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A GENERALIZATION OF MULHOLLAND'S INEQUALITY

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By introducing three parameters r, s, and λ, we give a generalization of Mulholland's inequality with a best constant factor involving the β function. As its applications, we also consider its equivalent form and some particular results.

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If $p > 1$, $1/p + 1/q = 1$, and $\{a_n\}$ and $\{b_n\}$ are nonnegative sequences of real numbers such that $0 < \sum_{n=2}^{\infty}(1/n)a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty}(1/n)b_n^q < \infty$, then the Mulholland's inequality is (cf. [4])

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{mn\ln mn} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=2}^{\infty}\frac{1}{n}a_n^p\right\}^{1/p}\left\{\sum_{n=2}^{\infty}\frac{1}{n}b_n^q\right\}^{1/q}. \tag{1}$$

Inequality (1) is similar to the well-known Hardy-Hilbert's inequality as (cf. [3])

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{1/q}. \tag{2}$$

In this paper, we introduce three parameters r, s, and λ to generalize Mulholland's inequality and then derive several equivalent forms of our generalized results with special cases.

Theorem 1. *If $p > 1$, $1/p + 1/q = 1$, $\{a_n\}$ and $\{b_n\}$ are nonnegative sequences of real numbers, $2 - \min\{p,q\} < \lambda \le 2$ and $r, s \in \mathbb{R}$, such that $0 < \sum_{n=2}^{\infty}((\ln n)^{1-\lambda}/n)(n^{1-r}a_n)^p < \infty$ and $0 < \sum_{n=2}^{\infty}((\ln n)^{1-\lambda}/n)(n^{1-s}b_n)^q < \infty$, then*

$$\begin{aligned}\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}&\frac{a_m b_n}{m^r n^s(\ln mn)^{\lambda}}\\ &< B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)\left\{\sum_{n=2}^{\infty}\frac{(\ln n)^{1-\lambda}}{n}(n^{1-r}a_n)^p\right\}^{1/p}\\ &\quad\times\left\{\sum_{n=2}^{\infty}\frac{(\ln n)^{1-\lambda}}{n}(n^{1-s}b_n)^q\right\}^{1/q},\end{aligned} \tag{3}$$

where the constant factor $B((p+\lambda-2)/p,(q+\lambda-2)/q)$ is the best possible. In particular,

(i) *for $r=1/q$ and $s=1/p$,*

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{m^{1/q}n^{1/p}(\ln mn)^{\lambda}} < B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)\left\{\sum_{n=2}^{\infty}(\ln n)^{1-\lambda}a_n^p\right\}^{1/p}\left\{\sum_{n=2}^{\infty}(\ln n)^{1-\lambda}b_n^q\right\}^{1/q}; \tag{4}$$

(ii) *for $\lambda=1$,*

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{m^{r}n^{s}\ln mn} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=2}^{\infty}\frac{1}{n}(n^{1-r}a_n)^p\right\}^{1/q}\left\{\sum_{n=2}^{\infty}\frac{1}{n}(n^{1-s}b_n)^q\right\}^{1/q}; \tag{5}$$

(iii) *for $r=s=0$,*

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{(\ln mn)^{\lambda}} < B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)\times\left\{\sum_{n=2}^{\infty}\frac{(\ln n)^{1-\lambda}}{n}(na_n)^p\right\}^{1/p}\left\{\sum_{n=2}^{\infty}\frac{(\ln n)^{1-\lambda}}{n}(nb_n)^q\right\}^{1/q}. \tag{6}$$

