+B(\lambda)A^*(\lambda)=0, \\ A(0)=I, \qquad B(0)=0, \end{gathered} \tag{2.9}$$

and let $H(A,B)$ be the associated reproducing kernel Hilbert space of $\mathbb{C}_{n\times n}$-valued functions. The operator $z \xrightarrow{H} zf(z)$ is a closed symmetric operator in $H(A,B)$ and, for any $\omega \neq \omega^*$,

$$\operatorname{Dim}(H-\omega I)^{\perp}=n. \tag{2.10}$$

Moreover, if F is in $H(A,B)$ and $E(\omega)=(A,B)(\omega)$ is invertible, then

$$\frac{F(z)-E(z)E^{-1}(\omega)F(\omega)}{z-\omega} \tag{2.11}$$

belongs to $H(A,B)$.

Define

$$\begin{gathered} \mathbb{R}(\omega)F=\frac{F(z)-E(z)E^{-1}(\omega)F(\omega)}{z-\omega}, \\ \mathbb{R}(\omega)F=E(z)R_{\omega}E^{-1}F. \end{gathered} \tag{2.12}$$

$\mathbb{R}(\omega)$ satisfies the resolvent equation. Moreover, $\mathbb{R}(\omega)$ is bounded from $H(A,B)$ into $H(A,B)$ and thus there is a relation T such that

$$(T-\omega I)^{-1}=\mathbb{R}(\omega). \tag{2.13}$$

Then,

$$\begin{gathered} (T^*-\omega I)^{-1}=\frac{F(z)-(A+B)(z)(A+B)^{-1}(\omega^*)F(\omega^*)}{z-\omega^*}, \\ i(T^*-\omega)^{-1}-i(T-\omega)^{-1}-i(\omega^*-\omega)(T^*-\omega^*)^{-1}(T-\omega)^{-1}\geq 0. \end{gathered} \tag{2.14}$$

$\operatorname{Graph} H=\operatorname{Graph} T\cap \operatorname{Graph} T^*$ and $\rho(T)\subset\mathbb{C}^-$.

In general, $H(A,B)$ is not resolvent invariant and it is of interest to look for $n\times n$-valued functions $S(z)$ such that

$$F\longrightarrow \frac{F(z)-S(z)S^{-1}(\omega)F(\omega)}{z-\omega}. \tag{2.15}$$

We denote by $\mathbb{R}_S$ the above operator (2.15) (see [6, 7, 8]) which is a bounded operator from $H(A,B)$ into itself. We then call the space $\mathbb{R}_S$-invariant.

Associated to such a pair (A,B) will be the following structure problem.

Definition 2.1. Given a pair (A,B) of $\mathbb{C}_{n\times n}$-valued functions as above, the structure problem associated to (A,B) consists in finding all pairs (A',B') with the same properties and such that

$$(A,B)=(A',B')\theta' \tag{2.16}$$

for some entire $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ inner θ'.

This problem is a particular case of the inverse scattering problem defined in [6]. Note that $H(A',B')$ is contractively included in $H(A,B)$ since

$$\begin{aligned}\frac{A(\lambda)B^*(\omega)+B(\lambda)A^*(\omega)}{i\pi(\lambda-\omega^*)} &= \frac{A'(\lambda)B'(\omega)^*+B'(\lambda)A'^*(\omega)}{-i\pi(\lambda-\omega^*)} \\ &\quad + (A'(\lambda),B'(\lambda))\frac{J-\theta'(\lambda)J\theta'(\omega)^*}{-i\pi(\lambda-\omega^*)}\begin{pmatrix}A'\\ B'\end{pmatrix}.\end{aligned} \tag{2.17}$$

Theorem 2.2. *Let (A,B) and $H(A',B')$ be as in (2.16) and suppose the space $H(A',B')\mathbb{R}_S$ is invariant. Then $H(A,B)$ is $\mathbb{R}_S$-invariant.*

Proof. The structure problem associated to a given space $H(A,B)$ thus consists in finding a family of invariant subspaces of $H(A,B)$. We distinguish two cases.

