IJMMS 2003:9, 549–556
PII. S0161171203110277
http://ijmms.hindawi.com

FUZZY STRUCTURES OF $\mathrm{PI}(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$-IDEALS IN HYPER BCK-ALGEBRAS

YOUNG BAE JUN and WOOK HWAN SHIM

Received 28 October 2001

The fuzzification of $\mathrm{PI}(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$-ideals is considered. Using the notion of α-cut, characterization of fuzzy $\mathrm{PI}(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$-ideals is given. Conditions for a fuzzy set to be a fuzzy $\mathrm{PI}(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$-ideal are provided.

2000 Mathematics Subject Classification: 06F35, 08A72, 20N20.

1. Introduction. The study of BCK-algebras was initiated by Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then, a great deal of literature has been produced on the theory of BCK-algebras. In particular, emphasis seems to have been put on the ideal theory of BCK-algebras. The hyperstructure theory (called also multialgebras) was introduced in 1934 by Marty [7]. Around the 1940's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia, and Japan. Over the following decades, many important results appeared, but above all, from the 1970's onwards, the most luxuriant flourishing of hyperstructures have been seen. Hyperstructures have many applications to several sectors of both pure and applied sciences. In [6], Jun et al. applied the hyperstructures to BCK-algebras, introduced the concept of a hyper BCK-algebra, which is a generalization of a BCK-algebra, and investigated some related properties. They also introduced the notion of a hyper BCK-ideal and a weak hyper BCK-ideal and gave relations between hyper BCK-ideal and weak hyper BCK-ideals. Jun and Shim [1] displayed several types of positive implicative hyper BCK-ideals in hyper BCK-algebras. In this paper, we consider the fuzzy structure of $\mathrm{PI}(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$-ideals in hyper BCK-algebras. We introduce the notion of fuzzy $\mathrm{PI}(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$-ideals, and then we investigates the relations between fuzzy hyper BCK-ideals and fuzzy $\mathrm{PI}(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$-ideals. We give a characterization of fuzzy $\mathrm{PI}(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$-ideals using α-cut, provide conditions for a fuzzy set to be a fuzzy $\mathrm{PI}(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$-ideal and establish a fuzzy weak hyper BCK-ideal of a hyper BCK-algebra using the concept of $\mathrm{PI}(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$-ideals.

2. Preliminaries. We include some elementary aspects of hyper BCK-algebras that are necessary for this paper, and, for more details, we refer to [2, 6].

Let H be a nonempty set endowed with a hyper operation "$\circ$"; that is, $\circ$ is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$. For two subsets A and B of H, denote by $A \circ B$ the set $\bigcup_{a\in A,\, b\in B} a \circ b$.

By a *hyper BCK-algebra*, we mean a nonempty set H endowed with a hyper operation "$\circ$" and a constant 0 satisfying the following axioms:

(K1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(K2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(K3) $x \circ H \ll \{x\}$,
(K4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$, and for every $A, B \subseteq H$, $A \ll B$ is defined by for all $a \in A$, $\exists b \in B$ such that $a \ll b$.

In any hyper BCK-algebra H, the following hold (see [2, 6]):

(P1) $0 \circ 0 = \{0\}$,
(P2) $0 \ll x$,
(P3) $x \ll x$,
(P4) $A \ll A$,
(P5) $A \subseteq B$ implies $A \ll B$,
(P6) $0 \circ x = \{0\}$,
(P7) $0 \circ A = \{0\}$,
(P8) $A \ll \{0\}$ implies $A = \{0\}$,
(P9) $x \in x \circ 0$,
(P10) $x \circ 0 \ll \{y\}$ implies $x \ll y$,
(P11) $y \ll z$ implies $x \circ z \ll x \circ y$,
(P12) $x \circ y = \{0\}$ implies $(x \circ z) \circ (y \circ z) = \{0\}$ and $x \circ z \ll y \circ z$,
(P13) $A \circ \{0\} = \{0\}$ implies $A = \{0\}$,
(P14) if $(x \circ y) \circ z \ll A$, then $a \circ z \ll A$ for all $a \in x \circ y$,
(P15) if $a \circ b \subseteq A$ for all $a, b \in A$, then $0 \in A$,
(P16) $(A \circ B) \circ C = (A \circ C) \circ B$, $A \circ B \ll A$ and $0 \circ A \ll \{0\}$,
(P17) $x \circ 0 = \{x\}$ and $A \circ 0 = A$

for all $x, y, z \in H$ and for all nonempty subsets A, B, and C of H.

