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# ON FURTHER STRENGTHENED HARDY-HILBERT'S INEQUALITY

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We obtain an inequality for the weight coefficient $\omega(q,n)$ $(q>1,\ 1/p+1/q=1,\ n\in\mathbb{N})$ in the form $\omega(q,n) =: \sum_{m=1}^{\infty}(1/(m+n))(n/m)^{1/q} < \pi/\sin(\pi/p) - 1/(2n^{1/p}+(2/a)n^{-1/q})$ where $0<a<147/45$, as $n\geq 3$; $0<a<(1-C)/(2C-1)$, as $n=1,2$, and $C$ is an Euler constant. We show a generalization and improvement of Hilbert's inequalities. The results of the paper by Yang and Debnath are improved.

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**1. Introduction.** The following inequalities are well known as Hardy-Hilbert's inequalities:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{n=1}^{\infty}a_n^p\right)^{1/p}\left(\sum_{n=1}^{\infty}b_n^q\right)^{1/q}, \tag{1.1}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^p < \left(\frac{\pi}{\sin(\pi/p)}\right)^p\sum_{n=1}^{\infty}a_n^p. \tag{1.2}$$

In recent years, Gao [1, 2], Xu and Guo [4], Hsu and Wang [3], Yang [6], and Yang and Gao [7] gave some distinct improvements of (1.1). Yang and Debnath [5] gave a strengthened version by the following inequality:

$$\omega(q,n) =: \sum_{m=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{1/q} < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p}+n^{-1/q}}. \tag{1.3}$$

In this paper, we show a new generalization and improvement of (1.1) by improving (1.3).

First we introduce some lemmas.

**Lemma 1.1** [2]. *Let $f(x)>0$, $f^{(2r-1)}(x)<0$, $f^{(2r)}(x)\geq 0$, $x\in[1,\infty)$, $r=1,2$, $f^{(r)}(\infty)=0$, $r=0,1,2,3,4$, and $\int_1^{\infty}f(x)dx<\infty$, then*

$$\sum_{m=1}^{\infty}f(m) \leq \int_1^{\infty}f(x)dx + \frac{1}{2}f(1) - \frac{1}{12}f'(1). \tag{1.4}$$

**Lemma 1.2** [5]. *Let $q>1$, $1/p+1/q=1$, $n\in\mathbb{N}$, then*

$$\omega(q,n) < \frac{\pi}{\sin(\pi/p)} - \frac{1}{n^{1/p}}\left[f_n(p)+g_n(p)\right], \tag{1.5}$$

*where $\omega(q,n)$ is defined by (1.3), and*

$$f_n(p) := p + \frac{1}{12p} + \frac{1}{(1+p)n} + \frac{1}{12pn^2} + \frac{1}{3(1+3p)n^3},$$
$$g_n(p) := -\frac{1}{12pn} - \frac{1}{2(1+2p)n^2} - \frac{7}{12} - \frac{1}{2n} + \frac{1}{12n^2} - \frac{7}{12n^3}. \tag{1.6}$$

**Lemma 1.3** [5]**.** *Let $p > 1$, $n \in \mathbb{N}$, then*

$$f_n(p) + g_n(p) > \frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}. \tag{1.7}$$

**Lemma 1.4.** *Let $q > 1$, $1/p + 1/q = 1$, $n \in \mathbb{N}$, then*

$$\omega(q,n) =: \sum_{m=1}^{\infty} \frac{1}{m+n}\left(\frac{n}{m}\right)^{1/q} < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + (2/a)n^{-1/q}}, \tag{1.8}$$

$$\omega(p,n) =: \sum_{m=1}^{\infty} \frac{1}{m+n}\left(\frac{n}{m}\right)^{1/p} < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/q} + (2/a)n^{-1/p}}, \tag{1.9}$$

*where $0 < a < 147/45$ as $n \geq 3$; $0 < a < (1-C)/(2C-1)$ as $n = 1,2$, and $C$ is an Euler constant.*

**Proof.** For $n \geq 3$,

$$\left(\frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}\right)\left(x + \frac{1}{ny}\right) = \frac{x}{2} + \frac{1}{n}\left(\frac{1}{2y} - \frac{x}{12} - \frac{1}{12yn} - \frac{x}{2n^2} - \frac{1}{2yn^3}\right), \tag{1.10}$$

where $x > 0$, $y > 0$, $xy = a$.

