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ON LONG EXACT $(\bar{\pi},\text{Ext}_\Lambda)$-SEQUENCES IN MODULE THEORY

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In (2003), we proved the injective homotopy exact sequence of modules by a method that does not refer to any elements of the sets in the argument, so that the duality applies automatically in the projective homotopy theory (of modules) without further derivation. We inherit this fashion in this paper during our process of expanding the homotopy exact sequence. We name the resulting doubly infinite sequence the *long exact* $(\bar{\pi},\text{Ext}_\Lambda)$-*sequence in the second variable*—it links the (injective) homotopy exact sequence with the long exact Ext_Λ-sequence in the second variable through a connecting term which has a structure containing traces of both a $\bar{\pi}$-homotopy group and an Ext_Λ-group. We then demonstrate the nontriviality of the injective/projective relative homotopy groups (of modules) based on the results of Su (2001). Finally, by inserting three $(\bar{\pi},\text{Ext}_\Lambda)$-sequences into a one-of-a-kind diagram, we establish the long exact $(\bar{\pi},\text{Ext}_\Lambda)$-sequence of a triple, which is an extension of the homotopy sequence of a triple in module theory.

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1. Introduction. It is well known that, in topology, for a path-connected space Y and a closed, path-connected subspace Y_0, there exists a homotopy exact sequence

$$\begin{gathered}\cdots \longrightarrow \pi_n(Y_0) \longrightarrow \pi_n(Y) \longrightarrow \pi_n(Y,Y_0) \longrightarrow \pi_{n-1}(Y_0) \longrightarrow \cdots \\ \longrightarrow \pi_1(Y_0) \longrightarrow \pi_1(Y) \longrightarrow \pi_1(Y,Y_0) \longrightarrow 0.\end{gathered} \tag{1.1}$$

Analogously, in module theory, let Λ be a unitary ring, and A, B_1, B_2 right Λ-modules. Suppose that given a Λ-module homomorphism $\beta : B_1 \to B_2$, then, for each A, there exists an (injective) homotopy exact sequence

$$\begin{gathered}\cdots \xrightarrow{\partial} \bar{\pi}_n(A,B_1) \xrightarrow{\beta_*} \bar{\pi}_n(A,B_2) \xrightarrow{J} \bar{\pi}_n(A,\beta) \xrightarrow{\partial} \bar{\pi}_{n-1}(A,B_1) \xrightarrow{\beta_*} \cdots \\ \xrightarrow{\partial} \bar{\pi}_1(A,B_1) \xrightarrow{\beta_*} \bar{\pi}_1(A,B_2) \xrightarrow{J} \bar{\pi}_1(A,\beta) \xrightarrow{\partial} \bar{\pi}(A,B_1) \xrightarrow{\beta_*} \bar{\pi}(A,B_2)\end{gathered} \tag{1.2}$$

(see [1]). As stated in [1, 4], in the relative homotopy theory of modules, for a given Λ-module homomorphism $\beta : B_1 \to B_2$ and a given Λ-module A, one considers the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Sigma^{n-1}A & \xhookrightarrow{\iota_{n-1}} & C\Sigma^{n-1}A & \xrightarrow{\epsilon_n} & \Sigma^n A \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ker\beta & \rightarrowtail & B_1 & \xrightarrow{\beta} & B_2 & \twoheadrightarrow & \operatorname{coker}\beta,
\end{array} \tag{1.3}$$

where ι_0 is the inclusion map which embeds A into an injective container CA, ϵ_1 is the quotient map with ΣA, called the suspension of A, as the quotient, and so on. We then say that the map $(\rho,\sigma) : \iota_{n-1} \to \beta$ is *i-null-homotopic*, denoted $(\rho,\sigma) \simeq_i 0$, if it can be extended to an injective container of ι_{n-1}, and define the nth (*injective*) *relative homotopy group*, $n \geq 1$, as $\bar{\pi}_n(A,\beta) = \operatorname{Hom}(\iota_{n-1},\beta)/\operatorname{Hom}_0(\iota_{n-1},\beta)$, where $\operatorname{Hom}(\iota_{n-1},\beta)$ is the abelian group of maps of ι_{n-1} to β, and $\operatorname{Hom}_0(\iota_{n-1},\beta)$ is the subgroup consisting of i-null-homotopic maps.

In addition, by duality, suppose that given a Λ-module homomorphism $\alpha : A_1 \to A_2$ (here, the modules are left Λ-modules), then, for each B, there exists a (projective) homotopy exact sequence

$$\begin{array}{l}
\cdots \xrightarrow{\partial} \underline{\pi}_n(A_2,B) \xrightarrow{\alpha^*} \underline{\pi}_n(A_1,B) \xrightarrow{J'} \underline{\pi}_n(\alpha,B) \xrightarrow{\partial} \underline{\pi}_{n-1}(A_2,B) \xrightarrow{\alpha^*} \cdots \\
\xrightarrow{\partial} \underline{\pi}_1(A_2,B) \xrightarrow{\alpha^*} \underline{\pi}_1(A_1,B) \xrightarrow{J'} \underline{\pi}_1(\alpha,B) \xrightarrow{\partial} \underline{\pi}(A_2,B) \xrightarrow{\alpha^*} \underline{\pi}(A_1,B)
\end{array} \tag{1.4}$$

(see [4]).

In this paper, by putting together a projective resolution and an injective resolution of the randomly given right Λ-module A and adopting the method introduced in [4], we extend the injective homotopy exact sequence, (1.2), by linking it with the long exact $\operatorname{Ext}_\Lambda$-sequence in the second variable in a somewhat unusual yet expectable way. We name the resulting doubly infinite sequence the *long exact* $(\bar{\pi},\operatorname{Ext}_\Lambda)$*-sequence in the second variable* (see Theorem 2.2).

Since our argument involves no reference to the elements of the sets, by duality, the existence of the *long exact* $(\underline{\pi},\operatorname{Ext}_\Lambda)$*-sequence in the first variable*, (2.23), which is an extension of (1.4) and the dual of (2.2), follows automatically.

2. The long exact $(\bar{\pi},\operatorname{Ext}_\Lambda)$-sequence in the second variable. In [4], we proved the injective homotopy exact sequence, (1.2), by a method which does not refer to any elements of the sets in the argument, so that the existence of the projective homotopy exact sequence, (1.4), is automatic by duality. We inherit this fashion in our process of expanding (1.2) to (2.2).

We first state a lemma which is easily checked.

Lemma 2.1. *In a commutative diagram of short exact sequences*

$$\begin{array}{ccccc}
A' & \xrightarrow{\mu} & A & \xrightarrow{\epsilon} & A'' \\
\downarrow{\scriptstyle\alpha} & \swarrow{\scriptstyle\theta} & \downarrow{\scriptstyle\beta} & \swarrow{\scriptstyle\eta} & \downarrow{\scriptstyle\gamma} \\
B' & \xrightarrow{\mu'} & B & \xrightarrow{\epsilon'} & B'',
\end{array} \tag{2.1}$$

(i) *the map* $\alpha = 0$ *if and only if* β *factors through* ϵ; *that is,* $\beta = \eta\epsilon$ *for some (unique)* $\eta : A'' \to B$;

(ii) *the map* $\gamma = 0$ *if and only if* β *factors through* μ'; *that is,* $\beta = \mu'\theta$ *for some (unique)* $\theta : A \to B'$.

