

## EFFECTS OF MODULATION ON RAYLEIGH-BENARD CONVECTION. PART I

B. S. BHADAURIA and LOKENATH DEBNATH

Received 10 November 2002 and in revised form 9 February 2003

The linear stability of a horizontal layer of fluid heated from below and above is considered. In addition to a steady temperature difference between the walls of the fluid layer, a time-dependent periodic perturbation is applied to the wall temperatures. Only infinitesimal disturbances are considered. Numerical results for the critical Rayleigh number are obtained at various Prandtl numbers and for various values of the frequency. Some comparisons have been made with the known results.

2000 Mathematics Subject Classification: 76E15, 76E06, 76R10.

**1. Introduction.** This paper deals with the stability of a fluid layer confined between two horizontal planes and heated from below and above periodically with time. Considerable attention has been given to this problem during the last thirty years. Chandrasekhar [6] has given a comprehensive review of this stability problem with steady heating.

Since the problem of Taylor stability and Benard stability are very similar, Venezian [18] investigated the thermal analogue of Donnelly's experiment [7], using free-free surfaces, and compared his results with the results of Donnelly. Venezian's theory does not find any such finite frequency, as obtained by Donnelly, but finds that for the case of modulation only at the lower surface, the modulation would be stabilizing with maximum stabilization occurring as the frequency goes to zero. However, in his explanation, it was suggested by Venezian that linear stability theory ceases to be applicable when the frequency of modulation is sufficiently small.

Rosenblat and Herbert [15] have investigated the linear stability problem for free-free surfaces using low-frequency modulation and found an asymptotic solution. Periodicity and amplitude criteria were employed to calculate the critical Rayleigh number. Rosenblat and Tanaka [16] have used Galerkin procedure to solve the linear problem for more realistic boundary conditions, that is, rigid walls. A similar problem has been considered earlier by Gershuni and Zhukhovitskii [9] for a temperature profile obeying rectangular law. Yih and Li [19] have investigated the formation of convective cells in a fluid between two horizontal rigid boundaries with time-periodic temperature distribution using Floquet theory. They found that the disturbances (or convection cells) oscillate either synchronously or with half frequency.

Gresho and Sani [10] have treated the case of linear stability problem with rigid boundaries and found that gravitational modulation can significantly affect the stability limits of the system. Finucane and Kelly [8] have carried out an analytical experimental

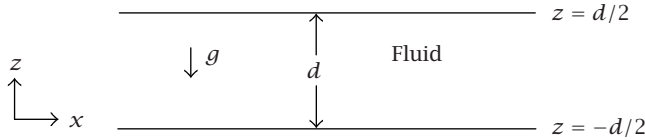


FIGURE 2.1. Benard configuration.

investigation to confirm the results of Rosenblat and Herbert. Besides investigating the linear stability, Roppo et al. [14] have also carried out the weakly nonlinear analysis of the problem. Aniss et al. [1] have worked out a linear problem of the convection parametric instability in the case of a Newtonian fluid confined in a Hele-Shaw cell and subjected to the vertical periodic motion. In their asymptotic analysis, they have investigated the influence of the gravitational modulation on the instability threshold. Recently, Bhatia and Bhadauria [3, 4] Bhadauria and Bhatia [2] have studied the linear stability problem for sawtooth, step-function, and day-night profiles.

The object of the present study is to find the critical conditions under which thermal convection starts. To modulate the wall temperatures, sinusoidal profile has been considered. The temperature modulation between the plates is out of phase. Only even solutions have been considered. The results have their relevance with convective flows in the terrestrial atmosphere.

**2. Formulation.** Consider a fluid layer of a viscous, incompressible fluid confined between two parallel horizontal walls, one at  $z = -d/2$  and the other at  $z = d/2$ . The walls are infinitely extended and rigid. The configuration is shown in Figure 2.1.

