NESTED SYMMETRIC REPRESENTATION OF ELEMENTS OF THE SUZUKI CHAIN GROUPS

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We demonstrate an economic and concise method for representing the elements of groups involved in the Suzuki chain. For example, we represent each element of Suz : 2 by a permutation on 14 letters from $L_3(2) : 2$ followed by four words, each of length at most two, in 14, 36, 100, and 416 *involutory symmetric generators*, respectively. Such expressions will have an obvious advantage over permutations on 1782 provided that it is reasonably simple to multiply and invert them. We refer to this as *nested symmetric representation* of an element of the group.

1. Introduction. An element of order three in the class $3D$ (see [2]) of Conway's group Co_1 is centralized by $3 \times A_9$ and the chain of subgroups K_i (of Co_1) which are the centralizers of the subgroups A_i obtained by fixing all but i points in this A_9 is called Suzuki chain, see [2, 8, 9]. This chain of subgroups in Co_1, discovered by Thompson (unpublished), see [6], has been of interest in finite groups.

In the present work, each of the Suzuki chain groups emerges as a group G generated by a set of $|(K_i : 2) : (K_{i+1} : 2)|$ involutions whose set normalizer in G is isomorphic to $K_i : 2$. Most of these groups are constructed by hand using the double coset enumeration technique shown in [5].

The main purpose of this paper is to introduce the concept of *nested symmetric representation* of elements of a group. In general, if we wish to multiply and invert elements in a straightforward manner, we must represent them as either permutations or matrices. The two operations are particularly easy to perform on permutations. Moreover, the cycle shape of an element immediately yields its order, and often its conjugacy class. However, for large sporadic groups, the lowest degree of permutation representations are unmanageable. Operations on matrices are much more difficult and basic information about an element is not readily recovered from its matrix representation. The approach illustrated in this paper combines conciseness with acceptable ease of manipulation and makes hand calculations with the elements possible. Inversion and multiplication can be performed manually or mechanically [1] by means of short recursive algorithms.

2. Involutory symmetric generators of groups. Let G be a group and let $T = \{t_0, t_1, \ldots, t_{n-1}\}$ be a set of elements of order m in G. Making the definitions $T_i = \langle t_i \rangle$ and $\overline{T} = \{T_0, T_1, \ldots, T_{n-1}\}$ allows us to define $N = \mathcal{N}_G(\overline{T})$, the set normalizer in G of $\overline{T}$. We say that T is a *symmetric generating set* for G if the following two conditions hold:

(i) $G = \langle T \rangle$,

(ii) N permutes $\overline{T}$ transitively.

We call N the *control subgroup.* Conditions (i) and (ii) imply that G is a homomorphic image of the *progenitor*

$$m^{*n} : N, \tag{2.1}$$

where m^{*n} represents a free product of n copies of the cyclic group C_m and N is a group of automorphisms of m^{*n} which permutes the n cyclic subgroups by conjugation, see [3, 4, 5].

Since in this paper we are only concerned with involutory symmetric generators, we restrict our attention to the case $m = 2$ (while N will simply act by conjugation as permutations of the n involutory symmetric generators).

THEOREM 2.1. *All non-abelian finite simple groups can arise as finite homomorphic images of progenitors of the form* $2^{*n} : N$.

PROOF. Let H be a maximal subgroup of a finite simple group G. Suppose that $1 \neq \mathbf{t} \in G$, $\mathbf{t}^2 = 1$. Under the subgroup H, $\mathbf{t}^G$, the conjugacy class of $\mathbf{t}$ in G, splits into orbits as

$$\mathbf{t}^G = \mathcal{T}_1 \dot{\cup} \mathcal{T}_2 \dot{\cup} \cdots \dot{\cup} \mathcal{T}_r. \tag{2.2}$$

Without loss of generality, we may assume that $\mathcal{T}_1 = \{\mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_{n-1}\}$ is not a subset of H. It is clear that

$$\mathcal{N}_G(\langle \mathcal{T}_1 \rangle) \geq \langle H, \mathcal{T}_1 \rangle = G \tag{2.3}$$

since H is maximal in G and $\mathcal{T}_1$ is not a subset of H. Therefore,

$$1 \neq \langle \mathcal{T}_1 \rangle \lhd G, \tag{2.4}$$

and, since G is simple, we have

$$\langle \mathcal{T}_1 \rangle = G. \tag{2.5}$$

Moreover, if $\pi \in H$ and $\mathbf{t}_i^\pi = \mathbf{t}_i$ $(i = 0, 1, \ldots, n-1)$, then $\pi \in \mathcal{Z}(G)$ and so $\pi = 1$, that is, H permutes the elements of $\mathcal{T}_1$ faithfully (and transitively). Now, let 2^{*n} denote a free product of n copies of the cyclic group C_2 with involutory generators $t_0, t_1, \ldots, t_{n-1}$ and let $N \cong H$ consist of all automorphisms of 2^{*n} which permute the t_i as H permutes the $\mathbf{t}_i$:

$$\pi^{-1} t_i \pi = t_i^\pi = t_{\pi(i)} \quad \text{for } \pi \in N. \tag{2.6}$$

Then, clearly G is a homomorphic image of $2^{*n} : N$, a split extension of 2^{*n} by the permutation automorphisms N. $\qquad \square$

Since the progenitor is a semidirect product (of $\langle T \rangle$ with N), it follows that, in any homomorphic image G, we may use the equation

$$t_i \pi = \pi t_i^\pi = \pi t_{\pi(i)} \tag{2.7}$$

or $i\pi = \pi i^\pi$ as we will more commonly write (see below) to gather the elements of N over to the left. Another consequence of this is that a relation of the form $(\pi t_i)^n = 1$ for some $\pi \in N$ in a permutation progenitor becomes

$$\pi^n = t_i t_{\pi(i)} \cdots t_{\pi^{n-1}(i)}. \tag{2.8}$$

Each element of the progenitor can be represented as πw, where $\pi \in N$ and w is a word in the symmetric generators. Indeed, this representation is unique provided that w is simplified so that those adjacent symmetric generators are distinct. Thus any additional relator by which we must factor the progenitor to obtain G must have the form $\pi w(t_0, t_1, \ldots, t_{n-1})$, where $\pi \in N$ and w is a word in T.

Now, if NxN is a double coset of N in G, we have

$$NxN = N\pi w N = NwN, \tag{2.9}$$

where $x = \pi w \in G$, with $\pi \in N$, and w a word in the symmetric generators. We denote this double coset by $[w]$; for example, $[01]$ denotes the double coset $Nt_0 t_1 N$. The double coset $NeN = N$, where e is the identity element, is denoted by $[*]$.