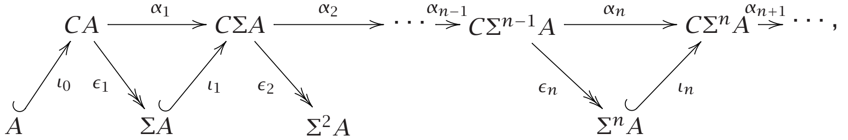
Theorem 2.2. *Suppose that given a map* $\beta : B_1 \to B_2$, *then there exists, for each* A, *a doubly infinite long exact sequence*

$$\begin{aligned}
\cdots &\xrightarrow{\partial} \bar{\pi}_n(A,B_1) \xrightarrow{\beta_*} \bar{\pi}_n(A,B_2) \xrightarrow{J} \bar{\pi}_n(A,\beta) \xrightarrow{\partial} \bar{\pi}_{n-1}(A,B_1) \xrightarrow{\beta_*} \cdots \\
&\xrightarrow{J} \bar{\pi}_1(A,\beta) \xrightarrow{\partial} \bar{\pi}(A,B_1) \xrightarrow{\beta_*} \bar{\pi}(A,B_2) \xrightarrow{J} \frac{\mathrm{Hom}_\Lambda(A,B_{12})}{\iota_0^*\kappa_*\mathrm{Hom}_\Lambda(CA, CB_1 \oplus B_2)} \\
&\xrightarrow{\delta_*} \mathrm{Ext}_\Lambda(A,B_1) \xrightarrow{\beta_*} \mathrm{Ext}_\Lambda(A,B_2) \xrightarrow{\kappa_*} \mathrm{Ext}_\Lambda(A,B_{12}) \xrightarrow{\delta} \mathrm{Ext}^2_\Lambda(A,B_1) \xrightarrow{\beta_*} \cdots \\
&\xrightarrow{\delta} \mathrm{Ext}^n_\Lambda(A,B_1) \xrightarrow{\beta_*} \mathrm{Ext}^n_\Lambda(A,B_2) \xrightarrow{\kappa_*} \mathrm{Ext}^n_\Lambda(A,B_{12}) \xrightarrow{\delta} \mathrm{Ext}^{n+1}_\Lambda(A,B_1) \xrightarrow{\beta_*} \cdots,
\end{aligned} \tag{2.2}$$

where $\iota_0 : A \hookrightarrow CA$ *is the inclusion of* A *into an injective container* CA, $\iota : B_1 \hookrightarrow CB_1$ *is the inclusion of* B_1 *into an injective container* CB_1, κ *is the quotient map in the short exact sequence* $B_1 \overset{\{\iota,\beta\}}{\rightarrowtail} CB_1 \oplus B_2 \overset{\kappa}{\twoheadrightarrow} B_{12}$, *and* $B_{12} = \mathrm{coker}\{\iota,\beta\}$. *This sequence is independent of the choices of* CA, CB_1, ι_0, *and* ι *and is named the long exact* $(\bar{\pi}, \mathrm{Ext}_\Lambda)$*-sequence in the second variable.*

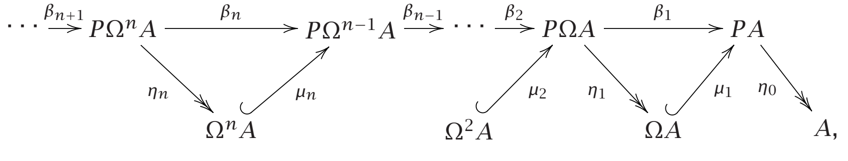
Proof. We first prove the special case when β is monomorphic, then the general case, (2.2), by exploiting the mapping cylinder of β.

Suppose that given a right Λ-module A, one constructs an injective resolution of A,



(2.3)

by first embedding A into an injective container CA, naming the quotient of the inclusion ι_0 the suspension, ΣA, of A, embedding the suspension into an injective container $C\Sigma A$, and so forth. Similarly, we construct a projective resolution of A,



(2.4)

by first choosing a projective ancestor PA of A, naming the kernel of the epimorphism η_0 the loop space, ΩA, of A, choosing a projective ancestor $P\Omega A$ of the loop space, and so forth.

Putting together (2.3) and (2.4) yields a doubly infinite long exact sequence,

$$
\begin{array}{ccccccccccccc}
\underline{C} : \cdots & \xrightarrow{\beta_2} & P\Omega A & \xrightarrow{\beta_1} & & PA & \xrightarrow{\alpha_0} & CA & \xrightarrow{\alpha_1} & & C\Sigma A & \xrightarrow{\alpha_2} & \cdots . \\
\Omega^2 A \hookrightarrow\nearrow & & & \searrow\ \Omega A \hookrightarrow\nearrow & & & \searrow^{\eta_0}\ A \hookrightarrow\nearrow^{\iota_0} & & \searrow\ \Sigma A \hookrightarrow\nearrow & & & \searrow\ \Sigma^2 A &
\end{array}
\tag{2.5}
$$

When applying the functor $\operatorname{Hom}_\Lambda(\underline{C},-)$ to $\beta : B_1 \to B_2$, whereas we assume that β is monomorphic, this leads to a short exact sequence of complexes,

$$
\begin{array}{c}
\operatorname{Hom}_\Lambda(\underline{C},B_1) \\
\big\downarrow \beta_* \\
\operatorname{Hom}_\Lambda(\underline{C},B_2) \\
\big\downarrow \text{quotient map} \\
\operatorname{Hom}_\Lambda(\underline{C},B_2)/\operatorname{image}\beta_*.
\end{array}
\tag{2.6}
$$

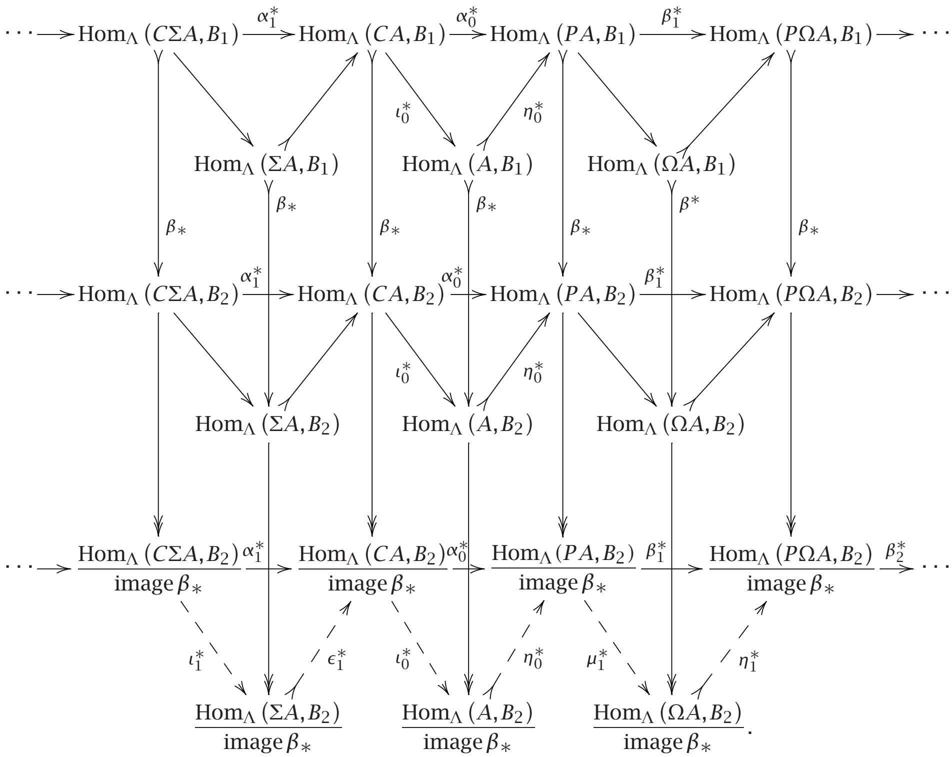
To say that the homology/cohomology sequence induced from (2.6) coincides with the long exact $(\bar{\pi},\operatorname{Ext}_\Lambda)$-sequence in the second variable when β is monomorphic,