The governing equations in the Boussinesq approximation are

$$\begin{aligned} \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} &= -\frac{1}{\rho_m} \nabla p + [1 - \alpha(T - T_m)]X + \nu \nabla^2 \mathbf{V}, \\ \nabla \cdot \mathbf{V} &= 0, \\ \frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T &= \kappa \nabla^2 T, \end{aligned} \tag{2.1}$$

where  $\rho_m, T_m$  are (constants) reference density and temperature, respectively,  $X = (0, 0, -g)$  or  $g$  is the acceleration due to gravity,  $\nu$  is the kinematic viscosity,  $\kappa$  is the thermal diffusivity,  $\alpha$  is the coefficient of volume expansion,  $\mathbf{V} = (u, v, w)$ ,  $p$ , and  $T$  are, respectively, the fluid velocity, pressure, and temperature fields, while  $t$  is the time. The relation between  $\rho_m$  and  $T_m$  is given by

$$\rho = \rho_m [1 - \alpha(T - T_m)]. \tag{2.2}$$

To modulate the wall temperatures, the boundary conditions are

$$\begin{aligned} T(t) &= \beta d (1 + \varepsilon \cos \omega t) \quad \text{at } z = -\frac{d}{2}, \\ T(t) &= -\beta d \varepsilon \cos \omega t \quad \text{at } z = \frac{d}{2}. \end{aligned} \tag{2.3}$$

Here  $\omega$  is the modulating frequency,  $2\pi/\omega$  is the period of oscillation,  $\varepsilon$  represents the amplitude of modulation, and  $\beta (= \Delta T/d)$  is the thermal gradient. Equations (2.1), (2.2), and (2.3) admit an equilibrium solution in which

$$\mathbf{V} = (u, v, w) = 0, \quad T = \bar{T}(z, t), \quad p = \bar{p}(z, t). \quad (2.4)$$

The equation for pressure  $\bar{p}(z, t)$ , which balances the buoyancy force, is not required explicitly; however, the temperature  $\bar{T}(z, t)$  can be given by the diffusion equation

$$\frac{\partial \bar{T}}{\partial t} = \kappa \frac{\partial^2 \bar{T}}{\partial z^2}. \quad (2.5)$$

The differential equation (2.5) can be solved with the help of the boundary conditions (2.3). We write

$$\bar{T}(z, t) = T_S(z) + \varepsilon T_1(z, t), \quad (2.6)$$

where  $T_S(z)$  is the steady temperature field and  $T_1(z, t)$  is the oscillating part. Then the solution can be given by

$$\begin{aligned} T_S(z) &= \Delta T \left( \frac{1}{2} - \frac{z}{d} \right), \\ T_1(z, t) &= -\Delta T \operatorname{Re} \left\{ \frac{F(z, t)}{\sinh(\lambda/2)} \right\}, \end{aligned} \quad (2.7)$$

where

$$\begin{aligned} F(z, t) &= \sinh \left( \frac{\lambda z}{d} \right) e^{i\omega t}, \\ \lambda^2 &= \frac{i\omega d^2}{\kappa}. \end{aligned} \quad (2.8)$$

Here the objective is to examine the behaviour of infinitesimal disturbances to the basic solution (2.4). With this in view, substitute

$$\mathbf{V} = (u, v, w), \quad T = \bar{T}(z, t) + \theta, \quad p = \bar{p}(z, t) + p_1 \quad (2.9)$$

into (2.1) and linearize with respect to the perturbation quantities  $\mathbf{V}$ ,  $\theta$ , and  $p_1$ . These quantities are Fourier analyzed with respect to their variations in the  $xy$ -plane; we write

$$\begin{aligned} w &= w(z, t) \exp [i(a_x x + a_y y)], \\ \theta &= \theta(z, t) \exp [i(a_x x + a_y y)]. \end{aligned} \quad (2.10)$$