NOTATION 2.2. We will allow i to stand for the coset Nt_i, ij for the coset $Nt_i t_j$, and so on. We will also let i stand for the symmetric generator t_i when there is no danger of confusion. Thus we write, for instance, $ij \sim k$ to mean $Nt_i t_j = Nt_k$ and $ij = k$ to mean $t_i t_j = t_k$.

We define the subgroups $N^i, N^{ij}, N^{ijk}, \ldots$ (for $i, j,$ and k distinct) as follows:

$$N^i = \mathscr{C}_N(\langle t_i \rangle), \qquad N^{ij} = \mathscr{C}_N(\langle t_i, t_j \rangle), \qquad N^{ijk} = \mathscr{C}_N(\langle t_i, t_j, t_k \rangle) \tag{2.10}$$

or, more generally,

$$N^{i_1 i_2 \cdots i_m} = \mathscr{C}_N(\langle t_{i_1}, t_{i_2}, \ldots, t_{i_m} \rangle) \tag{2.11}$$

for $i_1, i_2, \ldots, i_m$ distinct.

Let g be an element of G. Then we define

$$N^{(g)} = \{\pi \in N \mid Ng\pi = Ng\}, \tag{2.12}$$

where $N^{(g)}$ is called the *coset stabilizing subgroup* (of Ng in N). Clearly, $N^w \le N^{(w)}$ for w being a word in the symmetric generators and the number of cosets in the double coset $[w] = NwN$ is given by $|N|/|N^{(w)}|$, see [5, 7].

LEMMA 2.3. *In* $2^{*}n : N$, $\langle t_i, t_j \rangle \cap N \le \mathscr{C}_N(N^{ij})$.

PROOF. Suppose that some element of G is in $\langle t_i, t_j \rangle$ and N; then it must centralize everything in N^{ij}—since, by definition, everything in N^{ij} commutes with everything in $\langle t_i, t_j \rangle$—and also in N, see [4]. Of course, this result can be readily generalized to more than two symmetric generators, and the general result is

$$\langle t_{i_1}, t_{i_2}, \ldots, t_{i_m} \rangle \cap N \le \mathscr{C}_N(N^{i_1 i_2 \cdots i_m}). \tag{2.13}$$
$\square$

LEMMA 2.4. *In* $(2^n : N)/(\pi_{ab} = aba)$ *(ab is called a special pair),*

$$N^{(ij)} \ge \langle N^{ij}, \pi_{kj} \text{ with } ik \text{ and } kj \text{ special pairs} \rangle. \tag{2.14}$$

PROOF. Since $ij \sim ik \cdot kj \sim i \cdot \pi_{kj} \cdot k \sim i^{\pi_{kj}} \cdot k = (ij)^{\pi_{kj}}$, then π_{kj} fixes the coset ij. Then $\pi_{kj} \in N^{(ij)}$. $\square$

3. Manual double coset enumeration.

It is now clear that we intend to take our progenitor of shape $2^{*n} : N$, where N is a transitive permutation group on n letters. A canonical presentation for this progenitor is

$$\langle x, y, t \mid \langle x, y \rangle \cong N, \ t^2 = 1 = [N^0, t] \rangle, \tag{3.1}$$

where x and y generate N, and t corresponds to t_0. The meaning of $1 = [N^0, t]$ is that we adjoin the relations $1 = [x_1, t] = [x_2, t] = \cdots = [x_p, t]$, where $x_1, x_2, \ldots, x_p$ generate N^0, and $\langle x, y \rangle \cong N$ means that we adjoin sufficient relations between x and y to define N.

3.1. The progenitor $2^{*(7+7)} : (L_3(2) : 2)$. A presentation for the progenitor is

$$\langle x, y, s \mid x^2 = y^3 = (xy)^8 = [x, y]^4 = s^2 = [s, y] = [s, [x, y]^2] = 1 \rangle, \tag{3.2}$$

where the action of the elements of the control subgroup $N \cong L_3(2) : 2$ on the 14 symmetric generators may be given by $x = (0, 0)(1, 1)(2, 2)(3, 3)(4, 4)(5, 5)(6, 6)$ and $y = (0, 5, 6)(1, 2, 4)(1, 6, 5)(2, 4, 3)$. Here the symmetric generators are denoted by seven points 0,1,2,3,4,5, and 6, and seven lines 0,1,2,3,4,5, and 6 in the projective plane shown in Figure 3.1. In order to obtain a finite homomorphic image of such a progenitor, we must factor by some additional relations. There are three two-point stabilizers N^{ij}, depending on whether i is a point and j is a line not through it, i is a point and j is a line through it, or i and j are two different points or two different lines. Now consider the first case

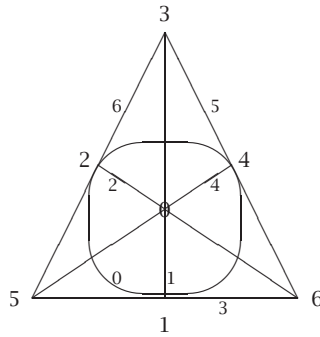$$N^{00} = \langle (2, 4)(5, 6)(2, 4)(5, 6), (1, 2)(3, 6)(1, 2)(3, 6) \rangle \cong S_3, \tag{3.3}$$

FIGURE 3.1. Seven-point projective plane.

which is centralized by the involution $\pi_{00} = (0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)$. Lemma 2.3 stated that π_{00} is the only permutation of N which can be written in terms of s_0 and s_0. We make the assumption that $\pi_{00} = s_0 s_0 s_0$, a word in the symmetric generators s_0 and s_0 of the shortest length that does not lead to collapse. Also

$$N^{65} = \langle (0,1)(2,5)(0,4)(2,3), (0,2)(1,5)(0,4)(1,6),$$
$$(0,1,2,5)(3,4)(0,6,4,1)(2,3) \rangle \tag{3.4}$$
$$\cong D_8,$$

which is centralized by the involution $\pi_{65} = (0,2)(1,5)(0,4)(1,6)$. If we can write the involution π_{65} as a word in the symmetric generators s_6 and s_5, the shortest possibility is $\pi_{65} = (s_6 s_5)^2$. The third relation is $\pi_{01} = (s_0 s_1)^2$, where $\pi_{01} = (2,5)(4,6)(0,3)(2,4) \in \mathscr{C}_N(N^{01})$.