3. Fuzzy structures of $\mathrm{PI}(\ll, \subseteq, \subseteq)_{\mathrm{BCK}}$-ideals. In what follows, let H denote a hyper BCK-algebra unless otherwise specified. We place a bar over a symbol to denote a fuzzy set, so $\bar{A}$, $\bar{B}$, ... all represent fuzzy sets in H. We write $\bar{A}(x)$, a number in $[0,1]$, for the membership function of $\bar{A}$ evaluated at $x \in H$. An α-cut of $\bar{A}$, written $\bar{A}[\alpha]$, is defined as

$$\{x \in H \mid \bar{A}(x) \geq \alpha\} \quad \text{for } 0 < \alpha \leq 1. \tag{3.1}$$

We separately specify $\bar{A}[0]$ as the closure of the union of all the $\bar{A}[\alpha]$ for $0 < \alpha \leq 1$.

Definition 3.1 (Jun et al. [6, Definition 3.14]). A nonempty subset A of H is called a *hyper BCK-ideal* of H if it satisfies the following conditions:

(I1) $0 \in A$,

(I2) $x \circ y \ll A$ and $y \in A$ imply $x \in A$ for all $x, y \in H$.

Definition 3.2 (Jun and Xin [3, Definition 3.1]). A fuzzy set $\bar{A}$ in H is called a *fuzzy hyper BCK-ideal* of H if it satisfies, for all $x, y \in H$, the following:

(F1) $x \ll y$ implies $\bar{A}(x) \geq \bar{A}(y)$,

(F2) $\bar{A}(x) \geq \min\{\inf_{a \in x \circ y} \bar{A}(a), \bar{A}(y)\}$.

Lemma 3.3 (Jun and Xin [3, Theorem 3.17]). *A fuzzy set $\bar{A}$ in H is a fuzzy hyper BCK-ideal of H if and only if the α-cut $\bar{A}[\alpha]$ of $\bar{A}$ is a hyper BCK-ideal of H whenever $\bar{A}[\alpha] \neq \emptyset$ for $\alpha \in [0,1]$.*

Lemma 3.4 (Jun and Xin [2, Proposition 3.7]). *Let I be a subset of H. If A is a hyper BCK-ideal of H such that $I \ll A$, then I is contained in A.*

Definition 3.5 (Jun and Shim [1, Definition 3.1]). A nonempty subset A of H is called a PI$(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$-*ideal* of H if it satisfies (I1) and

(I3) $(x \circ y) \circ z \ll A$ and $y \circ z \subseteq A$ imply $x \circ z \subseteq A$ for all $x, y, z \in H$.

We first consider the fuzzification of PI$(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$-ideals.

Definition 3.6. A fuzzy set $\bar{A}$ in H is called a *fuzzy* PI$(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$-*ideal* of H if it satisfies (F1) and

(F3) $\inf_{a \in x \circ z} \bar{A}(a) \geq \min\{\inf_{b \in (x \circ y) \circ z} \bar{A}(b), \inf_{c \in y \circ z} \bar{A}(c)\}$ for all $x, y, z \in H$.

Example 3.7. Let $H = \{0, a, b\}$ be a hyper BCK-algebra with the Cayley table (Table 3.1).

Table 3.1

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a,b\}$	$\{0,a,b\}$

Define a fuzzy set $\bar{A}$ in H by $\bar{A}(0) = \bar{A}(a) = 0.8$ and $\bar{A}(b) = 0.2$. It is easily checked that $\bar{A}$ is a fuzzy PI$(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$-ideal of H.