Here  $a = (a_x^2 + a_y^2)^{1/2}$  is the horizontal wavenumber. The variables have been nondimensionalized according to

$$\begin{aligned} \mathbf{r} &= d\mathbf{r}', \quad t = t'/\omega, \quad \bar{T} = \beta d T_0, \quad \theta = \beta d \theta', \quad a_x^2 + a_y^2 = d^2 a'^2 \\ \mathbf{V} &= (\alpha g \beta d^3 a'^2 / \nu) \mathbf{V}', \quad p_1 = (\alpha g \beta \kappa d^2 \rho_m / \nu) p'; \end{aligned} \quad (2.11)$$

then the nondimensionalized linear governing equations are, dropping the primes,

$$a^2 \omega^* \frac{\partial \mathbf{V}}{\partial t} + \nabla p = P \theta \hat{k} + a^2 P \nabla^2 \mathbf{V}, \tag{2.12}$$

$$\nabla \cdot \mathbf{V} = 0, \tag{2.13}$$

$$\omega^* \frac{\partial \theta}{\partial t} + Ra^2 \left( \frac{\partial T_0}{\partial z} \right) w = \nabla^2 \theta, \tag{2.14}$$

where  $R = \alpha g \Delta T d^3 / \nu \kappa$  is the Rayleigh number,  $P = \nu / \kappa$  is the Prandtl number,  $\hat{k}$  is the vertical unit vector in the positive  $z$  direction, and  $\omega^* = \omega d^2 / \kappa$  is the nondimensional frequency, which is a measure of the thickness of the thermal boundary layer at the planes.

The temperature gradient  $\partial T_0 / \partial z$  obtained from the dimensionless form of (2.6) is

$$\frac{\partial T_0}{\partial z} = -1 - \epsilon \operatorname{Re} \left[ \frac{F'(z, t)}{\sinh(\lambda/2)} \right], \tag{2.15}$$

where

$$F'(z, t) = \lambda \cos(\lambda z) e^{it}, \tag{2.16}$$

$$\lambda^2 = i \omega^*. \tag{2.17}$$

Henceforth, the asterisk will be dropped and  $\omega$  will be considered as the nondimensional frequency. For convenience, the entire problem has been expressed in terms of  $w$  and  $\theta$ . Now taking the curl of (2.12) twice and using (2.10), the system of equations reduces to

$$\begin{aligned} \omega \left( \frac{\partial^2}{\partial z^2} - a^2 \right) \frac{\partial w}{\partial t} &= -P \theta + P \left( \frac{\partial^2}{\partial z^2} - a^2 \right)^2 w, \\ \omega \frac{\partial \theta}{\partial t} &= \left( \frac{\partial^2}{\partial z^2} - a^2 \right) \theta - Ra^2 \left( \frac{\partial T_0}{\partial z} \right) w. \end{aligned} \tag{2.18}$$

The boundary conditions on  $w$  and  $\theta$  are

$$\begin{aligned} w = \frac{\partial w}{\partial z} = 0 & \quad \text{at } z = \pm \frac{1}{2}, \\ \theta = 0 & \quad \text{at } z = \pm \frac{1}{2}. \end{aligned} \tag{2.19}$$

**3. Method.** From the expression (2.16), it is clear that  $F'(z, t)$  is an even function of  $z$ . By carefully analyzing (2.18) and the boundary conditions (2.19), one can see that the proper solution of (2.18) can be divided into two noncombining groups of even and odd solutions. Previous investigations (see [12, 13]) on thermal convection have shown that disturbances corresponding to even solutions are most unstable; therefore, here the stability of the disturbances corresponding to the even eigenfunctions have been considered.