Proof. By Hölder's inequality, we find

$$\begin{aligned}
\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{m^{r}n^{s}(\ln mn)^{\lambda}}
&= \sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\left[\frac{a_m}{(\ln mn)^{\lambda/p}}\left(\frac{\ln m}{\ln n}\right)^{(2-\lambda)/pq}\left(\frac{m^{1/q-r}}{n^{1/p}}\right)\right] \\
&\qquad\times\left[\frac{b_n}{(\ln mn)^{\lambda/q}}\left(\frac{\ln n}{\ln m}\right)^{(2-\lambda)/pq}\left(\frac{n^{1/p-s}}{m^{1/q}}\right)\right] \\
&\le \left\{\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m^p}{(\ln mn)^{\lambda}}\left(\frac{\ln m}{\ln n}\right)^{(2-\lambda)/q}\left(\frac{m^{p(1-r)-1}}{n}\right)\right\}^{1/p} \\
&\quad\times\left\{\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{b_n^q}{(\ln mn)^{\lambda}}\left(\frac{\ln n}{\ln m}\right)^{(2-\lambda)/p}\left(\frac{n^{q(1-s)-1}}{m}\right)\right\}^{1/q} \\
&= \left\{\sum_{m=2}^{\infty}\omega_{\lambda}(q,m)m^{p(1-r)-1}a_m^p\right\}^{1/p}\left\{\sum_{n=2}^{\infty}\omega_{\lambda}(p,n)n^{q(1-s)-1}b_n^q\right\}^{1/q},
\end{aligned} \tag{7}$$

where the weight coefficient $\omega_\lambda(y,n)$ is defined by

$$\omega_\lambda(y,n) = \sum_{m=2}^{\infty} \frac{1}{m(\ln mn)^\lambda}\left(\frac{\ln n}{\ln m}\right)^{(2-\lambda)/y} \quad (y = p,q,\ n \in \mathbb{N}\setminus\{1\}). \tag{8}$$

For $0 \le 2-y < \lambda \le 2$ $(y = p,q)$, setting $u = \ln t/\ln n$ in (8), we have

$$\begin{aligned}\omega_\lambda(y,n) &< \int_0^\infty \frac{1}{t(\ln nt)^\lambda}\left(\frac{\ln n}{\ln t}\right)^{(2-\lambda)/y} dt\\ &= (\ln n)^{1-\lambda}\int_0^\infty \frac{1}{(1+u)^\lambda} u^{(\lambda-2)/y} du.\end{aligned} \tag{9}$$

Since for the β function $B(p,q)$, we have (cf. [5])

$$B(p,q) = \int_0^\infty \frac{1}{(1+u)^{p+q}} u^{-1+p} du = B(q,p) \quad (p,q > 0), \tag{10}$$

then by (9) and (10), we obtain

$$\omega_\lambda(y,n) < B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)(\ln n)^{1-\lambda} \quad (y = p,q,\ n \in N\setminus\{1\}). \tag{11}$$

In view of (7), (8), and (11), we have (3).

For $\varepsilon > 0$, such that $(2-\lambda+\varepsilon)/p < 1$, setting

$$\tilde{a}_n = \frac{1}{n^{1-r}(\ln n)^{(2-\lambda+\varepsilon)/p}}, \quad \tilde{b}_n = \frac{1}{n^{1-s}(\ln n)^{(2-\lambda+\varepsilon)/q}} \quad (n \in N\setminus\{1\}), \tag{12}$$

then we have

$$\begin{aligned}&\varepsilon\left\{\sum_{n=2}^{\infty} n^{p(1-r)-1}(\ln n)^{1-\lambda}\tilde{a}_n^p\right\}^{1/p}\left\{\sum_{n=2}^{\infty} n^{q(1-s)-1}(\ln n)^{1-\lambda}\tilde{b}_n^q\right\}^{1/q}\\ &\quad= \varepsilon\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^{1+\varepsilon}} < \varepsilon\left[\frac{1}{2(\ln 2)^2} + \frac{1}{3\ln 3} + \sum_{n=4}^{\infty}\frac{1}{n(\ln n)^{(1+\varepsilon)}}\right]\\ &\quad< \varepsilon\left[1 + \int_e^\infty \frac{1}{t(\ln t)^{1+\varepsilon}} dt\right]\\ &\quad= \varepsilon + 1.\end{aligned} \tag{13}$$