Consider the group

$$G \cong \frac{2^{*(7+7)} : (L_3(2) : 2)}{\pi_{00} = s_0 s_0 s_0, \pi_{65} = (s_6 s_5)^2}, \tag{3.5}$$

from which a simple presentation follows:

$$\langle x, y, s \mid x^2 = y^3 = (xy)^8 = [x,y]^4 = s^2 = [s,y]$$
$$= [s,[x,y]^2] = (sx)^3 = (xys^{xyx})^4 = 1 \rangle. \tag{3.6}$$

We are now in a position to carry out the double coset enumeration of G over N. The set of all double cosets $[w] = NwN$, the coset stabilizing subgroups $N^{(w)}$, and the number of single cosets each contains are shown in Table 3.1. The double coset enumeration shows that the group defined by the symmetric presentation contains a homomorphic image of $L_3(2) : 2$ to index at most 36, and gives a convenient name to each of the 36 cosets in terms of 14 symmetric generators. Moreover, the action of the generators on the 36 cosets, by right

TABLE 3.1. Double coset enumeration for $U_3(3):2$.

| Label $[w]$	Coset stabilizing subgroup $N^{(w)}$	No. of cosets
$[*]$	N	1
$[0]$	$N^0 = N^{(0)} \cong S_4$, with orbits $1+6+3+4$ on the 14 points	14
$[00] = [0]$	Since $000 \sim * \Rightarrow 00 \sim 0$	
$[01]$	$N^{01} \cong V_4$. Since $01 \sim 05 \cdot 51 \sim 05 \cdot \pi_{15} \cdot 5 \sim 61 \cdot 5 \sim 65$, $\pi_{15} = (0,6)(1,5)(2,4)(3,1)(4,0)(5,2)(6,3)$ $\quad \Rightarrow N^{(01)} \geq \langle (2,4)(5,6)(2,4)(5,6), \rho \rangle \cong D_{12}$, $\rho = (0,6,1,5)(3,1)(2,0,4,4,5,3,6,2)$, with orbits $2+4+8$ on the 14 points	21
$[01] = [01]$	$01 \sim 06 \cdot 61 \sim 06 \cdot 616 \cdot 6 \sim 06 \cdot \pi_{61} \cdot 6$ $\quad \sim 216 \sim 1 \cdot 26 \sim 46 \cdot 2 \sim 42 \sim 13 \sim 31$, $\pi_{61} = (0,2)(1,3)(2,4)(3,5)(4,6)(5,0)(6,1)$	
$[012] = [01]$	$012 \sim 312 \sim 3 \cdot 121 \cdot 1 \sim 3 \cdot \pi_{21} \cdot 1 \sim 61 \sim 46$, $\pi_{21} = (0,2)(1,0)(2,1)(3,6)(4,3)(5,5)(6,4)$. Also $31 \sim 36 \cdot 61 \sim 36 \cdot 616 \cdot 6 \sim 36 \cdot \pi_{61} \cdot 6$ $\quad \sim 516 \sim 1 \cdot 56 \sim 66 \cdot 5 \sim 65 \sim 56 \sim 01 \sim 10$	
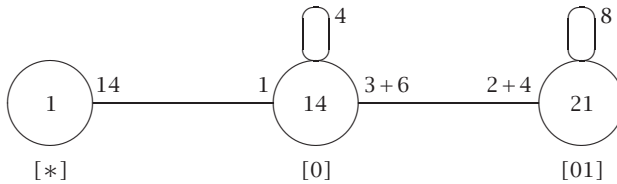
FIGURE 3.2

multiplication, is implicit in the enumeration and so it is readily checked that these permutations satisfy the given relations. Thus, $|G : N| \leq 36$, so $|G| \leq 12\,096 = |U_3(3):2|$, and the (relatively) easy task of finding generators for $U_3(3):2$ satisfying the required relations completes the identification of G with $U_3(3):2$. Table 3.1 shows that the Cayley graph of G over N has the form shown in Figure 3.2.

We conclude our work on G by giving symmetrically represented generators for each of the maximal subgroups of: $G \cong \langle (0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)$, $(0,5,6)(1,2,4)(1,6,5)(2,4,3), s_0 \rangle : U_3(3) \cong \langle (0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)s_0$, $(0,5,6)(1,2,4)(1,6,5)(2,4,3) \rangle$ is a simple subgroup of index 2 in G;

$$3_+^{1+2} : 8 : 2 \cong \langle (0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6),$$
$$(0,2,2,0,4,5,3,1)(1,4,6,6)(5,3)s_6 s_3 \rangle \tag{3.7}$$

is a subgroup of order 432 with index 28 in G and is the normalizer of

$$(0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)s_6, \tag{3.8}$$

an element of order 3 in class $3A$;

$$L_3(2):2 \cong \big\langle (0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6), \tag{3.9}$$
$$(0,5,6)(1,2,4)(1,6,5)(2,4,3)\big\rangle$$

is a subgroup of order 336 with index 36 in G and is the stabilizer of a point in the 36-point graph; and

$$4 \cdot S_4 : 2 \cong \big\langle s_0, (2,6)(4,5)(0,3)(5,6)s_1s_0 \big\rangle \tag{3.10}$$

is a subgroup of order 192 with index 63 in G and is the centralizer of

$$(2,4)(5,6)(2,4)(5,6), \tag{3.11}$$

an involution in class $2A$.

3.2. The progenitor $2^{*36}:(U_3(3):2)$. A presentation for the progenitor is

$$\big\langle x,y,s,r \mid x^2 = y^3 = (xy)^8 = [x,y]^4 = s^2 = [s,y] = [s,[x,y]^2]$$
$$= (xs)^3 = (xys^{xyx})^4 = r^2 = [r,x] = [r,y] = 1\big\rangle. \tag{3.12}$$

Here the symmetric generators are labeled with the vertices of the 36-point graph as follows. A single vertex is labeled ∞, 14 vertices are labeled 0,1,2,3,4, 5,6,0,1,2,3, 4,5, and 6, and the 21 vertices are labeled by the 21 flags (a point with a line through it) 0_1, 0_2, 0_4, 1_0, 1_1, 1_3, 2_0, 2_2, 2_6, 3_1, 3_5, 3_6, 4_0, 4_4, 4_5, 5_3, 5_4, 5_6, 6_2, 6_3, and 6_5. We would like to know which elements of the control subgroup $N \cong U_3(3):2$ can be written in terms of two symmetric generators. Lemma 2.3 says that $\mathscr{C}_N(N^{\infty 0}) = \langle s \rangle$. We make the assumption that $\pi_{\infty 0}(=s) = r_\infty r_0 r_\infty$, a word in the symmetric generators r_∞ and r_0 of the shortest length that does not lead to collapse.