Now since  $\theta$  vanishes at  $z = \pm 1/2$ , therefore, it is expanded in a series of  $\cos[(2n + 1)\pi z]$ . Also  $w$  is written in a series of  $\phi_n$  so that

$$\left(\frac{\partial^2}{\partial z^2} - a^2\right)^2 \phi_n = \cos[(2n + 1)\pi z], \tag{3.1}$$

where

$$\phi_n = \frac{\partial \phi_n}{\partial z} = 0 \quad \text{at } z = \pm \frac{1}{2}. \tag{3.2}$$

Then the general solution of (3.1) is given by (Chandrasekhar [6, page 56])

$$\phi_n = P_n \cosh az + Q_n z \sinh az + y_n^2 \cos[(2n + 1)\pi z], \tag{3.3}$$

where

$$\begin{aligned} P_n &= -(-1)^n \frac{(2n + 1)\pi y_n^2}{a + \sinh a} \sinh \frac{a}{2}, \\ Q_n &= (-1)^n \frac{2(2n + 1)\pi y_n^2}{a + \sinh a} \cosh \frac{a}{2}, \\ y_n &= \frac{1}{(2n + 1)^2 \pi^2 + a^2}. \end{aligned} \tag{3.4}$$

The expansions for  $w$  and  $\theta$  can be written as

$$\begin{aligned} w(z, t) &= \sum_{n=0}^{\infty} A_n(t) \phi_n(z), \\ \theta(z, t) &= \sum_{n=0}^{\infty} B_n(t) \cos[(2n + 1)\pi z]. \end{aligned} \tag{3.5}$$

Now substitute (3.5) into (2.18) and multiply by  $\cos[(2m + 1)\pi]$ . The resulting equations are then integrated with respect to  $z$  in the interval  $(-1/2, 1/2)$ . The outcome is a system of ordinary differential equations for the unknown coefficients  $A_n(t)$  and  $B_n(t)$ :

$$\omega \sum_{n=0}^{\infty} [K_{nm} - a^2 P_{nm}] \frac{dA_n}{dt} = -\frac{P}{2} B_m + P \sum_{n=0}^{\infty} [L_{nm} - 2a^2 K_{nm} + a^4 P_{nm}] A_n, \tag{3.6}$$

$$\begin{aligned} \frac{\omega}{2} \frac{dB_m}{dt} &= -\frac{1}{2} [(2m + 1)^2 \pi^2 + a^2] B_m \\ &+ Ra^2 \sum_{n=0}^{\infty} [P_{nm} + \epsilon \operatorname{Re} \{G_{nm} e^{it}\}] A_n \quad (m = 0, 1, 2, \dots). \end{aligned} \tag{3.7}$$

The other coefficients which occur in (3.6) and (3.7) are

$$P_{nm} = \int_{-1/2}^{1/2} \phi_n(z) \cos[(2m + 1)\pi z] dz, \tag{3.8}$$

$$K_{nm} = \int_{-1/2}^{1/2} D^2 \phi_n(z) \cos[(2m + 1)\pi z] dz, \tag{3.9}$$

$$L_{nm} = \int_{-1/2}^{1/2} D^4 \phi_n(z) \cos[(2m + 1)\pi z] dz, \tag{3.10}$$

$$G_{nm} = \int_{-1/2}^{1/2} \phi_n(z) \cosh(\lambda z) \cos[(2m + 1)\pi z] dz. \tag{3.11}$$

Here the values of the integrals (3.8), (3.9), and (3.10) have been obtained in their closed forms; however, (3.11) has been calculated numerically, using Simpson’s one-third rule (see [17, page 125]). Thus

$$\begin{aligned} P_{nm} &= \frac{1}{2} \gamma_n^2 \delta_{nm} \\ &\quad + (-1)^m (2m + 1) \pi \gamma_m \left[ 2P_n \cosh \frac{a}{2} \right. \\ &\quad \quad \left. + Q_n \left\{ \sinh \frac{a}{2} - 4a \gamma_m \cosh \frac{a}{2} \right\} \right], \\ K_{nm} &= -\frac{1}{2} \gamma_n^2 (2n + 1)^2 \pi^2 \delta_{nm} \\ &\quad + (-1)^m (2m + 1) \pi \gamma_m \left[ 2(a^2 P_n + 2a Q_n) \cosh \frac{a}{2} \right. \\ &\quad \quad \left. + a^2 Q_n \left\{ \sinh \frac{a}{2} - 4a \gamma_m \cosh \frac{a}{2} \right\} \right], \\ L_{nm} &= \frac{1}{2} \gamma_n^2 (2n + 1)^4 \pi^4 \delta_{nm} \\ &\quad + (-1)^m (2m + 1) \pi \gamma_m \left[ 2a^4 P_n \cosh \frac{a}{2} \right. \\ &\quad \quad \left. + Q_n \left\{ 4a^3 (2 - a^2 \gamma_m) \cosh \frac{a}{2} + a^4 \sinh \frac{a}{2} \right\} \right], \end{aligned} \tag{3.12}$$