Theorem 3.8. *Every fuzzy* PI$(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$*-ideal is a fuzzy hyper BCK-ideal.*

Proof. Let $\bar{A}$ be a fuzzy PI$(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$-ideal of H, and let $x, y \in H$. Taking $z = 0$ in (F3) and using (P17), we have

$$\bar{A}(x) = \inf_{a \in x \circ 0} \bar{A}(a) \geq \min\left\{\inf_{b \in (x \circ y) \circ 0} \bar{A}(b), \inf_{c \in y \circ 0} \bar{A}(c)\right\} = \min\left\{\inf_{b \in x \circ y} \bar{A}(b), \bar{A}(y)\right\}. \tag{3.2}$$

Hence, $\bar{A}$ is a fuzzy hyper BCK-ideal. □

The following example shows that the converse of Theorem 3.8 may not be true.

Example 3.9. Let $H = \{0, a, b\}$ be a hyper BCK-algebra with the Cayley table (Table 3.2).

Table 3.2

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a\}$	$\{0, a\}$

Define a fuzzy set $\bar{A}$ in H by $\bar{A}(0) = 0.5$ and $\bar{A}(a) = \bar{A}(b) = 0.3$. Then, $\bar{A}$ is a fuzzy hyper BCK-ideal but not a fuzzy $\mathrm{PI}(\ll, \subseteq, \subseteq)_{\mathrm{BCK}}$-ideal of H since $\inf_{a \in b \circ a} \bar{A}(a) = \bar{A}(a) = 0.3$ and

$$\min\left\{\inf_{b \in (b \circ a) \circ b} \bar{A}(b), \inf_{c \in a \circ b} \bar{A}(c)\right\} = \bar{A}(0) = 0.5. \tag{3.3}$$

Theorem 3.10. *If $\bar{A}$ is a fuzzy $\mathrm{PI}(\ll, \subseteq, \subseteq)_{\mathrm{BCK}}$-ideal of H, then the α-cut $\bar{A}[\alpha]$ of $\bar{A}$ is a $\mathrm{PI}(\ll, \subseteq, \subseteq)_{\mathrm{BCK}}$-ideal of H, where $\alpha \in \mathrm{Im}\,\bar{A}$.*

Proof. Let $\bar{A}$ be a fuzzy $\mathrm{PI}(\ll, \subseteq, \subseteq)_{\mathrm{BCK}}$-ideal of H. Both (P2) and (F1) induce the inequality $\bar{A}(0) \geq \bar{A}(x)$ for all $x \in H$, and so $0 \in \bar{A}[\alpha]$ for all $\alpha \in \mathrm{Im}(\bar{A})$. Let $x, y, z \in H$ be such that $(x \circ y) \circ z \ll \bar{A}[\alpha]$ and $y \circ z \subseteq \bar{A}[\alpha]$, where $\alpha \in \mathrm{Im}(\bar{A})$. Then, for every $a \in (x \circ y) \circ z$, there exists $a' \in \bar{A}[\alpha]$ such that $a \ll a'$, and therefore $\bar{A}(a) \geq \bar{A}(a')$ by (F1). Hence, $\bar{A}(a) \geq \alpha$ for all $a \in (x \circ y) \circ z$. It follows from (F3) that, for every $b \in x \circ z$,

$$\bar{A}(b) \geq \inf_{c \in x \circ z} \bar{A}(c) \geq \min\left\{\inf_{a \in (x \circ y) \circ z} \bar{A}(a), \inf_{d \in y \circ z} \bar{A}(d)\right\} \geq \alpha \tag{3.4}$$

so that $b \in \bar{A}[\alpha]$, that is, $x \circ z \subseteq \bar{A}[\alpha]$. Hence, $\bar{A}[\alpha]$ is a $\mathrm{PI}(\ll, \subseteq, \subseteq)_{\mathrm{BCK}}$-ideal of H. □

We now consider the converse of Theorem 3.10.