Case 1. (A,B) is the upper part of a $\left(\begin{smallmatrix}0 & I\\ I & 0\end{smallmatrix}\right)$ inner entire function. (This will happen if and only if $H(A,B)$ is resolvent invariant.)

Case 2. (A,B) is not a part of a $\left(\begin{smallmatrix}0 & I\\ I & 0\end{smallmatrix}\right)$ inner entire function.

In the first case, using Potapov's theorem, we associate to (A,B) a family (A_t,B_t) which is not enough to define a Fourier analysis (see [7, 8, 12, 15, 16, 17, 18, 20, 21, 22, 24, 30]). $\square$

3. De Branges space.

Let H be a Hermitian operator. We want to associate to H a Fourier analysis, which really means that we want a model for H in a space of analytic functions in terms of invariant subspaces (this last notion has to be precise).

The first step is to get a model for the operator H.

Theorem 3.1. *Let H be a Hermitian operator, closed, simple, and with equal and finite deficiency indices (n,n). Suppose that the graph of H is the intersection of the graph of T and T^*, where T is a relation, extending H, with spectrum in the open lower half-plane and such that, for ω in $\mathbb{C}^+$,*

$$i(T^*-\omega^*)^{-1/2}-i(T-\omega)^{-1}-i(\omega-\omega^*)(T^*-\omega^*)^{-1}(T-\omega)^{-1}\geq 0. \tag{3.1}$$

Proof. There exist $n\times n$ entire functions A and B such that $(A-B)$ is invertible in $\mathbb{C}^+$, $(A+B)$ is invertible in $\mathbb{C}^-$ such that $(A-B)^{-1}(A+B)$ is inner, and H is unitarily equivalent to multiplication by λ in $H(A,B)$, the reproducing kernel Hilbert space with reproducing kernel

$$\frac{A(\lambda)B^*(\omega)+B(\lambda)A^*(\omega)}{i\pi(\lambda-\omega^*)}. \tag{3.2}$$

An interesting feature of the proof of the theorem is that

$$(A-B)^{-1}(A+B)(\lambda)=I+2\pi i\lambda J(\lambda)J(0)^*, \tag{3.3}$$

where

$$J(\lambda)=J_\omega(I+(\alpha-\omega)(T-\alpha))^{-1} \tag{3.4}$$

and J_ω is a first part of the operator in (3.1).

The space $H(A,B)$ is not in general resolvent invariant, and it is of interest, in order to define invariant subspaces, to characterize functions S, entire and $n\times n$-valued, such that $H(A,B)$ is closed under the operator $\mathbb{R}_S(u)$,

$$(\mathbb{R}_S(u)f) = \frac{f(\lambda)-S(\lambda)S^{-1}(\omega)f(\omega)}{\lambda-\omega}. \tag{3.5}$$

Such conditions are stated in [6, 8]. □

To define the Fourier analysis associated to H, we first define the following structure problem for A, B.

Definition 3.2. Let A, B be entire $n\times n$-valued functions as above. The structure problem associated to $H(A,B)$ consists in finding all pairs (A',B') such that an $H(A',B')$ space exists and such that $(A,B)=(A',B')\theta'$, for some $\left(\begin{smallmatrix}0 & I\\ I & 0\end{smallmatrix}\right)$ inner entire function θ'.

Note that (3.3) implies that $H(A',B')$ is contractively included in $H(A,B)$.

The space $H(A,B)$ will be $\mathbb{R}_S$-invariant as soon as $H(A',B')$ is $\mathbb{R}_S$-invariant.

The question at hand can thus be described as follow. Given a medium of the operator H when enough $\mathbb{R}_S$-invariant subspaces are known. The answer to this question is different according to if S can be chosen to be I, there or not, see [3, 6, 13, 14, 17, 18, 19].

When S is equal to I, there exist matrix-valued functions C and D such that $M=\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)$ is $\left(\begin{smallmatrix}0 & I\\ I & 0\end{smallmatrix}\right)$ inner and entire.