where  $\delta_{nm}$  is the Kronecker delta. It is convenient for computational purpose to take  $m = 0, 1, 2, \dots, N - 1$ , that is, total  $2N$  equations, and then rearrange them. For this, first multiply (3.6) by the inverse of the matrix  $(K_{nm} - a^2 P_{nm})$  and then introduce the notations  $x_1 = A_0, x_2 = B_0, x_3 = A_1, x_4 = B_1, \dots$ . Now combine (3.6) and (3.7) to the form

$$\frac{dx_i}{dt} = H_{i1}x_1 + H_{i2}x_2 + \dots + H_{iL}x_L \quad (i = 1, 2, 3, \dots, 2N, L = 2N), \tag{3.13}$$

where  $H_{ij}(t)$  is the matrix of the coefficients in (3.6) and (3.7). Since the coefficients  $H_{ij}(t)$  are periodic in  $t$  with period  $2\pi$ , therefore we can discuss the stability of the solution of (3.13) on the basis of the classical Floquet theory (see [5, page 55]). Let

$$x_n(t) = x_{in}(t) = \text{col}[x_{1n}(t), x_{2n}(t), \dots, x_{Ln}(t)] \quad (n = 1, 2, 3, \dots, 2N) \tag{3.14}$$

be the solutions of (3.13) which satisfy the initial conditions

$$x_{in}(0) = \delta_{in}. \quad (3.15)$$

The solutions (3.14) with the conditions (3.15) form  $2N$  linearly independent solutions of (3.13). Once these solutions are found, one can get the values of  $x_{in}(2\pi)$  and then arrange them in the constant matrix

$$C = [x_{in}(2\pi)]. \quad (3.16)$$

The eigenvalues  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_L$  of the matrix  $C$  are also called the characteristic multipliers of system (3.13) and the numbers  $\mu_r$ , defined by the relations

$$\lambda_r = \exp(2\pi\mu_r), \quad r = 1, 2, 3, \dots, 2N, \quad (3.17)$$

are the characteristic exponents.

The values of the characteristic exponents determine the stability of the system. We assume that the  $\mu_r$  are ordered so that

$$\text{Re}(\mu_1) \geq \text{Re}(\mu_2) \geq \dots \geq \text{Re}(\mu_L). \quad (3.18)$$

Then the system is stable if  $\text{Re}(\mu_1) < 0$ , while  $\text{Re}(\mu_1) = 0$  corresponds to one periodic solution and represents a stability boundary. This periodic disturbance is the only disturbance which will manifest itself at a marginal stability.

To obtain the matrix  $C$ , we have integrated system (3.13) using Runge-Kutta-Gill Procedure (see [17, page 217]). The eigenvalues  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_L$  of the matrix  $C$  are found with the help of Rutishauser method (see [11, page 116]).

**4. Results and discussion.** The first approximation to the critical Rayleigh number in the absence of modulation ( $\varepsilon = 0$ ) is found by setting  $n = 0$  and  $m = 0$  in (3.6) and (3.7). This corresponds to  $\cos \pi z$ , a trial function for  $\theta$ . The corresponding value for  $R$  is

$$R = \frac{(\pi^2 + a^2)^3}{a^2 [1 - 16a\pi^2 \cosh^2(a/2) / \{(\pi^2 + a^2)^2 (\sinh a + a)\}]}. \quad (4.1)$$

This gives  $R = 1715.08$  for  $a = 3.117$ , while the exact value of  $R$  is 1707.76 at the same wavenumber. By including more terms in the expansion of  $w$  and  $\theta$ , one can achieve a higher degree of accuracy. The second approximation to the Rayleigh number is found to be 1707.93756 at  $a = 3.116846$ , which is obtained by setting  $m, n = 0$  and 1. These values are same, as they should, as the Chandrasekhar values [6].