Putting $u = \ln x/\ln y$ in the following, we find

$$\begin{aligned}
&\varepsilon\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{\tilde{a}_m\tilde{b}_n}{m^r n^s(\ln mn)^{\lambda}}\\
&\quad=\varepsilon\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{1}{mn(\ln mn)^{\lambda}}\cdot\frac{1}{(\ln m)^{(2-\lambda+\varepsilon)/p}}\cdot\frac{1}{(\ln n)^{(2-\lambda+\varepsilon)/q}}\\
&\quad>\varepsilon\int_e^{\infty}\left[\int_e^{\infty}\frac{1}{xy(\ln xy)^{\lambda}}\cdot\frac{1}{(\ln x)^{(2-\lambda+\varepsilon)/p}}\cdot\frac{1}{(\ln y)^{(2-\lambda+\varepsilon)/q}}dx\right]dy\\
&\quad=\varepsilon\int_e^{\infty}\frac{1}{y(\ln y)^{1+\varepsilon}}\left[\int_{1/\ln y}^{\infty}\frac{1}{(1+u)^{\lambda}}\left(\frac{1}{u}\right)^{(2-\lambda+\varepsilon)/p}du\right]dy\\
&\quad=\varepsilon\int_e^{\infty}\frac{1}{y(\ln y)^{1+\varepsilon}}\left[\int_0^{\infty}\frac{1}{(1+u)^{\lambda}}\left(\frac{1}{u}\right)^{(2-\lambda+\varepsilon)/p}du\right]dy\\
&\qquad-\varepsilon\int_e^{\infty}\frac{1}{y(\ln y)^{1+\varepsilon}}\left[\int_0^{1/\ln y}\frac{1}{(1+u)^{\lambda}}\left(\frac{1}{u}\right)^{(2-\lambda+\varepsilon)/p}du\right]dy\\
&\quad\geq B\left(\frac{p+\lambda-2}{p}-\frac{\varepsilon}{p},\frac{q+\lambda-2}{q}+\frac{\varepsilon}{q}\right)\\
&\qquad-\varepsilon\int_e^{\infty}\frac{1}{y\ln y}\left[\int_0^{1/\ln y}\left(\frac{1}{u}\right)^{(2-\lambda+\varepsilon)/p}du\right]dy\\
&\quad=B\left(\frac{p+\lambda-2}{p}-\frac{\varepsilon}{p},\frac{q+\lambda-2}{q}+\frac{\varepsilon}{q}\right)-\frac{\varepsilon}{[1-(2-\lambda+\varepsilon)/p]^2}.
\end{aligned}\tag{14}$$

If the constant factor $B((p+\lambda-2)/p,(q+\lambda-2)/q)$ in (3) is not the best possible, then there exists a positive number K (where $K < B((p+\lambda-2)/p,(q+\lambda-2)/q)$), such that (3) is valid if we change $B((p+\lambda-2)/p,(q+\lambda-2)/q)$ to K. In particular, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{\tilde{a}_m\tilde{b}_n}{m^r n^s(\ln mn)^{\lambda}}\\
&\quad<K\left\{\sum_{n=2}^{\infty}\frac{(\ln n)^{1-\lambda}}{n}\left(n^{1-r}\tilde{a}_n\right)^p\right\}^{1/p}\left\{\sum_{n=2}^{\infty}\frac{(\ln n)^{1-\lambda}}{n}\left(n^{1-s}\tilde{b}_n\right)^q\right\}^{1/q}.
\end{aligned}\tag{15}$$

In virtue of (13), (14), and (15), we have

$$B\left(\frac{p+\lambda-2}{p}-\frac{\varepsilon}{p},\frac{q+\lambda-2}{q}+\frac{\varepsilon}{q}\right)-\frac{\varepsilon}{[1-(2-\lambda+\varepsilon)/p]^2}<K(\varepsilon+1).\tag{16}$$

Setting $\varepsilon\to 0^+$, it follows that $B((p+\lambda-2)/p,(q+\lambda-2)/q)\leq K$, which contradicts the fact that $K<B((p+\lambda-2)/p,(q+\lambda-2/q))$. Hence the constant factor $B((p+\lambda-2)/p,(q+\lambda-2)/q)$ in (3) is the best possible. This proves the theorem. □

Remark 2. (i) For $\lambda = r = s = 1$, (3) changes to (1), it follows that (3) is a generalization of (1) with three parameters and (4), (5), and (6) are its some particular results with the best constant factors.