Consider the group

$$G \cong \frac{2^{*36}:(U_3(3):2)}{\pi_{\infty 0} = r_\infty r_0 r_\infty}, \tag{3.13}$$

from which a simple presentation follows:

$$\big\langle x,y,s,r \mid x^2 = y^3 = (xy)^8 = [x,y]^4 = s^2 = [s,y] = [s,[x,y]^2]$$
$$= (xs)^3 = (xys^{xyx})^4 = r^2 = [r,x] = [r,y] = (sr)^3 = 1\big\rangle. \tag{3.14}$$

The double cosets and coset stabilizing subgroups are shown in Table 3.2. The double coset enumeration yields a Cayley diagram of G over N (see Figure 3.3).

TABLE 3.2. Double coset enumeration for $J_2 : 2$.

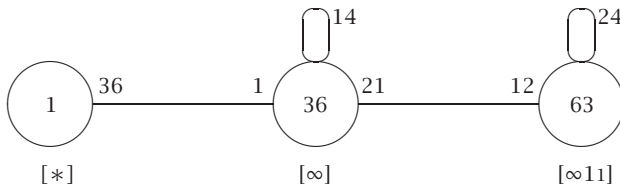| Label $[w]$	Coset stabilizing subgroup $N^{(w)}$	No. of cosets
$[*]$	N	1
$[\infty]$	$N^\infty = N^{(\infty)} \cong L_3(2) : 2$, with orbits $1 + 14 + 21$ on the 36 points	36
$[\infty 0] = [\infty]$	Since $\infty 0 \infty \sim * \Rightarrow \infty 0 \sim \infty$	
$[\infty 11]$	$N^{\infty 11} = \langle (0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6), (0,3) (2,5,4,6)(0,3)(2,6,4,5), (2,4)(5,6)(0,3)(2,4) \rangle \cong D_{16}$. Each element i of the $(2+4)$-orbits of D_{16} is joined to both ∞ and 11, that is, ∞i and $i11$ are special pairs $\Rightarrow N^{(\infty 11)} \geq \langle \pi_{011}, \pi_{111}, \pi_{311}, \overline{\pi}_{011}, \overline{\pi}_{111}, \overline{\pi}_{311}, (0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6), (2,4)(5,6)(0,3) (2,4), (0,3)(2,5,4,6)(0,3)(2,6,4,5) \rangle \cong 4 \cdot S_4 : 2$, $\pi_{011} = s_0 s_3 s_0 = (2,4)(5,6)(2,4)(5,6) s_3$, $\pi_{111} = s_1 s_1 s_1 = (2,4)(5,6)(2,4)(5,6) s_1$, $\pi_{311} = s_3 s_0 s_3 = (2,4)(5,6)(2,4)(5,6) s_0$, $\overline{\pi}_{011} = \overline{s}_0 \overline{s}_3 \overline{s}_0 = (2,4)(5,6)(2,4)(5,6) \overline{s}_3$, $\overline{\pi}_{111} = \overline{s}_1 \overline{s}_1 \overline{s}_1 = (2,4)(5,6)(2,4)(5,6) \overline{s}_1$, $\overline{\pi}_{311} = \overline{s}_3 \overline{s}_0 \overline{s}_3 = (2,4)(5,6)(2,4)(5,6) \overline{s}_0$, with orbits $12 + 24$ on the 36 points	63
$[\infty 112] = [\infty 11]$	$r_\infty r_{11} r_2 = r_{11} r_\infty r_2 = r_{11} \cdot \pi_{\infty 2} \cdot r_\infty = r_{11} s_2 r_\infty$ $\sim r_{13} r_\infty$ since $11 \cdot 2 \sim 112 \sim 1 \cdot 1212 \cdot 21$ $\sim 1 \cdot (0,5)(3,6)(1,3)(2,6) \cdot 21 \sim 321 \sim 31$	

FIGURE 3.3

One should note that the graph obtained above is not the regular graph, but the Cayley one. The regular graph whose automorphism group is G is obtained from the above one by joining the coset w to the coset iw. We may readily construct our symmetric generators as permutations of $1 + 36 + 63 = 100$ letters and verify that they do indeed satisfy the relations we assumed, thus proving that the group G has order $12\,096 \times 100 = 1\,209\,600 = |J_2 : 2|$. Identifying G with $J_2 : 2$ is straightforward and follows immediately from the construction of $J_2 : 2$ as the automorphism group of a rank-3 graph on 100 points, with suborbits 1, 36, and 63, and point stabilizer $U_3(3) : 2$, see [2].

Every element of G can be represented by a permutation on 36 letters followed by a word in the symmetric generators of length at most two. Alternatively and more concisely, we can represent each element of G by an expression of the form $\pi u v$, where π is a permutation on 14 letters (element of $L_3(2):2$) and u and v are words of length at most two in the symmetric generators s's and r's, respectively. The *nested* symmetrical representation of generators for each of the maximal subgroups of $G \cong \langle (0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6),$
$(0,5,6)(1,2,4)(1,6,5)(2,4,3), s_0, r_\infty \rangle$

$$
\begin{aligned}
J_2 \cong \langle &(0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)s_0, (0,5,6)(1,2,4)(1,6,5)(2,4,3), \\
&(0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)r_\infty \rangle
\end{aligned}
\tag{3.15}
$$

is a simple subgroup of index 2 in G;

$$
\begin{aligned}
U_3(3):2 \cong \langle &(0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6), \\
&(0,5,6)(1,2,4)(1,6,5)(2,4,3), s_0 \rangle
\end{aligned}
\tag{3.16}
$$

is a subgroup of order 12096 with index 100 in G and is the stabilizer of a point in the 100-point graph;

$$
3 \cdot A_6 \cdot 2^2 \cong \langle (0,3)(1,1)(2,5)(3,0)(4,6)(5,4)(6,2)s_6 s_2 r_3 r_{02}, s_0 \rangle
\tag{3.17}
$$

is a subgroup of order 4320 with index 280 in G and is the normalizer of

$$
(0,3)(1,1)(2,6)(3,0)(4,5)(5,2)(6,4)s_3,
\tag{3.18}
$$

an element of order 3 in class $3A$;

$$
\begin{aligned}
2_-^{1+4}:S_5 \cong \langle &(0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)s_0, \\
&(0,3)(5,6)(2,6)(4,5)s_3 s_1 r_\infty \rangle
\end{aligned}
\tag{3.19}
$$