When  $\varepsilon \neq 0$ , we calculate the modified value of  $Rc$ , with variation in other parameters. We also check the critical value of the wavenumber  $a$ . Here the results have been obtained by solving (3.13) for  $x_1, x_2, x_3$ , and  $x_4$ . The results are calculated for moderate values of  $\varepsilon$  as we are interested only in the modulating effect of the oscillation; there seems to be no reason why this theory cannot be applied for large values of the parameters.

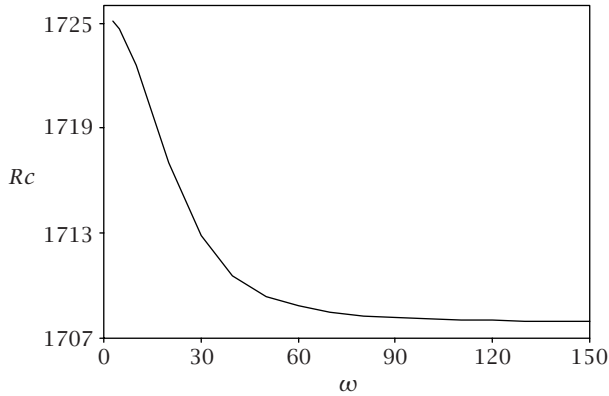


FIGURE 4.1. Variation of  $R_c$  with  $\omega$ :  $P = 0.1$ ,  $\varepsilon = 0.1$ , and  $a = 3.1168$ .

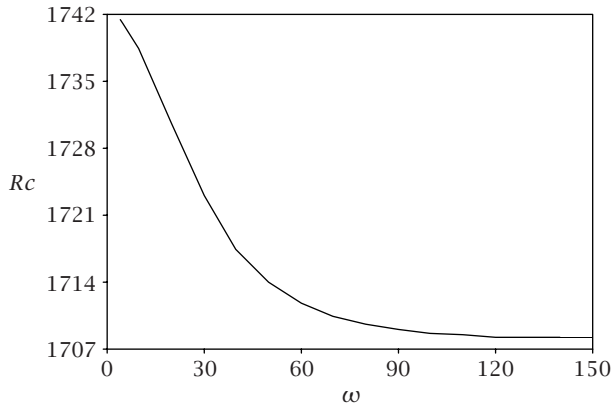


FIGURE 4.2. Variation of  $R_c$  with  $\omega$ :  $P = 0.73$ ,  $\varepsilon = 0.1$ , and  $a = 3.1168$ .

One can see that the effect of the unsteady part of the primary temperature is one of stabilization, decreasing with increasing frequency  $\omega$  (Figures 4.1, 4.2, and 4.3). The stabilization is greatest near  $\omega = 0$  and disappears altogether when the frequency  $\omega$  is sufficiently large. This agrees with the results of Venezian [18], Rosenblat and Tanaka [16], Bhatia and Bhadauria [3], and Bhadauria and Bhatia [2]. This also agrees with the results of Yih and Li [19], who found while studying the instability of unsteady flows that the effect of modulation is stabilizing. Similarly the results also agree with Donnelly's findings [7] for the related problem of Taylor vortices that oscillation of one cylinder can only stabilize the Couette flow.