Theorem 3.3. *Let $M(\lambda)$ be a J-inner entire function. Then*

$$M(\lambda)=\int_0^t \exp i\lambda H(u)\,du, \tag{3.6}$$

where

$$H(u)J\ge 0,\qquad T_T H(u)J=1; \tag{3.7}$$

and let

$$M(t,\lambda)=\int_0^t \exp i\lambda H(u)\,du,\qquad \frac{\partial M}{\partial t}=i\lambda M\cdot H(u). \tag{3.8}$$

Then $(A(t,\lambda),B(t,\lambda))$ is a solution of the structure problem associated to A, B.

Proof. When $J=\left(\begin{smallmatrix}0 & I\\ I & 0\end{smallmatrix}\right)$,

$$\frac{J-M(t,\lambda)JM(t,\omega)}{i\pi(\lambda-\omega^*)2\pi}=\int_0^t M(u,\lambda)\frac{H(u)J}{2\pi}M^*(u,\omega)\,du \tag{3.9}$$

and so the map

$$f\longrightarrow \int_0^\ell M(u,\lambda)\frac{H(u)J}{2\pi}f(u)\,du \tag{3.10}$$

is a partial isometry from $L^2(H,[0,\ell))$ into $H(M)$.

Let

$$F(z)=\int_0^{\ell} M(u,\lambda)\frac{H(u)J}{2\pi}f(u)du. \tag{3.11}$$

R_0F belongs to $H(M)$ and thus there exists an element g in $L^2(H,[0,\ell])$ such that

$$(R_0F)(z)=\int_0^{\ell} M(u,\lambda)\frac{H(u)J}{2\pi}g(u)du \tag{3.12}$$

(see [5, 9, 10, 11, 15, 20]).

The functions f and g are linked by (g is chosen in the orthogonal of the kernel of the partial isometry)

$$\int_c^1 H(u)Jg(u)du=\int_c^1 (H(c)-H(u))H(u)Jf(u)du. \tag{3.13}$$

When S cannot be chosen to I (and thus the space $H(A,B)$ is not resolvent invariant), the situation is more involved. One cannot construct from (A,B) the mass function $H(u)J$ and has to state as a hypothesis the enough solutions to the structure problem for A, B, that is, the existence of a family $H(A(t),B(t))$, $t>0$, such that for every $t>0$, $H(A(t),B(t))$ is contractively included in $H(A,B)$. One has also to state as a hypothesis the existence of a mass function $H(u)$ such that a weakened version of (3.9) holds, namely, for any $a,b>0$, z,ω in $\mathbb{C}$,

$$\begin{aligned}&(A(b,z),B(b,z))J(A(b,z),B(b,z)^*)-(A(a),B(a))JA((a,\omega),B(a,\omega)^*)\\&\quad=-i(z-\omega^*)\int_a^b (A(t,z),B(t,z))H(u)J(A(t,\omega),B(t,\omega))dt.\end{aligned} \tag{3.14}$$

□

Theorem 3.4. *Under such hypothesis, the map*

$$f\longrightarrow\int_a^b (A(t,z),B(t,z))H(u)du \tag{3.15}$$

is a partial isometry from $L^2(HJ,[a,b])$ onto the completion for $H(A(b),B(b))$ of $H(A(a),B(a))$.

When a may be chosen to be zero (i.e., when some limit process is justified), a partial isometry from $L^2(HJ,[a,b])$ onto $H(A,B)=H(A(b),B(b))$ is obtained.

Proof. Let

$$F(z)=\int_a^b (A(t,z),B(t,z))H(t)Jf(t)du \tag{3.16}$$

be in the orthogonal complement of the partial isometry and suppose $F(0)=0$. Then $F(z)/z$ belongs to $H(A,B)$ and thus there exists a g such that

$$\theta(z)=\int_0^b (A(t,z),B(t,z))H(t)Jf(t)du, \tag{3.17}$$

f and g are linked by (3.13) (see [6, 23, 30]). □

4. Theory of $H(A,B)$ spaces. In order to prove the expansion theorem described in Section 3, a more detailed analysis of $H(A,B)$ spaces is needed (see [3, 5, 7, 8]).