is a subgroup of order 3840 with index 315 in G and is the normalizer of

$$
(2,4)(5,6)(2,4)(5,6),
\tag{3.20}
$$

an involution in class $2A$;

$$
\begin{aligned}
&2^{2+4}:(3 \times S_3):2 \\
&\cong \langle (0,5,6)(1,2,4)(1,6,5)(2,4,3), (1,2)(3,6)(1,2)(3,6), s_0, r_1 \rangle
\end{aligned}
\tag{3.21}
$$

is a subgroup of order 2304 with index 525 in G and is the normalizer of

$$
\langle (0,5)(3,6)(1,3)(2,6), (0,6)(3,5)(1,3)(4,5) \rangle,
\tag{3.22}
$$

a four-group whose involutions are in class $2A$;

$$
\begin{aligned}
(A_4 \times A_5):2 \cong \langle &(0,0)(1,4)(2,2)(3,5)(4,1)(5,3)(6,6)s_6 r_6, \\
&(1,2)(3,6)(1,2)(3,6)s_0 r_\infty \rangle
\end{aligned}
\tag{3.23}
$$

is a subgroup of order 1440 with index 840 in G;

$$(A_5 \times D_{10}) \cdot 2 \cong \langle (0,0,3,4,2,1,4,6)(1,5,5,2)(6,3)s_6 s_5, \\ (0,1)(1,2,3,4)(2,5,5,0,6,6,4,3)s_6 r_3 r_{53} \rangle \tag{3.24}$$

is a subgroup of order 1200 with index 1008 in G and is the normalizer of

$$(0,2,6)(1,3,5)(0,5,4)(1,6,3)s_2 s_2 r_4 r_{36}, \tag{3.25}$$

an element of order 5 in class $5A$;

$$L_3(2):2\times 2 \cong \langle (0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6), \\ (0,5,6)(1,2,4)(1,6,5)(2,4,3),r_\infty \rangle \tag{3.26}$$

is a subgroup of order 672 with index 1800 in G and is the centralizer of r_∞, an involution in class $2C$;

$$5^2:(4 \times S_3) \cong \langle (0,6,5,4,3,2,1)(0,1,2,3,4,5,6)s_4 r_0, \\ (0,3,2)(1,5,6)(0,4,5)(1,6,2)s_6 s_3, \\ (0,6)(1,3)(2,2)(3,4)(4,5)(5,1)(6,0)s_6 \rangle \tag{3.27}$$

is a subgroup of order 600 with index 2016 in G; and

$$S_5 \cong \langle (0,2,6)(1,3,5)(0,5,4)(1,6,3)s_2 s_2 r_4 r_{36}, r_\infty \rangle \tag{3.28}$$

is a subgroup of order 120 with index 10080 in G.

There are 63 such sets of six couples corresponding to the choices of $N^{(ij)} (\cong 4 \cdot S_4 : 2)$ in N. Now we label each set of six couples corresponding to $N^{(ij)}$ by $i.j$. It is clear that $i.j$ has 6 (unordered) images under $N^{(ij)}$. Examples of set of six couples are

$$\infty.1_1 \equiv \begin{cases} (\infty, 1_1), \\ (1,1), \\ (0,3), \\ (0,3), \\ (1_0,1_3), \\ (0_1,3_1), \end{cases} \qquad 2.1_3 \equiv \begin{cases} (2,1_3), \\ (4,1_1), \\ (1,4_0), \\ (3,2_0), \\ (0_4,3_5), \\ (5_6,6_2). \end{cases} \tag{3.29}$$

3.3. The progenitor $2^{*100}:(J_2:2)$. A presentation for the progenitor is

$$\langle x,y,s,r,q \mid x^2 = y^3 = (xy)^8 = [x,y]^4 = s^2 = [s,y] = [s,[x,y]^2] \\ = (xs)^3 = (xys^{xyx})^4 = r^2 = [r,x] = [r,y] = (sr)^3 \tag{3.30} \\ = q^2 = [q,x] = [q,y] = [q,s] = 1 \rangle.$$

TABLE 3.3. Double coset enumeration for $G_2(4):2$.

| Label $[w]$	Coset stabilizing subgroup $N^{(w)}$	No. of cosets
$[*]$	N	1
$[x]$	$N^x = N^{(x)} \cong U_3(3):2$, with orbits $1+36+63$ on the 100 points	100
$[x\infty] = [x]$	Since $x\infty x \sim * \Rightarrow x\infty \sim x$	
$[x0.5]$	$N^{x0.5} = \langle (0,5)(1,6)(2,3)(3,1)(4,4)(5,0)(6,2)s_4s_4,$ $(1,2)(3,6)(1,2)(3,6),(1,6)(2,3)(0,5)(1,2),(1,3)$ $(2,6)(0,5)(3,6)s_4s_5 \rangle \cong 2 \cdot S_4:2.$ Each element i of the 12-orbits of $2 \cdot S_4:2$ is joined to both of x and 0.5, that is, xi and i0.5 are special pairs $\Rightarrow N^{(x0.5)} \geq \langle \pi_{\infty 0.5}, \pi_{00.5}, \pi_{40.5},$ $\pi_{50.5}\pi_{00.5}, \pi_{40.5}, \pi_{50.5}, \pi_{040.5}, \pi_{400.5}, \pi_{440.5}, \pi_{450.5},$ $\pi_{540.5}, (1,3)(2,6)(0,5)(3.6)s_4s_5, (0,5)(1,6)(2,3)$ $(3,1)(4,4)(5,0)(6,2)s_4s_4 \rangle \cong 2^{1+4}_-.S_5,$ with $20+80$ orbits on 100 points	315
$[x0.51] = [x0.5]$	$q_x q_{0.5} q_1 = q_{0.5} q_x q_1 = q_{0.5} \pi_{x1} q_x = q_{0.5} r_1 q_x \sim q_{0.4} q_x$ since $r_0 r_5 r_1 = r_0 r_5 r_1 r_5 r_5 = r_0 \pi_{15} r_5 \sim r_4 r_5 \sim r_0 r_4$	
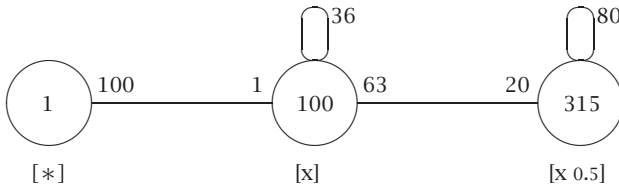
FIGURE 3.4