$$
\begin{aligned}
&\cdots \xrightarrow{\partial} \bar{\pi}_n(A,B_1) \xrightarrow{\beta_*} \bar{\pi}_n(A,B_2) \xrightarrow{J} \bar{\pi}_n(A,\beta) \xrightarrow{\partial} \bar{\pi}_{n-1}(A,B_1) \xrightarrow{\beta_*} \cdots \\
&\xrightarrow{J} \bar{\pi}_1(A,\beta) \xrightarrow{\partial} \bar{\pi}(A,B_1) \xrightarrow{\beta_*} \bar{\pi}(A,B_2) \xrightarrow{J} \frac{\operatorname{Hom}_\Lambda(A,B_{12})}{\iota_0^*\kappa_*\operatorname{Hom}_\Lambda(CA,B_2)} \\
&\xrightarrow{\delta_*} \operatorname{Ext}_\Lambda(A,B_1) \xrightarrow{\beta_*} \operatorname{Ext}_\Lambda(A,B_2) \xrightarrow{\kappa_*} \operatorname{Ext}_\Lambda(A,B_{12}) \xrightarrow{\delta} \operatorname{Ext}_\Lambda^2(A,B_1) \xrightarrow{\beta_*} \cdots \\
&\xrightarrow{\delta} \operatorname{Ext}_\Lambda^n(A,B_1) \xrightarrow{\beta_*} \operatorname{Ext}_\Lambda^n(A,B_2) \xrightarrow{\kappa_*} \operatorname{Ext}_\Lambda^n(A,B_{12}) \xrightarrow{\delta} \operatorname{Ext}_\Lambda^{n+1}(A,B_1) \xrightarrow{\beta_*} \cdots ,
\end{aligned}
\tag{2.7}
$$

where κ is the quotient map in the short exact sequence $B_1 \overset{\beta}{\rightarrowtail} B_2 \overset{\kappa}{\twoheadrightarrow} B_{12}$ and $B_{12} = \operatorname{coker}\beta$, one should show that, in the third complex of (2.6),

$$
\ker\beta_{n+1}^*/\operatorname{image}\beta_n^* \cong \operatorname{Ext}_\Lambda^n(A,B_{12}), \quad n \geq 1, \tag{2.8}
$$

$$
\ker\beta_1^*/\operatorname{image}\alpha_0^* \cong \operatorname{Hom}_\Lambda(A,B_{12})/\iota_0^*\kappa_*\operatorname{Hom}_\Lambda(CA,B_2), \quad \text{naturally}, \tag{2.9}
$$



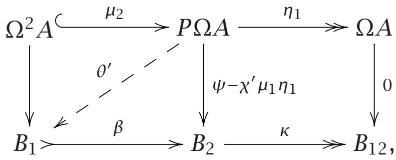
(2.10)

Regarding (2.8), it suffices to show that $\ker\beta_2^*/\operatorname{image}\beta_1^* \cong \mathrm{Ext}_\Lambda(A,B_{12})$: first we pick $\psi : P\Omega A \to B_2$ whose equivalence class $[\psi] \in \ker\beta_2^*$. Since $\ker\beta_2^* \cong \ker\mu_2^*$ and β is monomorphic, there is a unique $\psi| : \Omega^2 A \to B_1$ such that $\psi\mu_2 = \beta\circ\psi|$. Therefore, the map ψ yields a commutative diagram

$$\begin{array}{ccccc} \Omega^2 A & \xrightarrow{\mu_2} & P\Omega A & \xrightarrow{\eta_1} & \Omega A \\ \downarrow{\scriptstyle \psi|} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi'} \\ B_1 & \xrightarrow{\beta} & B_2 & \xrightarrow{\kappa} & B_{12} = \operatorname{coker}\beta, \end{array} \tag{2.11}$$

where ψ' is the induced map of ψ and represents an element in $\mathrm{Ext}_\Lambda(A,B_{12})$. We thus define $\zeta : \ker\beta_2^*/\operatorname{image}\beta_1^* \to \mathrm{Ext}_\Lambda(A,B_{12})$ by $\zeta[\psi] = [\psi']$.

To prove that ζ is monomorphic, suppose that given $[\psi] \in \ker\zeta$, then the induced map ψ' factors through the projective PA by a map ν'. Moreover, since $\kappa : B_2 \twoheadrightarrow B_{12}$ is epimorphic, ν' factors through B_2 by a map χ', so that $\psi' = \nu'\mu_1 = \kappa\chi'\mu_1$. Hence, one has a commutative diagram



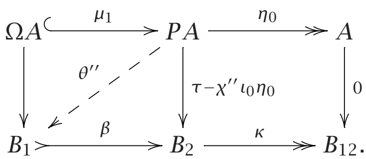
(2.12)

where the induced map of $\psi - \chi'\mu_1\eta_1$ is 0 because $\kappa(\psi - \chi'\mu_1\eta_1) = \kappa\psi - \kappa\chi'\mu_1\eta_1 = \psi'\eta_1 - \psi'\eta_1 = 0$. By Lemma 2.1(ii), $\psi - \chi'\mu_1\eta_1$ factors through β by a map θ', so $\psi = \beta\theta' + \chi'\mu_1\eta_1 = \beta\theta' + \chi'\beta_1 = \beta_*(\theta') + \beta_1^*(\chi')$. Thus, $[\psi] = 0$ and ζ is monomorphic.

The map ζ is epimorphic because, for any $\psi' : \Omega A \to B_{12}$, the map $\psi'\eta_1$ always factors through B_2 since $P\Omega A$ is projective. This completes the proof of (2.8).

The proof of (2.9) proceeds analogously: we define $\omega : \ker \beta_1^* / \operatorname{image} \alpha_0^* \to \operatorname{Hom}_\Lambda(A, B_{12})/\iota_0^*\kappa_* \operatorname{Hom}_\Lambda(CA, B_2)$ by $\omega[\tau] = [\tau']$, where $\tau : PA \to B_2$, and $\tau' : A \to B_{12}$ is the induced map of τ.

To show that ω is monomorphic, suppose that given $[\tau] \in \ker\omega$, then $\tau' = \iota_0^*\kappa_*(\chi'') = \kappa\chi''\iota_0$ for some $\chi'' : CA \to B_2$. Since $\kappa(\tau - \chi''\iota_0\eta_0) = \kappa\tau - \kappa\chi''\iota_0\eta_0 = \tau'\eta_0 - \tau'\eta_0 = 0$, we have a commutative diagram



(2.13)

By Lemma 2.1(ii), $\tau - \chi''\iota_0\eta_0$ factors through β by a map θ'', so $\tau = \beta\theta'' + \chi''\iota_0\eta_0 = \beta\theta'' + \chi''\alpha_0 = \beta_*(\theta'') + \alpha_0^*(\chi'')$. Thus, $[\tau] = 0$ and ω is monomorphic.

The map ω is epimorphic because, for arbitrary $\tau' : A \to B_{12}$, the map $\tau'\eta_0$ always factors through B_2 since PA is projective. This completes the proof of (2.9).

Hence, we assure the existence and the exactness of (2.7), a special case of the long exact $(\bar{\pi}, \operatorname{Ext}_\Lambda)$-sequence in the second variable when β is monomorphic. We then show its uniqueness, namely, the sequence (2.7) is independent of the choices of the injective container CA and the inclusion $\iota_0 : A \hookrightarrow CA$.