We first recall that a function analytic in the upper half-plane $\mathbb{C}^+$ is said of bounded type if it is a quotient of two bounded analytic functions in $\mathbb{C}^+$. It can be then written as

$$f(z) = e^{izh}B(z)\cdot\exp\left(i\int\frac{d\mu(t)}{t^2+1}\frac{tz+1}{z-t}\right), \tag{4.1}$$

where B is Blaschke product $h \in \mathbb{R}$, and $\int(|d\mu|/(t^2+1)) < \infty$.

Definition 4.1. The number h is called the mean type of f. When h is negative, then it is said to be of nonpositive mean type. The number h is negative if and only if, for any $\varepsilon > 0$,

$$\lim_{y\to+\infty} e^{-\varepsilon y} f(iy) = 0. \tag{4.2}$$

The interest in the functions of bounded type with nonpositive mean type is that the Cauchy formula holds for such functions, provided some regularity is satisfied on the real line.

Theorem 4.2. *Let f be analytic in $\mathbb{C}^+$ of bounded type and nonpositive mean type in $\mathbb{C}^+$ and suppose f has a continuous extension to the closure of $\mathbb{C}^+$. Then*

$$\int |f|^2(t)\,dt < \infty. \tag{4.3}$$

Proof. We have

$$2\pi i f(z) = \int_{-\infty}^{\infty}\frac{f(t)}{t-z}dt = 0 \quad \text{for } z \text{ in } \mathbb{C}^+ \tag{4.4}$$

(see [1, 2, 3, 4, 6, 7, 8]). □

5. Spectral function. We defined the Nevanlinna class $\mathbb{N}^n$ as the class of all $n\times n$ matrix functions $N(\ell)$, which are holomorphic in $\mathbb{C}_0$, satisfying $N(\ell)^* = N(\overline{\ell})$, $\ell \in \mathbb{C}_0$, and for which the kernel

$$K_N(\ell,\lambda) = \frac{N(\ell)-N(\lambda)^*}{\ell-\overline{\lambda}}, \quad \ell,\lambda \in \mathbb{C}_0,\ \ell \neq \overline{\lambda}, \tag{5.1}$$

is nonnegative. The reproducing kernel Hilbert space associated with the kernel $K_N(\ell,\lambda)$ (see [7, 8]) is denoted by $L(N)$. For general information concerning reproducing kernel Hilbert spaces, we refer to Aronszajn [3]. It is well known that for each $N(\ell) \in \mathbb{N}^n$, there exist $n\times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ with $\mathbf{A} = \mathbf{A}^*$, $\mathbf{B} = \mathbf{B}^* \geq 0$, and a nondecreasing $n\times n$ matrix function Σ on $\mathbb{R}$ with $\int_{\mathbb{R}}(t^2+1)^{-1}d\Sigma(t) < \infty$ such that

$$N(\ell) = \mathbf{A}+\mathbf{B}\ell+\int_{\mathbb{R}}\left(\frac{1}{t-\ell}-\frac{t}{t^2+\ell}\right)d\Sigma(t), \quad \ell \in \mathbb{C}_0. \tag{5.2}$$

With this so-called Riesz-Hergoltz representation, the kernel $K_N(\ell,\lambda)$ takes the form

$$K_N(\ell,\lambda) = \mathbf{B} + \int_R \frac{d\Sigma(t)}{(t-\ell)(t-\overline{\lambda})}, \quad \ell,\lambda \in \mathbb{C}_0 \tag{5.3}$$

(see [10, 13, 29, 30, 31, 32, 33]). We define the space $\mathbb{C}^n_{\tilde{B}}$ to be the range $R(\mathbf{B})$ of $\mathbf{B}$ in $\mathbb{C}_{n\times 1}$ endowed with the inner product $[\mathbf{B}c,\mathbf{B}d] = d^*\mathbf{B}c$. We note that $\mathbb{C}^n_{\tilde{B}}$ is equal to the space $L(\mathbf{B}\ell)$. As usual, $L^2(d\Sigma)$ is the Hilbert space of all $n\times 1$ vector functions f defined on $\mathbb{R}$ such that

$$\|f\|^2_\Sigma = \int_{\mathbb{R}} f(t)^* d\Sigma(t) f(t) < \infty. \tag{5.4}$$

Recall that by the Stieltjes-Liversion formula, the functions $(t-\ell)^{-1}c$, $c \in \mathbb{C}_{n\times 1}$ and $\ell \in \mathbb{C}_0$, are dense in $L^2(d\Sigma)$. The scalar version of the next result can be found in [8].