Here the symmetric generators are labeled with the vertices of the 100-point graph. A single vertex is labeled x, 36 vertices are labeled $\infty,0,1,2,3,4,5,6,0_1,2,3,$ $4,5,6,0_1, 0_2, 0_4, 1_0, 1_1, 1_3, 2_0, 2_2, 2_6, 3_1, 3_5, 3_6, 4_0, 4_4, 4_5, 5_3, 5_4, 5_6, 6_2, 6_3,$ and 6_5 and 63 vertices are labeled by the elements of the set of all six couples. We would like to know which elements of the control subgroup $N \cong J_2:2$ can be written in terms of two symmetric generators. Lemma 2.3 says that $\mathscr{C}_N(N^{x\infty}) = \langle r \rangle$. We make the assumption that $\pi_{x\infty}(=r) = q_x q_\infty q_x$, a word in the symmetric generators q_x and q_∞ of the shortest length that does not lead to collapse.

Consider the group

$$G \cong \frac{2^{*100}:(J_2:2)}{\pi_{x\infty} = q_x q_\infty q_x}. \tag{3.31}$$

The double cosets and coset stabilizing subgroups are shown in Table 3.3. The double coset enumeration yields a Cayley diagram of G over N (see Figure 3.4).

The coset enumeration shows that the group defined by the symmetric presentation contains a homomorphic image of N to index at most $1+100+315 = 416$, and gives a convenient name to each 416 cosets in terms of 100 symmetric generators. Moreover, the action of the generators on the 416 cosets, by right multiplication, is implicit in the enumeration and so it is readily checked that these permutations satisfy the given relation. Moreover, G has order $|J_2 : 2| \times 416 = 503\,193\,600$, and G is the automorphism group of a rank-3 graph (obtained from the above diagram by joining the coset w to the coset iw) of valence 100 on 416 points in which the point stabilizer is $J_2 : 2$. This is, of course, the group $G_2(4) : 2$, see [2].

Every element of G can be represented by a permutation on 100 letters (elements of $J_2 : 2$) followed by a word in the symmetric generators of length at most two. Also each element of G can be represented by the expression $\pi u v w$, where π is a permutation on 14 letters (element of $L_3(2) : 2$) and u, v, w are words of length at most two in the symmetric generators s's, r's, and q's, respectively. We refer to this as *nested* symmetric representation of an element of the group. The (nested) symmetrically represented generators for each of the maximal subgroups of $G \cong \langle (0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6), (0,5,6)(1,2,4)(1,6,5)(2,4,3), s_0, r_\infty, q_x \rangle$

$$G_2(4) \cong \langle (0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)s_0 r_\infty q_x, \tag{3.32}$$
$$(0,5,6)(1,2,4)(1,6,5)(2,4,3) \rangle$$

is a simple group of index 2 in G;

$$J_2 : 2 \cong \langle (0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6), \tag{3.33}$$
$$(0,5,6)(1,2,4)(1,6,5)(2,4,3), s_0, r_\infty \rangle$$

is a subgroup of index 416 in G and is the stabilizer of a point in the 416-point graph;

$$2^{2+8} : (3 \times A_5) : 2 \cong \langle (0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)r_2 r_{13} q_6 q_{1.65}, s_0 \rangle \tag{3.34}$$

is a subgroup of index 1365 in G;

$$2^{4+6} : (A_5 \times 3) : 2 \cong \langle (0,1,2,6,5,3,4)(0,2,3,6,5,4,1)r_1 q_1 q_{4.63}, \tag{3.35}$$
$$(0,1)(2,4,5,6)(0,4,3,2)(5,6)s_6 s_5 r_2 r_{54} q_{3.53} \rangle$$

is a subgroup of index 1365 in G;

$$U_3(4) : 4 \cong \langle (0,5,2)(3,6,4)(0,1,3)(2,6,4)s_5 r_{26}, \tag{3.36}$$
$$(0,1,4)(2,5,3)(0,4,1)(2,3,5)s_6 r_{22} q_{11} \rangle$$

is a subgroup of index 2016 in G;

$$
\begin{aligned}
3 \cdot L_3(4) : 2^2 \cong \langle &(0,0)(1,1)(2,4)(3,3)(4,2)(5,6)(6,5)q_x, \\
&(0,5)(1,6,2,3)(0,5)(1,3,2,6)s_4, \\
&(0,5,5,0)(1,1,3,6,2,2,6,3)(4,4)r_0 r_5 \rangle
\end{aligned}
\tag{3.37}
$$

is a subgroup of index 2080 in G and is the normalizer of

$$
(0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)s_0,
\tag{3.38}
$$

an element of order 3 in class $3A$;

$$
\begin{aligned}
U_3(3) : 2 \times 2 \cong \langle &(0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6), \\
&(0,5,6)(1,2,4)(1,6,5)(2,4,3), s_0, q_x \rangle
\end{aligned}
\tag{3.39}
$$

is a subgroup of index 20800 in G and is the centralizer of q_x, an involution in class $2C$;

$$
\begin{aligned}
(A_5 \times A_5) : 2 \cong \langle &(0,5,1,4,6,1,4,3)(2,6)(3,0,5,2)s_0 r_0 r_2 q_3, \\
&(0,4,3)(1,5,6)(0,6,2)(1,4,5)s_4 s_0 r_1 r_{45} q_5 q_{3.40} \rangle
\end{aligned}
\tag{3.40}
$$

is a subgroup of index 69888 in G; and

$$
\begin{aligned}
L_2(13) : 2 \cong \langle &(0,4,4,0,2,2)(1,6,6,1,5,5)(3,3)s_0 r_\infty q_x, \\
&(0,6)(1,4)(1,5)(3,4)q_{0.6} \rangle
\end{aligned}
\tag{3.41}
$$

is a subgroup of index 230400 in G.