Let CA and $C'A$ be two injective containers of A with inclusions $\iota_0 : A \hookrightarrow CA$ and $\iota_0' : A \hookrightarrow C'A$, respectively. We show that $\iota_0^* \operatorname{Hom}_\Lambda(CA, B_2) = \iota_0'^* \operatorname{Hom}_\Lambda(C'A, B_2)$: for one direction, let $\gamma : CA \to B_2$ be a map such that $\gamma\iota_0 = \iota_0^*(\gamma) \in \iota_0^* \operatorname{Hom}_\Lambda(CA, B_2)$. Since $\gamma\iota_0 \simeq_i 0$, it factors through $C'A$ (see [1, Proposition 13.2]). Thus, $\gamma\iota_0 \in \iota_0'^* \operatorname{Hom}_\Lambda(C'A, B_2)$ and this proves $\iota_0^* \operatorname{Hom}_\Lambda(CA, B_2) \subseteq \iota_0'^* \operatorname{Hom}_\Lambda(C'A, B_2)$. The other implication follows by symmetry.

The proof of this special case of Theorem 2.2 when the map $\beta : B_1 \to B_2$ is monomorphic is now complete. One more remark on the special case, before we move on to the general case when β is arbitrary, is that the maps in (2.7) are exactly those one expects.

Now that $\beta : B_1 \to B_2$ is arbitrary, we apply the mapping cylinder (see [1, 4]) of β, thus $\{\iota, \beta\} : B_1 \rightarrowtail CB_1 \oplus B_2$, where CB_1 is an injective container of B_1 and $\iota : B_1 \hookrightarrow CB_1$ the inclusion, so that the resulting monomorphism $\{\iota, \beta\} : B_1 \rightarrowtail CB_1 \oplus B_2$ induces a long exact $(\bar{\pi},\mathrm{Ext}_\Lambda)$-sequence

$$\begin{aligned}
\cdots &\xrightarrow{\partial} \bar{\pi}_n(A,B_1) \xrightarrow{\{\iota,\beta\}_*} \bar{\pi}_n(A,CB_1\oplus B_2) \xrightarrow{J} \bar{\pi}_n(A,\{\iota,\beta\}) \xrightarrow{\partial} \bar{\pi}_{n-1}(A,B_1) \xrightarrow{\{\iota,\beta\}_*} \cdots \\
&\xrightarrow{\partial} \bar{\pi}_1(A,B_1) \xrightarrow{\{\iota,\beta\}_*} \bar{\pi}_1(A,CB_1\oplus B_2) \xrightarrow{J} \bar{\pi}_1(A,\{\iota,\beta\}) \\
&\xrightarrow{\partial} \bar{\pi}(A,B_1) \xrightarrow{\{\iota,\beta\}_*} \bar{\pi}(A,CB_1\oplus B_2) \xrightarrow{J} \frac{\mathrm{Hom}_\Lambda(A,B_{12})}{\iota_0^*\kappa_*\mathrm{Hom}_\Lambda(CA,CB_1\oplus B_2)} \\
&\xrightarrow{\delta_*} \mathrm{Ext}_\Lambda(A,B_1) \xrightarrow{\{\iota,\beta\}_*} \mathrm{Ext}_\Lambda(A,CB_1\oplus B_2) \xrightarrow{\kappa_*} \mathrm{Ext}_\Lambda(A,B_{12}) \xrightarrow{\delta} \mathrm{Ext}^2_\Lambda(A,B_1) \xrightarrow{\{\iota,\beta\}_*} \cdots \\
&\xrightarrow{\delta} \mathrm{Ext}^n_\Lambda(A,B_1) \xrightarrow{\{\iota,\beta\}_*} \mathrm{Ext}^n_\Lambda(A,CB_1\oplus B_2) \xrightarrow{\kappa_*} \mathrm{Ext}^n_\Lambda(A,B_{12}) \xrightarrow{\delta} \mathrm{Ext}^{n+1}_\Lambda(A,B_1) \xrightarrow{\{\iota,\beta\}_*} \cdots,
\end{aligned} \tag{2.14}$$

where κ is the quotient map in the short exact sequence $B_1 \overset{\{\iota,\beta\}}{\rightarrowtail} CB_1\oplus B_2 \overset{\kappa}{\twoheadrightarrow} B_{12}$ and $B_{12} = \mathrm{coker}\{\iota,\beta\}$. The task is to show that the two exact sequences (2.14) and (2.2) are naturally isomorphic. The proof that the first halves of the sequences, namely, the homotopy exact sequences of the maps $\{\iota,\beta\}$ and β, respectively, are isomorphic is stated in [4]. As to the second halves of the sequences, namely, the Ext_Λ-sequences in the second variable, since $\mathrm{Ext}^n_\Lambda(A,CB_1\oplus B_2)\cong\mathrm{Ext}^n_\Lambda(A,B_2)$, $n\geq 1$, it remains to derive that the sequence (2.2) is unique regardless the choices of the injective container CB_1 and the inclusion $\iota: B_1\hookrightarrow CB_1$. That is, let CB_1 and $C'B_1$ be injective containers of B_1, $\iota: B_1\hookrightarrow CB_1$, $\iota': B_1\hookrightarrow C'B_1$ the inclusions, and $B_1 \overset{\{\iota,\beta\}}{\rightarrowtail} CB_1\oplus B_2 \overset{\kappa}{\twoheadrightarrow} B_{12} = \mathrm{coker}\{\iota,\beta\}$, $B_1 \overset{\{\iota',\beta\}}{\rightarrowtail} C'B_1\oplus B_2 \overset{\kappa'}{\twoheadrightarrow} B'_{12} = \mathrm{coker}\{\iota',\beta\}$ the induced short exact sequences. One ought to show that

$$\frac{\mathrm{Hom}_\Lambda(A,B_{12})}{\iota_0^*\kappa_*\mathrm{Hom}_\Lambda(CA,CB_1\oplus B_2)} \cong \frac{\mathrm{Hom}_\Lambda(A,B'_{12})}{\iota_0^*\kappa'_*\mathrm{Hom}_\Lambda(CA,C'B_1\oplus B_2)}, \tag{2.15}$$

$$\mathrm{Ext}^n_\Lambda(A,B_{12}) \cong \mathrm{Ext}^n_\Lambda(A,B'_{12}),\quad n\geq 1. \tag{2.16}$$

Regarding (2.15), since CB_1 and $C'B_1$ are injective, we have the commutative diagrams

$$\begin{array}{ccc} B_1 & \overset{\iota}{\hookrightarrow} & CB_1 \\ {\scriptstyle\iota'}\downarrow & \swarrow{\scriptstyle\exists\nu} & \\ C'B_1, & & \end{array} \qquad \begin{array}{ccc} B_1 & \overset{\iota'}{\hookrightarrow} & C'B_1 \\ {\scriptstyle\iota}\downarrow & \swarrow{\scriptstyle\exists\nu'} & \\ CB_1 & & \end{array} \tag{2.17}$$

which yield the diagram

$$\begin{array}{ccccc} B_1 & \overset{\{\iota,\beta\}}{\rightarrowtail} & CB_1\oplus B_2 & \overset{\kappa}{\twoheadrightarrow} & B_{12}\\ \| & & \downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle\xi'} \\ B_1 & \overset{\{\iota',\beta\}}{\rightarrowtail} & C'B_1\oplus B_2 & \overset{\kappa'}{\twoheadrightarrow} & B'_{12} \\ \| & & \downarrow{\scriptstyle\zeta} & & \downarrow{\scriptstyle\zeta'} \\ B_1 & \overset{\{\iota,\beta\}}{\rightarrowtail} & CB_1\oplus B_2 & \overset{\kappa}{\twoheadrightarrow} & B_{12}, \end{array} \tag{2.18}$$

where $\xi = \{\langle \nu, 0\rangle, \langle 0, 1_{B_2}\rangle\}$, $\zeta = \{\langle \nu', 0\rangle, \langle 0, 1_{B_2}\rangle\}$, and ξ', ζ' are the induced maps of ξ and ζ, respectively.