Theorem 3.11. *Let $\bar{A}$ be a fuzzy set in H such that $\bar{A}[\alpha]$, $\alpha \in \mathrm{Im}(\bar{A})$ is a $\mathrm{PI}(\ll, \subseteq, \subseteq)_{\mathrm{BCK}}$-ideal of H. Then, $\bar{A}$ is a fuzzy $\mathrm{PI}(\ll, \subseteq, \subseteq)_{\mathrm{BCK}}$-ideal of H.*

Proof. Assume that $\bar{A}[\alpha]$, $\alpha \in \mathrm{Im}(\bar{A})$ is a $\mathrm{PI}(\ll, \subseteq, \subseteq)_{\mathrm{BCK}}$-ideal of H. Then, $\bar{A}[\alpha]$ is a hyper BCK-ideal of H (see [4, Theorem 3.10]). It follows from Lemma 3.3 that $\bar{A}$ is a fuzzy hyper BCK-ideal of H, and so condition (F1) is valid. Now, let $\alpha = \min\{\inf_{b \in (x \circ y) \circ z} \bar{A}(b), \inf_{c \in y \circ z} \bar{A}(c)\}$. Then,

$$\bar{A}(b') \geq \inf_{b \in (x \circ y) \circ z} \bar{A}(b) \geq \alpha, \qquad \bar{A}(c') \geq \inf_{c \in y \circ z} \bar{A}(c) \geq \alpha \tag{3.5}$$

for all $b' \in (x\circ y)\circ z$ and $c' \in y\circ z$. Hence, $b', c' \in \bar{A}[\alpha]$, which implies that $(x\circ y)\circ z \subseteq \bar{A}[\alpha]$ and $y\circ z \subseteq \bar{A}[\alpha]$. It follows from (P5) and (I3) that $x\circ z \subseteq \bar{A}[\alpha]$ so that $\bar{A}(d) \geq \alpha$ for all $d \in x\circ z$. Consequently,

$$\inf_{a\in x\circ z} \bar{A}(a) \geq \alpha = \min\left\{\inf_{b\in(x\circ y)\circ z} \bar{A}(b), \inf_{c\in y\circ z} \bar{A}(c)\right\}. \tag{3.6}$$

Thus, $\bar{A}$ is a fuzzy PI$(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$-ideal of H. □

Proposition 3.12. *If $\bar{A}$ is a fuzzy* PI$(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$*-ideal of H in which $\bar{A}[\alpha]$ is reflexive for all $\alpha \in \mathrm{Im}(\bar{A})$, then it satisfies the inequality*

$$\inf_{a\in x\circ y} \bar{A}(a) \geq \inf_{b\in(x\circ y)\circ y} \bar{A}(b) \quad \forall x, y \in H. \tag{3.7}$$

Proof. Assume that $\bar{A}[\alpha]$ is reflexive for all $\alpha \in \mathrm{Im}(\bar{A})$. Then, $\bar{A}(a) \geq \alpha$ for all $a \in x\circ x$, and hence $\inf_{a\in x\circ x} \bar{A}(a) \geq \bar{A}(y)$ for all $x, y \in H$. It follows by taking $z = y$ in (F3) that

$$\inf_{a\in x\circ y} \bar{A}(a) \geq \min\left\{\inf_{b\in(x\circ y)\circ y} \bar{A}(b), \inf_{c\in y\circ y} \bar{A}(c)\right\} = \inf_{b\in(x\circ y)\circ y} \bar{A}(b) \tag{3.8}$$

for all $x, y \in H$. □

Lemma 3.13 (Jun and Xin [4, Proposition 3.13])**.** *In a hyper BCK-algebra H, the following axiom holds:*

$$((x\circ z)\circ(y\circ z))\circ u \ll (x\circ y)\circ u \quad \forall x, y, z, u \in H. \tag{3.9}$$