3.4. The progenitor $2^{*416} : (G_2(4) : 2)$. A presentation for the progenitor is

$$
\begin{aligned}
\big\langle x,y,s,r,q,p \mid &x^2 = y^3 = (xy)^8 = [x,y]^4 = s^2 = [s,y] = [s,[x,y]^2] \\
&= (xs)^3 = (xys^{xyx})^4 = r^2 = [r,x] = [r,y] = (sr)^3 \\
&= q^2 = [q,x] = [q,y] = [q,s] = (rq)^3 = [p,x] \\
&= [p,y] = [p,s] = [p,r] = 1 \big\rangle.
\end{aligned}
\tag{3.42}
$$

Here the symmetric generators which correspond to the vertices of the 416-point graph are denoted by $0, 1, \ldots, 415$, where $p_0 = p_{q*}$, $p_1 = p_{q_x}$, and $p_2 = p_{q_\infty}, \ldots$. We would like to know which elements of the control subgroup $N \cong G_2(4) : 2$ can be written in terms of two symmetric generators. Lemma 2.3 states that $\mathscr{C}_N(N^{01}) = \langle q \rangle$. We make the assumption that $\pi_{01} (= q) = p_0 p_1 p_0$, a word in the symmetric generators p_0 and p_1 of the shortest length that does not lead to collapse.

A nice way of looking at Suz : 2 is the way Suzuki constructed the group [2] as a rank-3 extension of $G_2(4) : 2$ of degree 1782 with suborbit sizes 1, 416, and 1365. We will be looking at the group from this point of view to identify our homomorphic image of the progenitor with Suz : 2.

TABLE 3.4. Double coset enumeration for $3 \cdot \mathrm{Suz} : 2$.

Label $[w]$	Coset stabilizing subgroup $N^{(w)}$	No. of cosets
$[*]$	N	1
$[0]$	$N^0 = N^{(0)} \cong J_2 : 2$, with orbits $1 + 100 + 315$ on the 416 points	416
$[01] = [0]$	Since $010 \sim * \Rightarrow 01 \sim 1$, 0 is an element of the 100-orbit	
$[0i]$	$N^{0i} \cong 2_-^{1+4} : S_5$, i is an element of the 315-orbit, $|N^{(0i)}| = 32 \times |\mathrm{Stab}_{N^{(0i)}}(0)| \geq 32 \times 3840$. $N^{(0i)} \cong 2^{2+8} : S_5$, thus the double coset $[0i]$ has at most 4095 single cosets, $N^{(0i)}$ has orbits $32 + 320 + 64$ on the 416 points	4095
$[0il] = [0i]$	$0il \sim 0\pi_{il}i \sim 0^{\pi_{il}}i$, i joined to l, l is an element of 320-orbit	
$[0ij]$	$N^{0ij} \cong 2_-^{1+4} : A_5$, j is an element of the 64-orbit since N^{0ij} is maximal in J_2, adding any permutation of N which fixes the coset $Np_0p_ip_j$ to $N^{0ij} \Rightarrow N^{(0ij)} \cong J_2$, with orbits $315 + 100 + 1$ on the 416 points	832
$[0ijl] = [0ij]$	$0ijl \sim 0i\pi_{jl}j \sim (0i)^{\pi_{jl}}j$, j joined to l, l is an element of 100-orbit	
$[0ijk]$	N^{0ijk} is maximal in $G_2(4)$, adding any permutation of N which fixes the coset $Np_0p_ip_jp_k$ to $N^{0ijk} \Rightarrow N^{(0ijk)} \cong G_2(4)$, which is transitive on the symmetric generators	2

Consider the group

$$G \cong \frac{2^{*416} : (G_2(4) : 2)}{\pi_{01} = p_0 p_1 p_0}. \tag{3.43}$$

The double cosets and coset stabilizing subgroups are given in Table 3.4. The double coset enumeration shown in Table 3.4 yields a Cayley diagram of G over N (see Figure 3.5).

The coset enumeration shows that the group defined by the symmetric presentation contains a homomorphic image of N to index at most $(1 + 416 + 4095 + 832 + 2) = 5346$, and gives a convenient name to each 5346 cosets in terms of 416 symmetric generators. Moreover, the action of the generators on the 5346 cosets, by right multiplication, is implicit in the enumeration and so it is readily checked that these permutations satisfy the given relation. Thus $|G| = |G_2(4) : 2| \times 5346 = 2\,690\,072\,965\,600$, and G is isomorphic to the group $3 \cdot \mathrm{Suz} : 2$. Finally, adding the relator $(\pi p_0)^{13}$ [2], where π induces a permutation of cyclic shape $1 \cdot 3 \cdot 4 \cdot 6^2 \cdot 12^{33}$ on 416 letters, to those of G gives $\mathrm{Suz} : 2$. Again the double coset enumeration over N gives a Cayley diagram (see Figure 3.6).
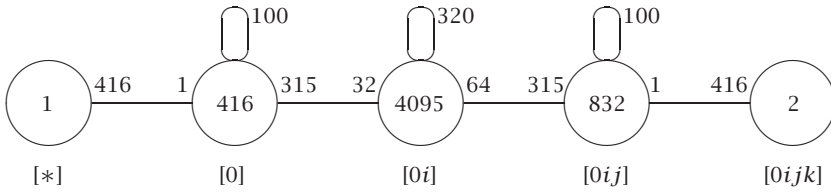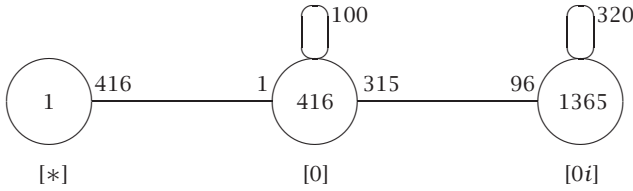
FIGURE 3.5

FIGURE 3.6

Every element of $G(\cong \mathrm{Suz}:2)$ can be represented by a permutation on 416 letters (elements of $G_2(4):2$) followed by a word in the symmetric generators of length at most two. Also each element of G can be represented by the expression $\pi u v w z$, where π is a permutation on 14 letters (element of $L_3(2):2$) and u, v, w, and z are words of length at most two in the symmetric generators s's, r's, q's, and p's, respectively. We refer to this as *nested* symmetric representation of an element of the group. The (nested) symmetrically represented generators for each of the maximal subgroups of $G \cong \langle (0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6), (0,5,6)(1,2,4)(1,6,5)(2,4,3), s_0, r_\infty, q_\mathrm{x}, p_0 \rangle$

$$
\begin{aligned}
\mathrm{Suz} \cong \langle & (0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)s_0, \\
& (0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)r_\infty, \\
& (0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)q_\mathrm{x}, \\
& (0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)p_0, \\
& (0,5,6)(1,2,4)(1,6,5)(2,4,3) \rangle
\end{aligned}
\tag{3.44}
$$

is a subgroup of index 2 in G;

$$
\begin{aligned}
G_2(4):2 \cong \langle & (0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6), \\
& (0,5,6)(1,2,4)(1,6,5)(2,4,3), s_0, r_\infty, q_\mathrm{x} \rangle
\end{aligned}
\tag{3.45}
$$