When the modulating frequency is small, the convective wave propagates across the fluid layer, thereby inhibiting the instability, and so the convection occurs at higher Rayleigh number than that predicted by the linear theory for steady temperature gradient. Here Figures 4.1, 4.2, and 4.3 present the variation of the critical Rayleigh number



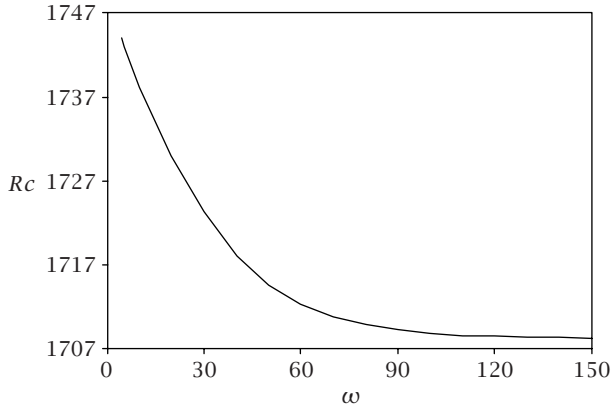


FIGURE 4.3. Variation of  $R_c$  with  $\omega$ :  $P = 1.0$ ,  $\varepsilon = 0.1$ , and  $a = 3.1168$ .

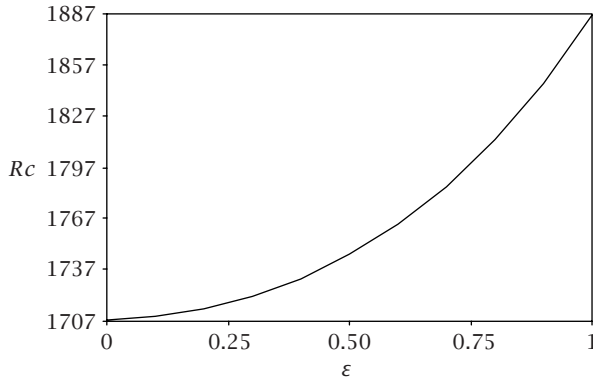


FIGURE 4.4. Variation of  $R_c$  with  $\varepsilon$ :  $P = 0.1$ ,  $\omega = 50.0$ , and  $a = 3.1168$ .

with respect to the modulating frequency for different Prandtl numbers. These figures are similar in every respect except that the values of the critical Rayleigh number are higher at higher Prandtl number. One of the reasons for this may be that for high viscous fluid, higher temperature gradient is required for thermal convection to occur. The results agree with that of Rosenblat and Tanaka [16] who found that the critical Rayleigh number increases as the Prandtl number increases and approaches to one, and beyond this value, the Rayleigh number decreases.

Figure 4.4 shows the variation of  $R_c$  with the amplitude of modulation. It is found here that as the amplitude of modulation increases,  $R_c$  also increases, showing the stabilizing effect.

Finally in Figure 4.5, the variation of  $R_c$  with the wavenumber  $a$  has been depicted. It is very clear from the figure that the critical value of the wavenumber  $a$  is found to be near 3.11.

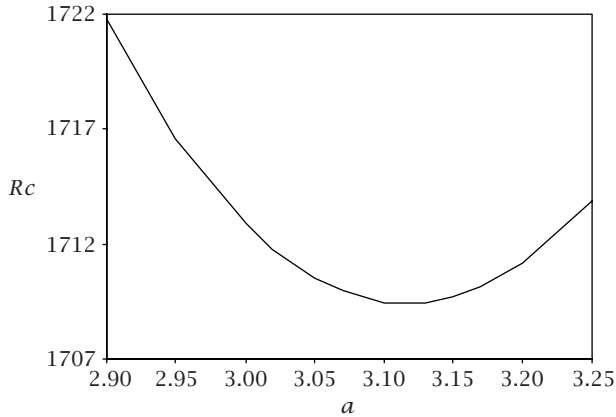


FIGURE 4.5. Variation of  $R_c$  with  $a$ :  $P = 0.1$ ,  $\omega = 50.0$ , and  $\varepsilon = 0.1$ .

**ACKNOWLEDGMENTS.** Part of this work was done during the tenure of a Major Research Project (no. F.8-10/2003(SR)) awarded by the University Grants Commission (UGC), India, to one of the authors, B. S. Bhadouria. The financial assistance to B. S. Bhadouria from UGC is gratefully acknowledged. The research of the second author was partially supported by the Faculty Research Council of the University of Texas - Pan American.