Lemma 3.14 (Jun and Xin [4, Lemma 3.17])**.** *Let A be a hyper BCK-ideal of H. Then, $I\circ J \subseteq A$ and $J \subseteq A$ imply that $I \subseteq A$ for every nonempty subsets I and J of H.*

Proposition 3.15. *If $\bar{A}$ is a fuzzy hyper BCK-ideal of H satisfying (3.7), then it satisfies the inequality*

$$\inf_{a\in(x\circ z)\circ(y\circ z)} \bar{A}(a) \geq \inf_{b\in(x\circ y)\circ z} \bar{A}(b) \quad \forall x, y, z \in H. \tag{3.10}$$

Moreover, if $\bar{A}[\alpha]$ is reflexive for all $\alpha \in \mathrm{Im}(\bar{A})$, then

$$\inf_{a\in x\circ y} \bar{A}(a) \geq \min\left\{\bar{A}(z), \inf_{b\in((x\circ y)\circ y)\circ z} \bar{A}(b)\right\} \quad \forall x, y, z \in H. \tag{3.11}$$

Proof. Let $\bar{A}$ be a fuzzy hyper BCK-ideal of H satisfying (3.7), and let $\alpha = \inf_{b\in(x\circ y)\circ z}\bar{A}(b)$. Then clearly, $(x\circ y)\circ z \subseteq \bar{A}[\alpha]$. Using (P16) and Lemma 3.13, we get

$$((x\circ(y\circ z))\circ z)\circ z = ((x\circ z)\circ(y\circ z))\circ z \ll (x\circ y)\circ z, \tag{3.12}$$

and hence $((x\circ(y\circ z))\circ z)\circ z \ll \bar{A}[\alpha]$. It follows from Lemma 3.4 that $((x\circ(y\circ z))\circ z)\circ z \subseteq \bar{A}[\alpha]$ so that $(t\circ z)\circ z \subseteq \bar{A}[\alpha]$ for every $t\in x\circ(y\circ z)$. Hence, by (3.7), $\inf_{a\in t\circ z}\bar{A}(a) \geq \inf_{b\in(t\circ z)\circ z}\bar{A}(b) \geq \alpha$ for all $t\in x\circ(y\circ z)$, which implies that $a\in\bar{A}[\alpha]$ for all $a\in t\circ z$, that is, $t\circ z\subseteq\bar{A}[\alpha]$ for every $t\in x\circ(y\circ z)$. This shows that

$$(x\circ z)\circ(y\circ z) = (x\circ(y\circ z))\circ z = \bigcup_{t\in x\circ(y\circ z)} t\circ z \subseteq \bar{A}[\alpha], \tag{3.13}$$

and therefore $\inf_{a\in(x\circ z)\circ(y\circ z)}\bar{A}(a) \geq \alpha = \inf_{b\in(x\circ y)\circ z}\bar{A}(b)$, which proves (3.10). Assume that $\bar{A}[\alpha]$ is reflexive for all $\alpha\in\operatorname{Im}(\bar{A})$. Let $x,y,z\in H$ and $\beta = \min\{\bar{A}(z), \inf_{b\in((x\circ y)\circ y)\circ z}\bar{A}(b)\}$. Then, $z\in\bar{A}[\beta]$ and

$$((x\circ z)\circ y)\circ y = ((x\circ y)\circ y)\circ z \subseteq \bar{A}[\beta]. \tag{3.14}$$