is a subgroup of index 1782 in G and is the stabilizer of a point in the 1782-point graph;

$$
3U_4(3):(2^2)_{133} \cong \langle (0,6,3,5)(2,4)(1,3)(2,5,6,4)s_0 s_2 r_0 r_6 p_0, s_0 q_\mathrm{x} \rangle
\tag{3.46}
$$

is a subgroup of index 22880 in G and is the normalizer of

$$(0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)s_0, \tag{3.47}$$

an element of order 3 in class $3A$;

$$2_-^{1+6} \cdot U_4(2):2 \cong \langle (0,0)(1,2)(2,1)(3,6)(4,4)(5,5)(6,3)s_0q_x,s_4r_\infty p_0 \rangle \tag{3.48}$$

is a subgroup of index 135135 in G and is the normalizer of $(1,2)(3,6)(1,2)(3,6)$, an involution in class $2A$;

$$J_2:2\times2 \cong \langle (0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6),$$
$$(0,5,6)(1,2,4)(1,6,5)(2,4,3),s_0,r_\infty,p_0 \rangle \tag{3.49}$$

is a subgroup of index 370656 in G and is the centralizer of p_0, an involution in class $2C$; and

$$M_{12}:2\times2 \cong \langle (0,6)(1,0,4,5,6,3)(2,4,3,2,5,1)r_5r_4q_xq_{4.5}p_0,$$
$$(1,6)(2,3)(0,5)(1,2)q_\infty p_0,(1,4)(3,5)(1,4)(3,5)s_6r_0 \rangle \tag{3.50}$$

is a subgroup of index 2358720 in G and is the centralizer of

$$(0,6)(1,5)(2,2)(3,1)(4,3)(5,4)(6,0)r_\infty p_0, \tag{3.51}$$

an involution in class $3D$.

3.5. The progenitor $(2^2)^{*1782}:(3\cdot\text{Suz}:2)$**.** A presentation for the progenitor is

$$\langle x,y,s,r,q,p,t \mid x^2 = y^3 = (xy)^8 = [x,y]^4 = s^2 = [s,y] = [s,[x,y]^2]$$
$$= (xs)^3 = (xys^{xyx})^4 = r^2 = [r,x] = [r,y]$$
$$= (sr)^3 = q^2 = [q,x] = [q,y] = [q,s] = (rq)^3$$
$$= [p,x] = [p,y] = [p,s] = [p,r] = (qp)^3$$
$$= [t,x] = [t,y] = [t,s] = [t,q] = 1 \rangle. \tag{3.52}$$

We seek a monomial semilinear 1782-dimensional representation of $N \cong 3 \cdot$ Suz$:2$ over GF$_4$, the Galois field of order 4. Elements in $3\cdot$Suz$:2$ can thus act as permutations of 1782 Klein four-groups, followed by the *field automorphism* σ of GF$_4$. The cyclic groups T_i are replaced by copies of the Klein four-groups which we label by $V_i = \langle t_{i_1}, t_{i_2}t_{i_3} \rangle$. The centralizer in N of V_1 is isomorphic to $G_2(4)$ with orbits $1 + 416 + 1365$. Centralizing a further four-group in the 416-orbit, say V_2, yeilds a subgroup isomorphic to J_2 whose centralizer in N
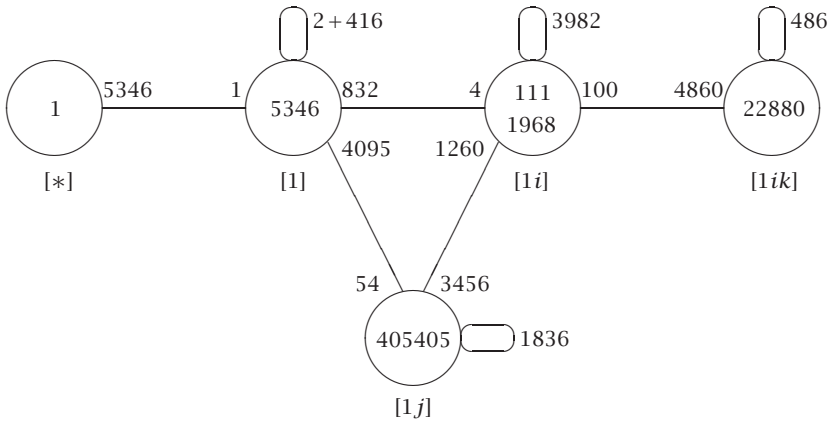
FIGURE 3.7

is a copy of S_3:

$$\langle V_1, V_2 \rangle \cap N \leq \mathscr{C}_N(N^{12}) \cong S_3. \tag{3.53}$$

Elements of order 3 in S_3 cycle the involutions in each of the two fixed four-groups, while its involutions interchange them and apply the field automorphism σ. We thus seek an image of $(2^2)^{*2} : S_3$, where

$$S_3 \cong \left\langle a = \begin{pmatrix} \omega & \cdot \\ \cdot & \omega \end{pmatrix}, b = \begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix} \sigma \right\rangle. \tag{3.54}$$

Factoring this by the relator $(bt_{1_1})^3$ is easily seen to give the image A_5. For example, if $a = (3,4,5)$, $b = (1,2)(4,5)$, $t_{1_1} = (1,3)(4,5)$, then

$$
\begin{aligned}
t_{1_1} &= (1,3)(4,5), & t_{2_1} &= (2,3)(4,5), \\
t_{1_2} &= \omega t_{1_1} = (1,4)(3,5), & t_{2_2} &= \omega t_{2_1} = (2,4)(3,5), \\
t_{1_3} &= \overline{\omega} t_{1_1} = (1,5)(3,4), & t_{2_3} &= \overline{\omega} t_{2_1} = (2,5)(3,4).
\end{aligned} \tag{3.55}
$$

If the progenitor $(2^2)^{*1782} : 3 \cdot \text{Suz} : 2$ is factored by a corresponding relation, we obtain

$$\text{Co}_1 \cong \frac{(2^2)^{*1782} : (3 \cdot \text{Suz} : 2)}{\left(\begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix} \sigma t_{1_1} \right)^3}. \tag{3.56}$$

The central element of order three [2], which fixes each of the 1782 four-groups while cycling its nontrivial element, is $([x,y]^2([x,y]^2)^y x([x,y]^2)^y xsrqp)^{13}$. With the help of the program in [5], the enumeration over N gives a Cayley diagram (see Figure 3.7).

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