Proposition 5.1. *Let $(\ell) \in \mathbb{N}^n$ have the integral representation (5.2). The Hilbert space $L(n)$ is isomorphic to the space of all $n\times 1$ vector functions $F(\ell)$ of the form*

$$F(\ell) = \mathbf{B}c + \int_R \frac{d\Sigma(t) f(t)}{t-\ell}, \quad \ell \in \mathbb{C}_0, \tag{5.5}$$

where $c \in \mathbb{C}_{n\times 1}$, $f \in L^2(d\Sigma)$ are uniquely determined by $F(\ell)$, with norm given by

$$\|F\|^2 = c^*\mathbf{B}c + \|f\|^2_\Sigma. \tag{5.6}$$

Proof. Let $F(\ell)$ be an element of the form (5.5). We first check that $\mathbf{B}c$ and f are uniquely determined by $F(\ell)$. Indeed, if $F(\ell)$ admits two such representations with $f \in L^2(d\Sigma)$, $c \in \mathbb{C}_{n\times 1}$ and $\tilde{f} \in L^2(d\Sigma)$, $\tilde{c} \in \mathbb{C}_{n\times 1}$, we have

$$\mathbf{B}(\tilde{c}-c) = \int_R \frac{d\Sigma(t)(f(t)-\tilde{f}(t))}{t-\ell} = \left[f(t)-\tilde{f}(t),(t-\overline{\ell})^{-1}\right]_\Sigma. \tag{5.7}$$

Letting $\ell \to \infty$, we obtain $\mathbf{B}c = \mathbf{B}\tilde{c}$ and $f = \tilde{f}$. The set of functions of the form (5.5) with norm (5.6) is easily seen to be a Hilbert space, which we denote by $\mathbf{K}$. The representation (5.3) shows that the function $\ell \to K_N(\ell,\lambda)d$ belongs to $\mathbf{K}$ for any $d \in \mathbb{C}_{n\times 1}$ and $\lambda \in \mathbb{C}_0$. Moreover, for $F(\ell)$ of the form (5.5), we have the reproducing property of the kernel $K_N(\ell,\lambda)$,

$$[F, K_N(\cdot,\lambda)d] = d^*F(\lambda), \quad d \in \mathbb{C}_{n\times 1}, \tag{5.8}$$

where $[\cdot,\cdot]$ denotes the inner product associated with the norm (5.6). The uniqueness of the reproducing kernel Hilbert space with reproducing kernel $K_N(\ell,\lambda)$ implies that $\mathbf{K}$ and $L(N)$ are isomorphic.

Note that we may write $L(N) = L(\mathbf{B}\ell) \oplus L(N-\mathbf{B}\ell) = \mathbb{C}^n_{\tilde{B}} \oplus L(N-\mathbf{B}\ell)$. The mapping defined by

$$f \longrightarrow \int_R \frac{d\Sigma(t) f(t)}{t-\ell}, \quad f \in L^2(d\Sigma), \tag{5.9}$$

is an isometry from $L^2(d\Sigma)$ onto $L(N-\mathbf{B}\ell)$ and the mapping defined by (5.5) is an isometry from $\mathbb{C}^n_{\tilde{B}} \oplus L(N-\mathbf{B}\ell)$ onto $L(N)$. The elements of $L(N)$ are $n\times 1$ vector functions, which are defined and holomorphic on $\mathbb{C}_0$. This follows from the fact that $L(N)$ is a reproducing kernel Hilbert space, and also from (5.5). We will identify $L(N)$ with the space $\mathbb{C}^n_{\tilde{B}} \oplus L^2(d\Sigma)$. For any $n\times 1$ vector function $F(\ell)$, holomorphic on $\mathbb{C}_0$, and any $\lambda \in \mathbb{C}_0$, we define the operator R_λ by

$$(R_\lambda F)(\ell) = \frac{F(\ell)-F(\lambda)}{\ell-\lambda}, \quad \ell \in \mathbb{C}_0,\ \ell \neq \lambda, \qquad (R_\lambda F)(\lambda) = F'(\lambda). \tag{5.10}$$