#### REFERENCES

- [1] S. Aniss, M. Souhar, and M. Belhaq, *Asymptotic study of the convective parametric instability in Hele-Shaw cell*, Phys. Fluids **12** (2000), no. 2, 262-268.
- [2] B. S. Bhadouria and P. K. Bhatia, *Time-periodic heating of Rayleigh-Benard convection*, Phys. Scripta **66** (2002), no. 1, 59-65.
- [3] P. K. Bhatia and B. S. Bhadouria, *Effect of modulation on thermal convection instability*, Z. Naturforsch. **55a** (2000), 957-966.
- [4] ———, *Effect of low-frequency modulation on thermal convection instability*, Z. Naturforsch. **56a** (2001), 509-522.
- [5] L. Cesari, *Asymptotic Behavior and Stability Problems*, Springer Verlag, Berlin, 1963.
- [6] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, The International Series of Monographs on Physics, Clarendon Press, Oxford, 1961.
- [7] R. J. Donnelly, *Experiments on the stability of viscous flow between rotating cylinders III. Enhancement of hydrodynamic stability by modulation*, Proc. Roy. Soc. Ser. A **281** (1964), 130-139.
- [8] R. G. Finucane and R. E. Kelly, *Onset of instability in a fluid layer heated sinusoidally from below*, Int. J. Heat Mass Transf. **19** (1976), 71-85.
- [9] G. Z. Gershuni and E. M. Zhukhovitskii, *On parametric excitation of convective instability*, J. Appl. Math. Mech. **27** (1963), 1197-1204.
- [10] P. M. Gresho and R. L. Sani, *The effects of gravity modulation on the stability of a heated fluid layer*, J. Fluid Mech. **40** (1970), no. 4, 783-806.
- [11] M. K. Jain, S. R. K. Iyengar, and R. K. Jain, *Numerical Methods for Scientific and Engineering Computation*, Wiley Eastern, New Delhi, 1985.
- [12] A. Pellew and R. V. Southwell, *On maintained convective motion in a fluid heated from below*, Proc. Roy. Soc. London Ser. A **176** (1940), 312-343.

- [13] W. H. Reid and D. L. Harris, *Some further results on the Bénard problem*, Phys. Fluids **1** (1958), 102-110.
- [14] M. N. Roppo, S. H. Davis, and S. Rosenblat, *Bénard convection with time-periodic heating*, Phys. Fluids **27** (1984), no. 4, 796-803.
- [15] S. Rosenblat and D. M. Herbert, *Low-frequency modulation of thermal instability*, J. Fluid Mech. **43** (1970), 385-398.
- [16] S. Rosenblat and G. A. Tanaka, *Modulation of thermal convection instability*, Phys. Fluids **14** (1971), 1319-1322.
- [17] S. S. Sastry, *Introductory Methods of Numerical Analysis*, 2nd ed., Prentice-Hall of India Private Limited, New Delhi, 1993.
- [18] G. Venezian, *Effect of modulation on the onset of thermal convection*, J. Fluid Mech. **35** (1969), 243-254.
- [19] C.-S. Yih and C.-H. Li, *Instability of unsteady flows or configurations. II: convective instability*, J. Fluid Mech. **54** (1972), no. 1, 143-152.

B. S. Bhadauria: Department of Mathematics and Statistics, Jai Narain Vyas University, Jodhpur 342001, India

*Current address:* Department of Mathematics, Eritrea Institute of Technology, Asmara, Eritrea  
*E-mail address:* [bsbhadauria@rediffmail.com](mailto:bsbhadauria@rediffmail.com)

Lokenath Debnath: Department of Mathematics, University of Texas - Pan American, Edinburg, TX 78539-2999, USA

*E-mail address:* [debnath1@panam.edu](mailto:debnath1@panam.edu)