Thus for any $t\in x\circ z$, we obtain $(t\circ y)\circ y\subseteq\bar{A}[\beta]$. Since $\bar{A}[\beta]$ is a hyper BCK-ideal of H, it follows from [4, Theorem 3.18] that $(t\circ y)\circ(y\circ y)\subseteq\bar{A}[\beta]$ for all $t\in x\circ z$ so that $((x\circ z)\circ y)\circ(y\circ y)\subseteq\bar{A}[\beta]$. Note that $y\circ y\subseteq\bar{A}[\beta]$ because $\bar{A}[\beta]$ is reflexive. Hence, by Lemma 3.14 and (K2), $(x\circ y)\circ z = (x\circ z)\circ y\subseteq\bar{A}[\beta]$. Since $\{z\}\subseteq\bar{A}[\beta]$, we have $x\circ y\subseteq\bar{A}[\beta]$ by Lemma 3.14. Therefore,

$$\inf_{a\in x\circ y}\bar{A}(a) \geq \beta = \min\left\{\bar{A}(z), \inf_{b\in((x\circ y)\circ y)\circ z}\bar{A}(b)\right\} \quad \forall x,y,z\in H. \tag{3.15}$$

This completes the proof. □

Definition 3.16 (Jun et al. [6, Definition 3.19]). A nonempty subset A of H is called a *weak hyper BCK-ideal* of H if it satisfies (I1) and

(I4) $x\circ y\subseteq A$ and $y\in A$ imply $x\in A$ for all $x,y\in H$.

Definition 3.17 (Jun and Xin [3, Definition 3.5]). A fuzzy set $\bar{A}$ in H is called a *fuzzy weak hyper BCK-ideal* of H if

$$\bar{A}(0) \geq \bar{A}(x) \geq \min\left\{\inf_{a\in x\circ y}\bar{A}(a), \bar{A}(y)\right\} \quad \forall x,y\in H. \tag{3.16}$$

Using the notion of $\mathrm{PI}(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$-ideal, we establish a fuzzy weak hyper BCK-ideal as follows.

Theorem 3.18. *Let A be a $\mathrm{PI}(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$-ideal of H, and let $w \in H$. Then the fuzzy set $\bar{A}$ in H defined by*

$$\bar{A}(x) = \begin{cases} \alpha & \text{if } x \in A_w = \{y \in H \mid y \circ w \subseteq A\}, \\ \beta & \text{otherwise}, \end{cases} \tag{3.17}$$

where $\alpha > \beta$ in $[0,1]$ is a fuzzy weak hyper BCK-ideal of H.

Proof. Obviously, $\bar{A}(0) \geq \bar{A}(x)$ for all $x \in H$. Let $x, y \in H$. If $y \notin A_w$, then $\bar{A}(y) = \beta$ and so

$$\bar{A}(x) \geq \beta = \min\Big\{\bar{A}(y), \inf_{a \in x \circ y} \bar{A}(a)\Big\}. \tag{3.18}$$

If $x \circ y \nsubseteq A_w$, then there exists $u \in x \circ y \setminus A_w$, and therefore $\bar{A}(u) = \beta$. Hence,

$$\min\Big\{\bar{A}(y), \inf_{a \in x \circ y} \bar{A}(a)\Big\} = \beta \leq \bar{A}(x). \tag{3.19}$$

Assume that $x \circ y \subseteq A_w$ and $y \in A_w$. Then, $(x \circ y) \circ w \subseteq A$ and $y \circ w \subseteq A$, which imply that $(x \circ y) \circ w \ll A$ and $y \circ w \subseteq A$. It follows from (I3) that $x \circ w \subseteq A$, that is, $x \in A_w$. Hence

$$\bar{A}(x) = \alpha \geq \min\Big\{\bar{A}(y), \inf_{a \in x \circ y} \bar{A}(a)\Big\}, \tag{3.20}$$

and therefore $\bar{A}$ is a fuzzy weak hyper BCK-ideal of H. □

Lemma 3.19 (Jun et al. [5, Corollary 3.15])**.** *Let A be a reflexive hyper BCK-ideal of H. Then,*

$$(x \circ y) \cap A \neq \emptyset \Longrightarrow x \circ y \subseteq A \quad \forall x, y \in H. \tag{3.21}$$

Theorem 3.20. *If $\bar{A}$ is a fuzzy $\mathrm{PI}(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$-ideal of H in which $\bar{A}[\alpha]$ is reflexive for all $\alpha \in \mathrm{Im}(\bar{A})$, then the set*

$$\bar{A}_w = \{x \in H \mid x \circ w \subseteq \bar{A}[\alpha],\ \alpha \in \mathrm{Im}(\bar{A})\} \tag{3.22}$$

is a hyper BCK-ideal of H for all $w \in H$.