Next, we define $\chi : \operatorname{Hom}_\Lambda(A, B_{12})/\iota_0^*\kappa_*\operatorname{Hom}_\Lambda(CA, CB_1\oplus B_2) \to \operatorname{Hom}_\Lambda(A, B'_{12})/\iota_0^*\kappa'_*$ $\operatorname{Hom}_\Lambda(CA, C'B_1\oplus B_2)$ by $\chi[\phi] = [\xi'\phi]$, where $\phi : A \to B_{12}$. To prove that χ is an isomorphism, we first show that $[\zeta'\xi'\phi] = [\phi]$ in $\operatorname{Hom}_\Lambda(A, B_{12})/\iota_0^*\kappa_*\operatorname{Hom}_\Lambda(CA, CB_1\oplus B_2)$: diagram (2.18) leads to

$$\begin{array}{ccccc} B_1 & \overset{\{\iota,\beta\}}{\rightarrowtail} & CB_1\oplus B_2 & \overset{\kappa}{\twoheadrightarrow} & B_{12}\\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle\zeta\xi-1} & \swarrow{\scriptstyle\eta} & \downarrow{\scriptstyle\zeta'\xi'-1} \\ B_1 & \overset{\{\iota,\beta\}}{\rightarrowtail} & CB_1\oplus B_2 & \overset{\kappa}{\twoheadrightarrow} & B_{12}, \end{array} \tag{2.19}$$

where $\zeta\xi - 1 = \eta\kappa$ for some $\eta : B_{12} \to CB_1 \oplus B_2$, due to Lemma 2.1(i). Let $\iota_{CB_1} : CB_1 \hookrightarrow CB_1 \oplus B_2$ and $p_{CB_1} : CB_1 \oplus B_2 \twoheadrightarrow CB_1$ be, respectively, the inclusion and the projection of the first factor. Since κ is epimorphic and $\zeta\xi - 1 = \iota_{CB_1} \circ p_{CB_1} \circ (\zeta\xi - 1)$, $\zeta'\xi' - 1 = \kappa\eta = \kappa \circ \iota_{CB_1} \circ p_{CB_1} \circ \eta$. In addition, since CB_1 is injective, the composite $p_{CB_1} \circ \eta \circ \phi$ extends to CA by a map γ:

$$\begin{array}{ccc} A & \overset{\iota_0}{\hookrightarrow} & CA \\ {\scriptstyle p_{CB_1}\circ\eta\circ\phi}\downarrow & \swarrow{\scriptstyle \exists\gamma} & \\ CB_1. & & \end{array} \tag{2.20}$$

Thus, $\zeta'\xi'\phi - \phi = (\zeta'\xi' - 1)\phi = \kappa \circ \iota_{CB_1} \circ p_{CB_1} \circ \eta \circ \phi = \kappa \circ \iota_{CB_1} \circ \gamma \circ \iota_0$, which means that $\zeta'\xi'\phi - \phi \in \iota_0^*\kappa_*\operatorname{Hom}_\Lambda(CA, CB_1 \oplus B_2)$, so $[\zeta'\xi'\phi] = [\phi]$.

Note that, for $\psi : A \to B'_{12}$, $[\xi'\zeta'\psi] = [\psi]$ in $\operatorname{Hom}_\Lambda(A, B'_{12})/\iota_0^*\kappa'_*\operatorname{Hom}_\Lambda(CA, C'B_1 \oplus B_2)$ follows by symmetry.

We now show that χ is an isomorphism: if $[\phi] \in \ker\chi$, where $\phi : A \to B_{12}$, then $\xi'\phi = \kappa'\theta\iota_0$ for some $\theta : CA \to C'B_1 \oplus B_2$ so that $[\phi] = [\zeta'\xi'\phi] = [\zeta'\kappa'\theta\iota_0] = [\kappa\zeta\theta\iota_0] = 0$.

Hence, χ is monomorphic. To assure that χ is epimorphic, suppose that given $\psi : A \to B'_{12}$, we have $\zeta'\psi : A \to B_{12}$ so that $\chi[\zeta'\psi] = [\xi'\zeta'\psi] = [\psi]$. This completes the proof of (2.15).

For (2.16), one compares the long exact Ext_Λ-sequences in the second variable induced from the short exact sequences $B_1 \overset{\{\iota,\beta\}}{\rightarrowtail} CB_1 \oplus B_2 \overset{\kappa}{\twoheadrightarrow} B_{12} = \mathrm{coker}\{\iota,\beta\}$ and $B_1 \overset{\{\iota',\beta\}}{\rightarrowtail} C'B_1 \oplus B_2 \overset{\kappa'}{\twoheadrightarrow} B'_{12} = \mathrm{coker}\{\iota',\beta\}$, respectively. Since CB_1 and $C'B_1$ are injective, $\mathrm{Ext}^n_\Lambda(A, CB_1 \oplus B_2) \cong \mathrm{Ext}^n_\Lambda(A, B_2) \cong \mathrm{Ext}^n_\Lambda(A, C'B_1 \oplus B_2)$, $n \geq 1$, and the isomorphism (2.16) follows from the five lemma. This completes the proof of Theorem 2.2. □

We remark that the term $\mathrm{Hom}_\Lambda(A, B_{12})/\iota_0^*\kappa_*\mathrm{Hom}_\Lambda(CA, CB_1 \oplus B_2)$ in (2.2), which links together the homotopy exact sequence of the map $\beta : B_1 \to B_2$ and the long exact Ext_Λ-sequence in the second variable, has a structure of both an (injective) homotopy group and an Ext_Λ-group. Moreover, there is a close connection, shown in the next commutative diagram (2.21), between the long exact Ext_Λ-sequence in the second variable (induced from the short exact sequence $B_1 \overset{\{\iota,\beta\}}{\rightarrowtail} CB_1 \oplus B_2 \overset{\kappa}{\twoheadrightarrow} B_{12} = \mathrm{coker}\{\iota,\beta\}$) and the long exact $(\bar{\pi},\mathrm{Ext}_\Lambda)$-sequence in the second variable:

$$
\begin{array}{ccccccccc}
 & & \mathrm{Hom}_\Lambda(A,B_1) & \rightarrowtail & \mathrm{Hom}_\Lambda(A,B_2) & \longrightarrow & \mathrm{Hom}_\Lambda(A,B_{12}) & \longrightarrow & \mathrm{Ext}_\Lambda(A,B_1) \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & \nearrow & \\
\cdots \to \bar{\pi}_1(A,\beta) & \longrightarrow & \bar{\pi}(A,B_1) & \longrightarrow & \bar{\pi}(A,B_2) & \longrightarrow & \dfrac{\mathrm{Hom}_\Lambda(A,B_{12})}{\iota_0^*\kappa_*\mathrm{Hom}_\Lambda(CA,CB_1\oplus B_2)}. & &
\end{array}
\tag{2.21}
$$

Note that there is a much analogous connection between the long exact Ext_Λ-sequence in the first variable and the long exact $(\bar{\pi},\mathrm{Ext}_\Lambda)$-sequence induced from a short exact sequence $A' \rightarrowtail A \twoheadrightarrow A''$ (we call it the long exact $(\bar{\pi},\mathrm{Ext}_\Lambda)$-sequence in the first variable),

$$
\begin{array}{ccccccccc}
 & & \mathrm{Hom}_\Lambda(A'',B) & \rightarrowtail & \mathrm{Hom}_\Lambda(A,B) & \to & \mathrm{Hom}_\Lambda(A',B) & \to & \mathrm{Ext}_\Lambda(A'',B) \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & \nearrow & \\
\cdots \to \bar{\pi}_1(A',B) & \longrightarrow & \bar{\pi}(A'',B) & \longrightarrow & \bar{\pi}(A,B) & \longrightarrow & \bar{\pi}(A',B), & &
\end{array}
\tag{2.22}
$$

in the study of (absolute) homotopy theory of modules (see [1]).