(ii) For $\lambda = 2$, (6) changes to

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{(\ln mn)^2} < \left\{\sum_{n=2}^{\infty}\frac{1}{n\ln n}(na_n)^p\right\}^{1/p}\left\{\sum_{n=2}^{\infty}\frac{1}{n\ln n}(nb_n)^q\right\}^{1/q}, \tag{17}$$

which is a new inequality with a best constant factor 1.

(iii) For $p = q = 2$, (5) and (6) change, respectively, to

$$\begin{gathered}\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{m^r n^s \ln mn} < \pi\left\{\sum_{n=2}^{\infty}n^{1-2r}a_n^2\sum_{n=2}^{\infty}n^{1-2s}b_n^2\right\}^{1/2};\\ \sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{(\ln mn)^{\lambda}} < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\sum_{n=2}^{\infty}n(\ln n)^{1-\lambda}a_n^2\sum_{n=2}^{\infty}n(\ln n)^{1-\lambda}b_n^2\right\}^{1/2},\end{gathered} \tag{18}$$

which are new inequalities similar to (cf. [1, 2])

$$\begin{gathered}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^r n^s \ln e^{3/4} mn} < \pi\left\{\sum_{n=1}^{\infty}n^{1-2r}a_n^2\sum_{n=1}^{\infty}n^{1-2s}b_n^2\right\}^{1/2};\\ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\sum_{n=1}^{\infty}n^{1-\lambda}a_n^2\sum_{n=1}^{\infty}n^{1-\lambda}b_n^2\right\}^{1/2} \quad (0<\lambda\le 2).\end{gathered} \tag{19}$$

Theorem 3. *If $p > 1$, $1/p + 1/q = 1$, $\{a_n\}$ is a nonnegative sequence of real numbers, $2 - \min\{p,q\} < \lambda \le 2$ and $r \in \mathbb{R}$, such that*

$$0 < \sum_{n=2}^{\infty}\frac{(\ln n)^{1-\lambda}}{n}(n^{1-r}a_n)^p < \infty, \tag{20}$$

then

$$\begin{aligned}&\sum_{n=2}^{\infty}\frac{(\ln n)^{(p-1)(\lambda-1)}}{n}\left[\sum_{m=2}^{\infty}\frac{a_m}{m^r(\ln mn)^{\lambda}}\right]^p\\ &\quad< \left[B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)\right]^p\sum_{n=2}^{\infty}\frac{(\ln n)^{1-\lambda}}{n}(n^{1-r}a_n)^p,\end{aligned} \tag{21}$$

where the constant factor $[B((p+\lambda-2)/p,(q+\lambda-2)/q)]^p$ is the best possible. Equation (21) is equivalent to (3). In particular,

(i) *for $r = 1/q$,*

$$\begin{aligned}&\sum_{n=2}^{\infty}\frac{(\ln n)^{(p-1)(\lambda-1)}}{n}\left[\sum_{m=2}^{\infty}\frac{a_m}{m^{1/q}(\ln mn)^{\lambda}}\right]^p\\ &\quad< \left[B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)\right]^p\sum_{n=2}^{\infty}(\ln n)^{1-\lambda}a_n^p;\end{aligned} \tag{22}$$

(ii) *for* $\lambda = 1$,

$$\sum_{n=2}^{\infty} \frac{1}{n} \left[\sum_{m=2}^{\infty} \frac{a_m}{m^r \ln mn} \right]^p < \left[\frac{\pi}{\sin(\pi/p)} \right]^p \sum_{n=2}^{\infty} \frac{1}{n} (n^{1-r} a_n)^p; \tag{23}$$