Proof. Clearly, $0 \in \bar{A}_w$. Let $x, y \in H$ be such that $x \circ y \subseteq \bar{A}_w$ and $y \in \bar{A}_w$. Then, $(x \circ y) \circ w \subseteq \bar{A}[\alpha]$ and $y \circ w \subseteq \bar{A}[\alpha]$ for all $\alpha \in \mathrm{Im}(\bar{A})$. Using (P5), we know that $(x \circ y) \circ w \ll \bar{A}[\alpha]$. Since $\bar{A}[\alpha]$ is a $\mathrm{PI}(\ll,\subseteq,\subseteq)_{\mathrm{BCK}}$-ideal of H (see Theorem 3.10), it follows from (I3) that $x \circ w \subseteq \bar{A}[\alpha]$, that is, $x \in \bar{A}_w$. This shows that $\bar{A}_w$ is a weak hyper BCK-ideal of H. Now, let $x, y \in H$ be such that $x \circ y \ll \bar{A}_w$ and $y \in \bar{A}_w$, and let $a \in x \circ y$. Then, there exists $b \in \bar{A}_w$ such that $a \ll b$, that is, $0 \in a \circ b$. Hence, $(a \circ b) \cap \bar{A}[\alpha] \neq \emptyset$. Since $\bar{A}[\alpha]$ is a reflexive hyper BCK-ideal of H, it follows from (K1) and Lemma 3.19 that $(a \circ w) \circ (b \circ w) \ll a \circ b \subseteq \bar{A}[\alpha]$ so that $a \circ w \subseteq \bar{A}[\alpha]$ since $b \circ w \subset \bar{A}[\alpha]$. Thus, $a \in \bar{A}_w$ and so $x \circ y \subseteq \bar{A}_w$. Since $\bar{A}_w$ is a weak hyper BCK-ideal, it follows that $x \in \bar{A}_w$. Consequently, $\bar{A}_w$ is a hyper BCK-ideal of H. □

Acknowledgments. The authors would like to thank the referees for their valuable suggestions and comments. The first author was supported by the Korea Research Foundation grant KRF-2001-005-D00002.

References

[1] Y. B. Jun and W. H. Shim, *Some types of positive implicative hyper BCK-ideals*, Sci. Math. Jpn. **56** (2002), no. 1, 63–68.

[2] Y. B. Jun and X. L. Xin, *Scalar elements and hyperatoms of hyper BCK-algebras*, Sci. Math. **2** (1999), no. 3, 303–309.

[3] ______, *Fuzzy hyper BCK-ideals of hyper BCK-algebras*, Sci. Math. Jpn. **53** (2001), no. 2, 353–360.

[4] ______, *Positive implicative Hyper BCK-algebras*, Sci. Math. Jpn. **5** (2001), 67–76.

[5] Y. B. Jun, X. L. Xin, M. M. Zahedi, and E. H. Roh, *Strong hyper BCK-ideals of hyper BCK-algebras*, Math. Japon. **51** (2000), no. 3, 493–498.

[6] Y. B. Jun, M. M. Zahedi, X. L. Xin, and R. A. Borzoei, *On hyper BCK-algebras*, Ital. J. Pure Appl. Math. **8** (2000), 127–136.

[7] F. Marty, *Sur une generalization de la notion de groupe*, 8iem Congres Math Scandinaves, Stockholm, 1934, pp. 45–49 (French).

Young Bae Jun: Department of Mathematics Education, Gyeongsang National University, Chinju 660-701, Korea

E-mail address: ybjun@nongae.gsnu.ac.kr

Wook Hwan Shim: Department of Mathematics Education, Gyeongsang National University, Chinju 660-701, Korea

E-mail address: wookhan@hotmail.com