One final remark of Theorem 2.2 is that our argument does not involve reference to the elements of the sets, so that one can pursue, by duality, the results in the projective relative homotopy theory without further derivation. As an example, the dual statement of Theorem 2.2 is presented as follows.

Theorem 2.3. *Suppose that, given a map $\alpha : A_1 \to A_2$, there exists, for each B, a doubly infinite long exact sequence*

$$\begin{aligned}\cdots &\xrightarrow{\partial} \underline{\pi}_n(A_2,B) \xrightarrow{\alpha^*} \underline{\pi}_n(A_1,B) \xrightarrow{J'} \underline{\pi}_n(\alpha,B) \xrightarrow{\partial} \underline{\pi}_{n-1}(A_2,B) \xrightarrow{\alpha^*} \cdots \\ &\xrightarrow{J'} \underline{\pi}_1(\alpha,B) \xrightarrow{\partial} \underline{\pi}(A_2,B) \xrightarrow{\alpha^*} \underline{\pi}(A_1,B) \xrightarrow{J'} \frac{\operatorname{Hom}_\Lambda(A_{12},B)}{\eta_{0*}\iota^*\operatorname{Hom}_\Lambda(A_1\oplus PA_2,PB)} \\ &\xrightarrow{\delta^*} \operatorname{Ext}_\Lambda(A_2,B) \xrightarrow{\alpha^*} \operatorname{Ext}_\Lambda(A_1,B) \xrightarrow{\iota^*} \operatorname{Ext}_\Lambda(A_{12},B) \xrightarrow{\delta} \operatorname{Ext}^2_\Lambda(A_2,B) \xrightarrow{\alpha^*} \cdots \\ &\xrightarrow{\delta} \operatorname{Ext}^n_\Lambda(A_2,B) \xrightarrow{\alpha^*} \operatorname{Ext}^n_\Lambda(A_1,B) \xrightarrow{\iota^*} \operatorname{Ext}^n_\Lambda(A_{12},B) \xrightarrow{\delta} \operatorname{Ext}^{n+1}_\Lambda(A_2,B) \xrightarrow{\alpha^*} \cdots,\end{aligned} \tag{2.23}$$

where $\eta_0 : PB \twoheadrightarrow B$ is the projection of a projective ancestor PB onto B, $\eta : PA_2 \twoheadrightarrow A_2$ is the projection of a projective ancestor PA_2 onto A_2, ι is the inclusion in the short exact sequence $A_{12} \overset{\iota}{\hookrightarrow} A_1 \oplus PA_2 \overset{\langle\alpha,\eta\rangle}{\twoheadrightarrow} A_2$, and $A_{12} = \ker\langle\alpha,\eta\rangle$. This sequence is independent of the choices of PB, PA_2, η_0, and η and is called the long exact $(\underline{\pi},\operatorname{Ext}_\Lambda)$-sequence in the first variable.

As expected, the diagram dual to (2.21),

$$\begin{array}{ccccccccc} & & \operatorname{Hom}_\Lambda(A_2,B) & \rightarrowtail & \operatorname{Hom}_\Lambda(A_1,B) & \longrightarrow & \operatorname{Hom}_\Lambda(A_{12},B) & \longrightarrow & \operatorname{Ext}_\Lambda(A_2,B) \to \cdots \\ & & \downarrow & & \downarrow & & \downarrow & \nearrow & \\ \cdots \to \underline{\pi}_1(\alpha,B) & \longrightarrow & \underline{\pi}(A_2,B) & \longrightarrow & \underline{\pi}(A_1,B) & \longrightarrow & \dfrac{\operatorname{Hom}_\Lambda(A_{12},B)}{\eta_{0*}\iota^*\operatorname{Hom}_\Lambda(A_1\oplus PA_2,PB)}, & & \end{array} \tag{2.24}$$

shows the connection between the long exact $\operatorname{Ext}_\Lambda$-sequence in the first variable and the long exact $(\underline{\pi},\operatorname{Ext}_\Lambda)$-sequence in the first variable.

3. Some examples of nontrivial relative homotopy groups of modules. We now construct a few nontrivial injective/projective relative homotopy groups (of modules) based on the study in [3]. That is, we will concentrate our attention on the case that Λ is the integral group ring of the finite cyclic group C_k, and all the modules are regarded as trivial C_k-modules.

First, it is obvious that the injective/projective relative homotopy groups of identity maps are trivial. Also, from [3, Theorem 2.2 and Corollary 2.3], we learn that, under the assumption that the abelian groups D, B_1, B_2 are regarded as trivial C_k-modules, if D is torsion-free and divisible, then, for arbitrary $\beta : B_1 \to B_2$, $\bar{\pi}_n(D,\beta) = 0$ for all $n \geq 1$. For instance, for arbitrary $\beta : B_1 \to B_2$, $\bar{\pi}_n(\mathbb{Q},\beta) = 0$ for all $n \geq 1$, if $\mathbb{Q}$ is treated as a trivial C_k-module. Next, since $\bar{\pi}_n(\mathbb{Q}/\mathbb{Z},\mathbb{Q}/\mathbb{Z})$, $\bar{\pi}_n(\mathbb{Z},\mathbb{Q}/\mathbb{Z})$, and $\underline{\pi}_n(\mathbb{Z},\mathbb{Z})$ are our examples of nontrivial homotopy groups (of modules) [3], it is natural to consider the short exact sequence $\mathbb{Z} \overset{\iota}{\hookrightarrow} \mathbb{Q} \overset{\kappa}{\twoheadrightarrow} \mathbb{Q}/\mathbb{Z}$ in the search.

Theorem 3.1. *Let $\mathbb{Z} \overset{\iota}{\hookrightarrow} \mathbb{Q} \overset{\kappa}{\twoheadrightarrow} \mathbb{Q}/\mathbb{Z}$ be a short exact sequence, where ι is the inclusion, κ is the quotient map, and $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$ are regarded as trivial C_k-modules. Then,*

(i)

$$\bar{\pi}_n(\mathbb{Q}/\mathbb{Z},\kappa) \cong \bar{\pi}_n(\mathbb{Q}/\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/k & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd,} \end{cases} \tag{3.1}$$

(ii)

$$\bar{\pi}_n(\mathbb{Z},\kappa) \cong \bar{\pi}_n(\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \cong \begin{cases} 0 & \text{for } n \text{ even,} \\ \mathbb{Z}/k & \text{for } n \text{ odd,} \end{cases} \tag{3.2}$$

(iii)

$$\underline{\pi}_n(\iota,\mathbb{Z}) \cong \underline{\pi}_n(\mathbb{Z},\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/k & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases} \tag{3.3}$$

Proof. The proofs follow from the results in [3] with suitable homotopy exact sequence (1.2): for (i), we use the homotopy exact sequence of the map κ,

$$\cdots \xrightarrow{\partial} \bar{\pi}_n(\mathbb{Q}/\mathbb{Z},\mathbb{Q}) \xrightarrow{\kappa_*} \bar{\pi}_n(\mathbb{Q}/\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \xrightarrow{J} \bar{\pi}_n(\mathbb{Q}/\mathbb{Z},\kappa) \xrightarrow{\partial} \bar{\pi}_{n-1}(\mathbb{Q}/\mathbb{Z},\mathbb{Q}) \xrightarrow{\kappa_*} \cdots. \tag{3.4}$$

Since $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z},\mathbb{Q}) = 0$, $\bar{\pi}_n(\mathbb{Q}/\mathbb{Z},\mathbb{Q}) = 0$ for all $n \geq 0$, so $\bar{\pi}_n(\mathbb{Q}/\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \xrightarrow{J} \bar{\pi}_n(\mathbb{Q}/\mathbb{Z},\kappa)$ is an isomorphism and the rest is presented in [3, Theorem 2.6].