(iii) *for* $r = 0$,

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{(\ln n)^{(p-1)(\lambda-1)}}{n} \left[\sum_{m=2}^{\infty} \frac{a_m}{(\ln mn)^{\lambda}} \right]^p \\ &\qquad < \left[B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q} \right) \right]^p \sum_{n=2}^{\infty} \frac{(\ln n)^{1-\lambda}}{n} (n a_n)^p. \end{aligned} \tag{24}$$

Proof. Setting

$$b_n = (\ln n)^{(p-1)(\lambda-1)} \left[\sum_{m=2}^{\infty} \frac{a_m}{m^r n^{1/p} (\ln mn)^{\lambda}} \right]^{p-1} \quad (n = 2, 3, \dots), \tag{25}$$

then by (3), for $s = 1/p$, we have

$$\begin{aligned} 0 &< \left[\sum_{n=2}^{\infty} (\ln n)^{1-\lambda} b_n^q \right]^p = \left\{ \sum_{n=2}^{\infty} \frac{(\ln n)^{(p-1)(\lambda-1)}}{n} \left[\sum_{m=2}^{\infty} \frac{a_m}{m^r (\ln mn)^{\lambda}} \right]^p \right\}^p \\ &= \left[\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{a_m b_n}{m^r n^{1/s} (\ln mn)^{\lambda}} \right]^p \\ &\le \left[B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q} \right) \right]^p \sum_{n=2}^{\infty} \frac{(\ln n)^{1-\lambda}}{n} (n^{1-r} a_n)^p \left\{ \sum_{n=2}^{\infty} (\ln n)^{1-\lambda} b_n^q \right\}^{p-1}. \end{aligned} \tag{26}$$

Hence, we have

$$\begin{aligned} 0 &< \sum_{n=2}^{\infty} (\ln n)^{1-\lambda} b_n^q = \sum_{n=2}^{\infty} \frac{(\ln n)^{(p-1)(\lambda-1)}}{n} \left[\sum_{m=2}^{\infty} \frac{a_m}{m^r (\ln mn)^{\lambda}} \right]^p \\ &\le \left[B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q} \right) \right]^p \sum_{n=2}^{\infty} \frac{(\ln n)^{1-\lambda}}{n} (n^{1-r} a_n)^p < \infty. \end{aligned} \tag{27}$$

In view of (3), neither (26) nor (27) keeps the form of equality. Hence, (21) is valid.

On the other hand, if (21) is valid, by Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{m^r n^s(\ln mn)^{\lambda}}
&= \sum_{n=2}^{\infty}\left[\sum_{m=2}^{\infty}\frac{a_m}{m^r n^{1/p}(\ln mn)^{\lambda}(\ln n)^{(1-\lambda)/q}}\right]\left[(\ln n)^{(1-\lambda)/q}n^{1/p-s}b_n\right]\\
&\le \left\{\sum_{n=2}^{\infty}\frac{(\ln n)^{(p-1)(\lambda-1)}}{n}\left[\sum_{m=2}^{\infty}\frac{a_m}{m^r(\ln mn)^{\lambda}}\right]^p\right\}^{1/p}\\
&\quad\times\left\{\sum_{n=2}^{\infty}\frac{(\ln n)^{1-\lambda}}{n}\left(n^{1-s}b_n\right)^q\right\}^{1/q}.
\end{aligned} \tag{28}$$

In view of (21), we have (3). Hence (21) is equivalent to (3). If the constant factor in (21) is not the best possible, then by (28), we can get a contradiction that the constant factor in (3) is not the best possible. This proves the theorem. □

Remark 4. For $r = 1$, (23) changes to

$$\sum_{n=2}^{\infty}\frac{1}{n}\left[\sum_{m=2}^{\infty}\frac{a_m}{m\ln mn}\right]^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^p\sum_{n=2}^{\infty}\frac{1}{n}a_n^p, \tag{29}$$

which is equivalent to (1). Both constant factors in (1) and (29) are the best possible.

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