For Hilbert spaces $\mathbf{K}$, R, we denote by $\mathbf{L}(\mathbf{K},R)$ the space of all bounded linear operators from $\mathbf{K}$ to R; we write $\mathbf{L}(\mathbf{K}) = \mathbf{L}(\mathbf{K},\mathbf{K})$. □

Proposition 5.2. *The following items are satisfying:*

(i) *for all $\lambda \in \mathbb{C}_0$, $R_\lambda \in \mathbf{L}(L(N))$. In fact, $\|R_\lambda F\| \le |\operatorname{Im}\lambda|^{-1}\|F\|$, $F \in L(N)$;*

(ii) *for all $\lambda \in \mathbb{C}_0$, $(R_\lambda)^* = R_{\overline{\lambda}}$;*

(iii) *the resolvent identity $R_\lambda - R_\mu = (\lambda-\mu)R_\mu R_\lambda$ holds for all $\lambda,\mu \in \mathbb{C}_0$.*

Proof. Let $F(\ell) \in L(N)$ have representation (5.5). Then for $\lambda \in \mathbb{C}_0$,

$$(R_\lambda F)(\ell) = \int_R \frac{d\Sigma(t) f(t)}{(t-\ell)(t-\lambda)}, \tag{5.11}$$

which on account of Proposition 5.1 implies that $R_\lambda F \in L(N)$, $\lambda \in \mathbb{C}_0$. By (5.11),

$$\begin{aligned} ||R_\lambda F||^2 &= \left\|\frac{f(t)}{t-\lambda}\right\|^2_\Sigma = \int_R \left(\frac{f(t)}{t-\lambda}\right)^* d\Sigma(t)\left(\frac{f(t)}{t-\lambda}\right) \\ &\le |\operatorname{Im}\lambda|^{-2}\int_R f(t)^* d\Sigma(t) f(t) = |\operatorname{Im}\lambda|^{-2}\|f\|^2_\Sigma \le |\operatorname{Im}\lambda|^{-2}\|F\|^2, \end{aligned} \tag{5.12}$$

for $\lambda \in \mathbb{C}_0$, which shows that R_λ is a bounded operator in $L(N)$; this proves (i). Items (ii) and (iii) also follow from (5.11).

Combining (ii) and (iii) of Proposition 5.2, we obtain the identity

$$R_\lambda - R^*_\mu = (\lambda-\overline{\mu})R^*_\mu, \quad \lambda,\mu \in \mathbb{C}_0. \tag{5.13}$$

Conversely, if $R_\lambda \in \mathbf{L}(L(N))$ satisfies (5.13), then the inequality in (i), (ii), and (iii) follow. For a selfadjoint relation (i.e., multivalued operator) A in a Hilbert space $\mathbf{K}$, $A(0) = \{\varphi \in \mathbf{K} \mid \{0,\varphi\} \in A\}$ stands for the multivalued part of A and $A_s = A \cap (\mathbf{K} \ominus A(0))^2$ is the operator part of A, which is a (densely defined) selfadjoint operator in $\mathbf{K} \ominus A(0)$ with $D(A_s) = D(A)$. In the following theorem, we assume that $N(\ell) \in \mathbb{N}^n$ and that it has the integral representation (5.2). □

Theorem 5.3. *The operator R_λ, $\lambda \in \mathbb{C}_0$, is the resolvent operator of the selfadjoint relation A in $L(N)$ given by*

$$A = \{\{F,G\} \in L(N)^2 \mid G(\ell) - \ell F(\ell) = c,\ \forall \ell \in \mathbb{C}_0,\ \textit{for some } c \in \mathbb{C}_{n\times 1}\}. \tag{5.14}$$