To prove (ii), we first show that $\bar{\pi}_n(\mathbb{Z},\mathbb{Q}) = 0$ for all $n \geq 0$: consider the short exact sequence $\mathbb{Z} \overset{\iota}{\hookrightarrow} \mathbb{Q} \overset{\kappa}{\twoheadrightarrow} \mathbb{Q}/\mathbb{Z}$ and its inducing long exact $(\bar{\pi},\mathrm{Ext}_\Lambda)$-sequence

$$\cdots \xrightarrow{\partial} \bar{\pi}_n(\mathbb{Q}/\mathbb{Z},\mathbb{Q}) \xrightarrow{\kappa^*} \bar{\pi}_n(\mathbb{Q},\mathbb{Q}) \xrightarrow{\iota^*} \bar{\pi}_n(\mathbb{Z},\mathbb{Q}) \xrightarrow{\partial} \bar{\pi}_{n-1}(\mathbb{Q}/\mathbb{Z},\mathbb{Q}) \xrightarrow{\kappa^*} \cdots. \tag{3.5}$$

Since $\bar{\pi}_n(\mathbb{Q}/\mathbb{Z},\mathbb{Q}) = 0$ and $\bar{\pi}_n(\mathbb{Q},\mathbb{Q}) = 0$ for all $n \geq 0$ [3, Corollary 2.4], $\bar{\pi}_n(\mathbb{Z},\mathbb{Q}) = 0$ for all $n \geq 0$.

Next, apply the homotopy exact sequence of the map κ,

$$\cdots \xrightarrow{\partial} \bar{\pi}_n(\mathbb{Z},\mathbb{Q}) \xrightarrow{\kappa_*} \bar{\pi}_n(\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \xrightarrow{J} \bar{\pi}_n(\mathbb{Z},\kappa) \xrightarrow{\partial} \bar{\pi}_{n-1}(\mathbb{Z},\mathbb{Q}) \xrightarrow{\kappa_*} \cdots. \tag{3.6}$$

Since $\bar{\pi}_n(\mathbb{Z},\mathbb{Q}) = 0$ for all $n \geq 0$, $\bar{\pi}_n(\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \xrightarrow{J} \bar{\pi}_n(\mathbb{Z},\kappa)$ is an isomorphism and the rest is presented in [3, Theorem 2.8].

The proof of (iii) is similar; we leave it to the reader. □

4. The long exact $(\bar{\pi}, \mathrm{Ext}_\Lambda)$-sequence of a triple and its interaction with three long exact $(\bar{\pi}, \mathrm{Ext}_\Lambda)$-sequences in the second variable. In topology, let X be a path-connected space, and Y, Z closed path-connected subspaces such that

$$Z \overset{i_1}{\hookrightarrow} Y \overset{i_2}{\hookrightarrow} X, \quad i_3 = i_2 \circ i_1 : Z \to X, \tag{4.1}$$

where i_1, i_2, i_3 are inclusions. Then, there exists an exact sequence of relative homotopy groups,

$$\begin{aligned} \cdots &\longrightarrow \pi_n(i_1) \longrightarrow \pi_n(i_3) \longrightarrow \pi_n(i_2) \longrightarrow \pi_{n-1}(i_1) \longrightarrow \cdots \\ &\longrightarrow \pi_1(i_1) \longrightarrow \pi_1(i_3) \longrightarrow \pi_1(i_2), \end{aligned} \tag{4.2}$$

where $\pi_n(i_1) = \pi_n(Y, Z)$, $\pi_n(i_2) = \pi_n(X, Y)$, and $\pi_n(i_3) = \pi_n(X, Z)$, $n \geq 1$. In the case that the maps in

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3, \quad f_3 = f_2 \circ f_1 : X_1 \to X_3 \tag{4.3}$$

are not necessarily injective, where X_1, X_2, and X_3 are topological spaces with base-points, one applies the character of mapping cylinders, and (4.2) is then

$$\begin{aligned} \cdots &\longrightarrow \pi_n(f_1) \longrightarrow \pi_n(f_3) \longrightarrow \pi_n(f_2) \longrightarrow \pi_{n-1}(f_1) \longrightarrow \cdots \\ &\longrightarrow \pi_1(f_1) \longrightarrow \pi_1(f_3) \longrightarrow \pi_1(f_2). \end{aligned} \tag{4.4}$$

It is called the *homotopy sequence of a triple*.

Analogously, in module theory, suppose that given

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3, \quad \beta_3 = \beta_2 \circ \beta_1 : B_1 \to B_3, \tag{4.5}$$

where the maps β_1, β_2, β_3 are not necessarily monomorphic, there arises, for each A, a sequence, which we will prove to be exact, of (injective) relative homotopy groups,

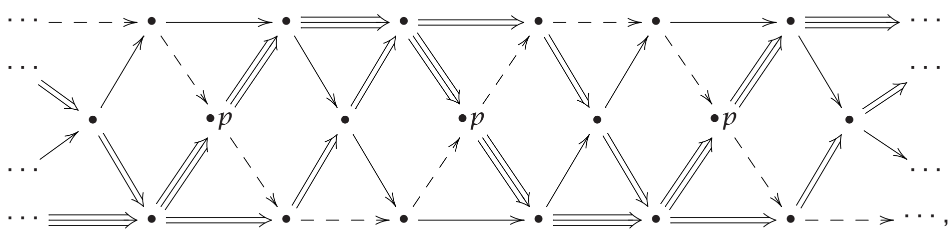
$$\begin{aligned} \cdots &\longrightarrow \bar{\pi}_n(A, \beta_1) \longrightarrow \bar{\pi}_n(A, \beta_3) \longrightarrow \bar{\pi}_n(A, \beta_2) \longrightarrow \bar{\pi}_{n-1}(A, \beta_1) \longrightarrow \cdots \\ &\longrightarrow \bar{\pi}_1(A, \beta_1) \longrightarrow \bar{\pi}_1(A, \beta_3) \longrightarrow \bar{\pi}_1(A, \beta_2). \end{aligned} \tag{4.6}$$

To show the exactness of (4.6), we use the following theorem, which not only grants us the exactness and thus the existence of the homotopy sequence of a triple in module theory, (4.6), but also allows us to expand it to a doubly infinite sequence named the *long exact* $(\bar{\pi}, \mathrm{Ext}_\Lambda)$*-sequence of a triple*, (4.11).

Theorem 4.1 [2]. *Suppose that given four sequences*

$$\begin{gathered}\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots,\\ \cdots \Longrightarrow \bullet \Longrightarrow \bullet \Longrightarrow \bullet \Longrightarrow \cdots,\\ \cdots \Rrightarrow \bullet \Rrightarrow \bullet \Rrightarrow \bullet \Rrightarrow \cdots,\\ \cdots \dashrightarrow \bullet \dashrightarrow \bullet \dashrightarrow \bullet \dashrightarrow \cdots,\end{gathered} \tag{4.7}$$

of which three are long exact, forming a commutative diagram

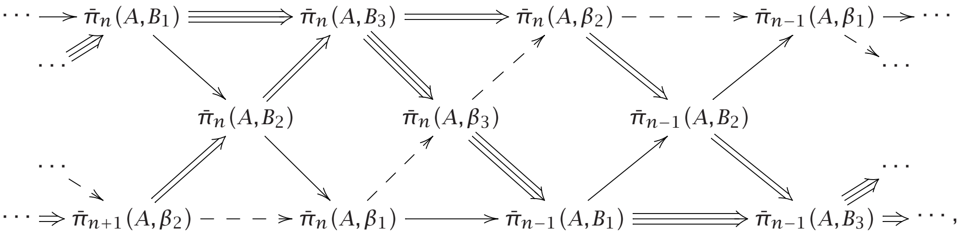


(4.8)

then the fourth is also long exact provided it is differential; that is, $\partial\partial = 0$ at its crossing points p.