We first prove

$$\frac{1}{2y} - \frac{x}{12} - \frac{1}{12yn} - \frac{x}{2n^2} - \frac{1}{2yn^3} = \frac{6n^3 - xyn^3 - n^2 - 6xyn - 6}{12yn^3} > 0. \tag{1.11}$$

Formula (1.11) is equivalent to $\psi(n) = 6n^3 - xyn^3 - n^2 - 6xyn - 6 > 0$.

Since $\psi'(x) = 18x^2 - 3ax^2 - 2x - 6a$, $\psi''(x) = 36x - 6ax - 2$. When $0 < a < 147/45$, $\psi''(x) = 36x - 6ax - 2 > 0$, $\psi'(3) = 156 - 33a > 0$, then $\psi'(x) > 0$ and $\psi(3) = 147 - 45a > 0$, hence $\psi(n) > 0$ for $n \geq 3$. Therefore $(1/2 - 1/12n - 1/2n^3)(x + 1/ny) > x/2$. Namely, $1/2 - 1/12n - 1/2n^3 > 1/(2 + 2(an)^{-1})$, for $n \geq 3$. By (1.5) and (1.7), we have

$$\omega(q,n) < \frac{\pi}{\sin(\pi/p)} - \frac{1}{n^{1/p}}\left(\frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}\right)$$
$$< \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + (2/a)n^{-1/q}}. \tag{1.12}$$

Since $\phi(x) = 1/(2x + 2(xn)^{-1})$ is strictly increasing on $(0,\infty)$ and $0 < a < (1-C)/(2C-1)$, by $\omega(q,n) < \pi/\sin(\pi/p) - (1-C)/n^{1/p}$ (see [7]), we have, when $n = 1$,

$$\begin{aligned}\omega(q,1) &< \frac{\pi}{\sin(\pi/p)} - \frac{1-C}{1} < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2+2(2C-1)/(1-C)} \\ &\le \frac{\pi}{\sin(\pi/p)} - \frac{1}{2+2/a};\end{aligned} \tag{1.13}$$

when $n = 2$,

$$\omega(q,2) < \frac{\pi}{\sin(\pi/p)} - \frac{1-C}{2^{1/p}} < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2^{1/p}}\frac{1}{2+(2/a)(1/2)}. \tag{1.14}$$

By (1.12), (1.13), and (1.14), (1.8) is valid for any $n \in \mathbb{N}$. Interchanging $p, q$ in (1.8), since $\pi/\sin(\pi/p) = \pi/\sin(\pi/q)$, we have (1.9). The lemma is proved. □

## 2. Main results.

Now we introduce main results.

**Theorem 2.1.** *Let $p > 1$, $1/p + 1/q = 1$, $a_n \ge 0$, $b_n \ge 0$, $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$. Then*

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} \\ &\quad< \left(\left(\frac{\pi}{\sin(\pi/p)} - \frac{b}{2+2b}\right)a_1^p + \left(\frac{\pi}{\sin(\pi/p)} - \frac{b}{2^{1/p}(2b+2)}\right)a_2^p\right. \\ &\qquad\left.+ \sum_{n=3}^{\infty}\left(\frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p}+(2/a)n^{-1/q}}\right)a_n^p\right)^{1/p} \\ &\qquad\times\left(\left(\frac{\pi}{\sin(\pi/p)} - \frac{b}{2+2b}\right)b_1^q + \left(\frac{\pi}{\sin(\pi/p)} - \frac{b}{2^{1/q}(2b+2)}\right)b_2^q\right. \\ &\qquad\left.+ \sum_{n=3}^{\infty}\left(\frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/q}+(2/a)n^{-1/p}}\right)b_n^q\right)^{1/q},\end{aligned} \tag{2.1}$$

$$\begin{aligned}&\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{a_n}{m+n}\right)^p \\ &\quad< \left(\frac{\pi}{\sin(\pi/p)}\right)^{p-1}\left(\left(\frac{\pi}{\sin(\pi/p)} - \frac{b}{2+2b}\right)a_1^p + \left(\frac{\pi}{\sin(\pi/p)} - \frac{b}{2^{1/p}(2b+2)}\right)a_2^p\right. \\ &\qquad\left.+ \sum_{n=3}^{\infty}\left(\frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p}+(2/a)n^{-1/q}}\right)a_n^p\right),\end{aligned} \tag{2.2}$$