The multivalued part of A is equal to $A(0) = L(N) \cap \mathbb{C}_{n\times 1} = R(\mathbf{B})$. Representing $F(\ell) \in L(N)$ by (5.5),

(i)

$$F \in D(A) \Longleftrightarrow F(\ell) = \int_R \frac{d\Sigma(t) f(t)}{t-\ell} \quad \textit{for some } f \in L^2(d\Sigma) \textit{ with } t f(t) \in L^2(d\Sigma), \tag{5.15}$$

(ii)

$$(A_s F)(\ell) = \ell F(\ell) + \int_R d\Sigma(t) f(t) = \int_R \frac{d\Sigma(t) t f(t)}{t-\ell}, \quad F \in D(A). \tag{5.16}$$

Proof. Proposition 5.2 implies that $B = \{\{R_\mu H, (I+\mu R_\mu)H\} \mid H \in L(N)\}$ is a selfadjoint relation in $L(N)$ which is independent of $\mu \in \mathbb{C}_0$. We have $R_\lambda = (B-\lambda)^{-1}$, that is, R_λ is the resolvent of B. It is easy to see that B is contained in the right-hand side of (5.14). To prove the inclusion $A \subset B$, take $\{F, G\} \in L(N)^2$ for which $G(\ell) - \ell F(\ell) = c$ for some $c \in \mathbb{C}_{n\times 1}$ and put $H(\ell) = G(\ell) - \mu F(\ell)$. Then $H(\ell) \in L(N)$, $(R_\mu H)(\ell) = F(\ell)$, and $((I+\mu R_\mu)H)(\ell) = G(\ell)$ and hence $\{F, G\} \in B$. This proves $B = A$ and the first part of the theorem. The item about the multivalued part of A is clear. Let $F(\ell) \in D(A)$ have the representation (5.5). Then $\mathbf{B}c = 0$ since $D(A)$ is orthogonal to $A(0)$. By (5.14), there exists a constant $d \in \mathbb{C}_{n\times 1}$ so that $\ell F(\ell) + d \in L(N)$ and hence for some $e \in \mathbb{C}_{n\times 1}$ and $g \in L^2(d\Sigma)$,

$$\int_R \frac{d\Sigma(t) \ell f(t)}{t-\ell} + d = \mathbf{B}e + \int_R \frac{d\Sigma(t) g(t)}{t-\ell}. \tag{5.17}$$

Applying R_α with $\alpha \in \mathbb{C}_0$ to both sides, we obtain

$$\int_R \frac{d\Sigma(t)(g(t) - t f(t))}{(t-\ell)(t-\alpha)} = 0 \tag{5.18}$$

which implies $g(t) = t f(t)$ and hence $t f(t) \in L^2(d\Sigma)$. Therefore, F has the representation in (i), the integral $\int_R d\Sigma(t) f(t)$ exists, and

$$\ell F(\ell) = \int_R \frac{d\Sigma(t) \ell f(t)}{t-\ell} = -\int_R d\Sigma(t) f(t) + \int_R \frac{d\Sigma(t) t f(t)}{t-\ell}. \tag{5.19}$$

This shows that $\ell F(\ell) + \int_R d\Sigma(t) f(t) \in L(N)$ and is orthogonal to $A(0)$ so that (ii) has been proved. As to the converse of (i), let $F(\ell) \in L(N)$ have the indicated representation for some $f \in L^2(d\Sigma)$ with $t f(t) \in L^2(d\Sigma)$. Then $\int_R d\Sigma(t) f(t)$ exists and

$$\ell F(\ell) + \int_R d\Sigma(t) f(t) = \int_R \frac{d\Sigma(t) t f(t)}{t-\ell}, \tag{5.20}$$

where the right-hand side belongs to $L(N)$. Therefore, by (5.14), $F(\ell) \in D(A)$ (see [6, 9]). Several observations and proofs in this section are due to A. Dijksma, H. de Snoo, and P. Bruinsma. □

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