Note that one derives Theorem 4.1 through diagram chasing, yet, unexpectedly, the assumption that the fourth sequence is differential at its crossing points is necessary when showing the exactness.

On inserting the (injective) homotopy exact sequence (1.2), of the maps β_1, β_2, and β_3, respectively, into (4.8), one produces a commutative diagram



(4.9)

which settles the exactness of (4.6), due to Theorem 4.1 (it is evident that the composite map

$$\begin{array}{ccccc} B_1 & \longrightarrow & B_1 & \longrightarrow & B_2 \\ \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta_3} \\ B_2 & \longrightarrow & B_3 & \longrightarrow & B_3 \end{array} \tag{4.10}$$

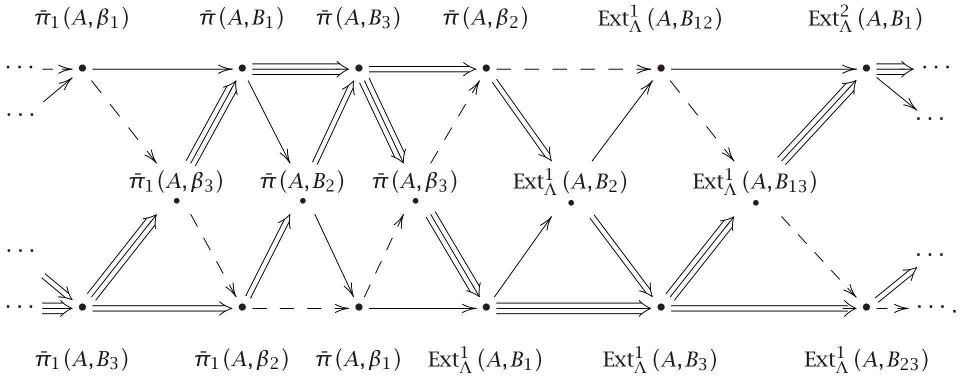
from β_1 to β_3 factors through 1_{B_2} and is thus null-homotopic). We thus call (4.6) the *homotopy sequence of a triple in module theory*. Furthermore, from Theorem 2.2, we learn that the homotopy exact sequence (1.2) extends to the long exact $(\bar{\pi}, \mathrm{Ext}_\Lambda)$-sequence in the second variable, (2.2). By Theorem 4.1, the homotopy sequence of a triple also expands and links with an Ext_Λ-sequence.

Theorem 4.2. *Suppose that given maps $\beta_1 : B_1 \to B_2$, $\beta_2 : B_2 \to B_3$, and $\beta_3 = \beta_2 \circ \beta_1 : B_1 \to B_3$, there exists, for each A, a doubly infinite long exact sequence*

$$\begin{aligned} \cdots &\longrightarrow \bar{\pi}_n(A,\beta_1) \longrightarrow \bar{\pi}_n(A,\beta_3) \longrightarrow \bar{\pi}_n(A,\beta_2) \longrightarrow \bar{\pi}_{n-1}(A,\beta_1) \longrightarrow \cdots \\ &\longrightarrow \bar{\pi}_1(A,\beta_1) \longrightarrow \bar{\pi}_1(A,\beta_3) \longrightarrow \bar{\pi}_1(A,\beta_2) \\ &\longrightarrow \mathrm{Hom}_\Lambda(A,B_{12})/\iota_0^*\kappa_{1*}\mathrm{Hom}_\Lambda(CA,CB_1\oplus B_2) \\ &\longrightarrow \mathrm{Hom}_\Lambda(A,B_{13})/\iota_0^*\kappa_{3*}\mathrm{Hom}_\Lambda(CA,CB_1\oplus B_3) \\ &\longrightarrow \mathrm{Hom}_\Lambda(A,B_{23})/\iota_0^*\kappa_{2*}\mathrm{Hom}_\Lambda(CA,CB_2\oplus B_3) \\ &\longrightarrow \mathrm{Ext}_\Lambda(A,B_{12}) \longrightarrow \mathrm{Ext}_\Lambda(A,B_{13}) \longrightarrow \mathrm{Ext}_\Lambda(A,B_{23}) \\ \cdots &\longrightarrow \mathrm{Ext}^n_\Lambda(A,B_{12}) \longrightarrow \mathrm{Ext}^n_\Lambda(A,B_{13}) \longrightarrow \mathrm{Ext}^n_\Lambda(A,B_{23}) \\ &\longrightarrow \mathrm{Ext}^{n+1}_\Lambda(A,B_{12}) \longrightarrow \cdots, \end{aligned} \tag{4.11}$$

where $\iota_0 : A \hookrightarrow CA$ is the inclusion of A into an injective container CA, $\iota' : B_1 \hookrightarrow CB_1$ is the inclusion of B_1 into an injective container CB_1, $\iota'' : B_2 \hookrightarrow CB_2$ is the inclusion of B_2 into an injective container CB_2, κ_1 is the quotient map in the short exact sequence $B_1 \overset{\{\iota',\beta_1\}}{\rightarrowtail} CB_1 \oplus B_2 \overset{\kappa_1}{\twoheadrightarrow} B_{12} = \mathrm{coker}\{\iota',\beta_1\}$, κ_2 is the quotient map in $B_2 \overset{\{\iota'',\beta_2\}}{\rightarrowtail} CB_2 \oplus B_3 \overset{\kappa_2}{\twoheadrightarrow} B_{23} = \mathrm{coker}\{\iota'',\beta_2\}$, and κ_3 is the quotient map in $B_1 \overset{\{\iota',\beta_3\}}{\rightarrowtail} CB_1 \oplus B_3 \overset{\kappa_3}{\twoheadrightarrow} B_{13} = \mathrm{coker}\{\iota',\beta_3\}$. This sequence is independent of the choices of CA, CB_1, CB_2, ι_0, ι', and ι''. It is called the long exact $(\bar{\pi},\mathrm{Ext}_\Lambda)$-sequence of a triple.

We abbreviate the terms $\mathrm{Hom}_\Lambda(A,B_{12})/\iota_0^*\kappa_{1*}\mathrm{Hom}_\Lambda(CA,CB_1\oplus B_2)$, $\mathrm{Hom}_\Lambda(A,B_{13})/\iota_0^*\kappa_{3*}\mathrm{Hom}_\Lambda(CA,CB_1\oplus B_3)$, and $\mathrm{Hom}_\Lambda(A,B_{23})/\iota_0^*\kappa_{2*}\mathrm{Hom}_\Lambda(CA,CB_2\oplus B_3)$, which put together the homotopy sequence of a triple (4.6) and a sequence of Ext_Λ-groups, as $\bar{\pi}(A,\beta_1)$, $\bar{\pi}(A,\beta_3)$, and $\bar{\pi}(A,\beta_2)$, respectively. Then, there is diagram (4.12) which not only is an extension of (4.9), but also indicates the interaction between the long exact $(\bar{\pi},\mathrm{Ext}_\Lambda)$-sequence of a triple and three long exact $(\bar{\pi},\mathrm{Ext}_\Lambda)$-sequences in the second variable:



(4.12)

A final note is that, as in Section 2, the dual statements, especially those for Theorem 4.2 and diagram (4.12), in the projective relative homotopy theory arise automatically without further derivation. In addition, as in [4], since our argument does not refer to elements of the sets, one can define the necessary homotopy concepts in arbitrary abelian categories with enough injectives and projectives and proceed accordingly.

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