*where $0 < a < 147/45$, $0 < b < (1-C)/(2C-1)$.*

*In particular, when $a = b = e$, $e$ is a constant,*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \left(\sum_{n=1}^{\infty}\left(\frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p}+(2/e)n^{-1/q}}\right)a_n^p\right)^{1/p} \times \left(\sum_{n=1}^{\infty}\left(\frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/q}+(2/e)n^{-1/p}}\right)b_n^q\right)^{1/q}, \tag{2.3}$$

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{a_n}{m+n}\right)^p < \left(\frac{\pi}{\sin(\pi/p)}\right)^{p-1}\sum_{n=1}^{\infty}\left(\frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p}+(2/e)n^{-1/q}}\right)a_n^p.$$

**Proof.** By Hölder's inequality, we have

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m}{(m+n)^{1/p}}\left(\frac{m}{n}\right)^{1/pq}\frac{b_n}{(m+n)^{1/q}}\left(\frac{n}{m}\right)^{1/pq}\\ &\le \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m^p}{m+n}\left(\frac{m}{n}\right)^{1/q}\right)^{1/p}\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{b_n^q}{m+n}\left(\frac{n}{m}\right)^{1/p}\right)^{1/q}\\ &= \left(\sum_{n=1}^{\infty}\omega(q,n)a_n^p\right)^{1/p}\left(\sum_{n=1}^{\infty}\omega(p,n)b_n^q\right)^{1/q}.\end{aligned} \tag{2.4}$$

By (1.8) and (1.9), (2.1) is valid.
By Hölder's inequality and (1.9), we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{a_m}{m+n} &= \sum_{n=1}^{\infty}\frac{a_n}{(m+n)^{1/p}}\left(\frac{n}{m}\right)^{1/pq}\frac{1}{(m+n)^{1/q}}\left(\frac{m}{n}\right)^{1/pq}\\ &\le \left(\sum_{n=1}^{\infty}\frac{a_n^p}{m+n}\left(\frac{m}{n}\right)^{1/q}\right)^{1/p}\left(\sum_{n=1}^{\infty}\frac{1}{m+n}\left(\frac{m}{n}\right)^{1/p}\right)^{1/q}\\ &< \left(\frac{\pi}{\sin(\pi/p)}\right)^{1/q}\left(\sum_{n=1}^{\infty}\frac{a_n^p}{m+n}\left(\frac{n}{m}\right)^{(2-\lambda)/q}\right)^{1/p}.\end{aligned} \tag{2.5}$$

Then

$$\begin{aligned}\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{a_n}{m+n}\right)^p &< \left(\frac{\pi}{\sin(\pi/p)}\right)^{p/q}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_n^p}{m+n}\left(\frac{n}{m}\right)^{1/q}\\ &= \left(\frac{\pi}{\sin(\pi/p)}\right)^{p-1}\sum_{n=1}^{\infty}\omega(q,n)a_n^p.\end{aligned} \tag{2.6}$$

By (1.8), (2.2) is valid. The theorem is proved. □

**Remark 2.2.** As $a = b = 0$, inequalities (2.1) and (2.2) change to (1.1) and (1.2), respectively, hence inequalities (2.1) and (2.2) are generalization and improvement of (1.1) and (1.2), respectively.

**Remark 2.3.** As $a = b = 2$, inequalities (2.1) and (2.2) change to (1.11) and (3.1) in [5], respectively, hence inequalities (2.1) and (2.2) are generalization and improvement of (1.11) and (3.1) in [5], respectively.

**Remark 2.4.** We give an open question: how to determine the constant $a$ such that $1/(2n^{1/p}+(2/a)n^{-1/q})$ is best possible.

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