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REDUCTIVE COMPACTIFICATIONS OF SEMITOPOLOGICAL SEMIGROUPS

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We consider the enveloping semigroup of a flow generated by the action of a semitopological semigroup on any of its semigroup compactifications and explore the possibility of its being one of the known semigroup compactifications again. In this way, we introduce the notion of E-algebra, and show that this notion is closely related to the reductivity of the semigroup compactification involved. Moreover, the structure of the universal $E\mathcal{F}$-compactification is also given.

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1. Introduction. A semigroup S is called *right reductive* if $a = b$ for each $a, b \in S$, since $at = bt$ for every $t \in S$. For example, all right cancellative semigroups and semigroups with a right identity are right reductive.

From now on, S will be a semitopological semigroup, unless otherwise is stipulated. By a *semigroup compactification* of S we mean a pair (ψ, X), where X is a compact Hausdorff right topological semigroup, and $\psi : S \to X$ is a continuous homomorphism with dense image such that, for each $s \in S$, the mapping $x \to \psi(s)x : X \to X$ is continuous. The C^*-algebra of all bounded complex-valued continuous functions on S will be denoted by $\mathscr{C}(S)$. For $\mathscr{C}(S)$, the left and right translations, L_s and R_t, are defined for each $s, t \in S$ by $(L_s f)(t) = f(st) = (R_t f)(s)$, $f \in \mathscr{C}(S)$. The subset $\mathcal{F}$ of $\mathscr{C}(S)$ is said to be left translation invariant if for all $s \in S$, $L_s \mathcal{F} \subseteq \mathcal{F}$. A left translation invariant unital C^*-subalgebra $\mathcal{F}$ of $\mathscr{C}(S)$ is called *m-admissible* if the function $s \to T_\mu f(s) = \mu(L_s f)$ is in $\mathcal{F}$ for all $f \in \mathcal{F}$ and $\mu \in S^{\mathcal{F}}$ (where $S^{\mathcal{F}}$ is the spectrum of $\mathcal{F}$). Then the product of $\mu, \nu \in S^{\mathcal{F}}$ can be defined by $\mu\nu = \mu \circ T_\nu$ and the Gelfand topology on $S^{\mathcal{F}}$ makes $(\epsilon, S^{\mathcal{F}})$ a semigroup compactification (called the $\mathcal{F}$-compactification) of S, where $\epsilon : S \to S^{\mathcal{F}}$ is the evaluation mapping.

Some m-admissible subalgebras of $\mathscr{C}(S)$, that we will need, are left multiplicatively continuous functions $\mathscr{LMC}$, distal functions $\mathscr{D}$, minimal distal functions $\mathscr{MD}$, and strongly distal functions $\mathscr{SD}$. We also write $\mathscr{GP}$ for $\mathscr{MD} \cap \mathscr{SD}$; and we define $\mathscr{LZ} := \{f \in \mathscr{C}(S);\ f(st) = f(s) \text{ for all } s, t \in S\}$. For a discussion of the universal property of the corresponding compactifications of these function algebras see [1, 2].

2. Reductive compactifications and E-algebras. Let (ψ, X) be a compactification of S, then the mapping $\sigma : S\times X \to X$, defined by $\sigma(s,x) = \psi(s)x$, is separately continuous and so (S, X, σ) is a flow. If Σ_X denotes the enveloping semigroup of the flow (S, X, σ) (i.e., the pointwise closure of semigroup $\{\sigma(s,\cdot) : s \in S\}$ in X^X) and the mapping $\sigma_X : S \to \Sigma_X$ is defined by $\sigma_X(s) = \sigma(s,\cdot)$ for all $s \in S$, then (σ_X, Σ_X) is a compactification of S (see [1, Proposition 1.6.5]).

One can easily verify that $\Sigma_X = \{\lambda_x : x \in X\}$, where $\lambda_x(y) = xy$ for each $y \in X$. If we define the mapping $\theta : X \to \Sigma_X$ by $\theta(x) = \lambda_x$, then θ is a continuous homomorphism with the property that $\theta \circ \psi = \sigma_X$. So (σ_X, Σ_X) is a factor of (ψ, X), that is $(\psi, X) \geq (\sigma_X, \Sigma_X)$. By definition, θ is one-to-one if and only if X is right reductive. So we get the next proposition, which is an extension of the Lawson's result [3, Lemma 2.4(ii)].

Proposition 2.1. *Let (ψ, X) be a compactification of S. Then $(\sigma_X, \Sigma_X) \cong (\psi, X)$ if and only if X is right reductive.*

A compactification (ψ, X) is called *reductive* if X is right reductive. For example, the $\mathcal{MD}$-, $\mathcal{GP}$-, and $\mathcal{LZ}$-compactifications are reductive.

An m-admissible subalgebra $\mathcal{F}$ of $\mathcal{C}(S)$ is called an *E-algebra* if there is a compactification (ψ, X) such that $(\sigma_X, \Sigma_X) \cong (\epsilon, S^{\mathcal{F}})$. In this setting (ψ, X) is called an *$E\mathcal{F}$-compactification* of S. Trivially for every reductive compactification (ψ, X), $\psi^*(\mathcal{C}(X))$ is an E-algebra. But the converse is not, in general, true. For instance, for any compactification (ψ, X), $\sigma_X^*(\mathcal{C}(\Sigma_X))$ is an E-algebra; however, it is possible that Σ_X would be nonreductive, as the next example shows.

Example 2.2. Let $S = \{a, b, c, d\}$ be the semigroup with the following multiplication table:

	a	b	c	d
a	a	a	a	a
b	a	a	a	c
c	a	a	a	a
d	a	c	a	b

Then for the identity compactification (i, X) of S, Σ_X is not right reductive; in fact, $\lambda_a \neq \lambda_b$, however, $\lambda_{at} = \lambda_{bt}$ for every $t \in S$.

Lemma 2.3. *If (ψ, X) is a compactification satisfying $X^2 = X$, then the compactification (σ_X, Σ_X) is reductive.*

Proof. Since $X^2 = X$, for each $x_1, x_2 \in X$, from $\lambda_{x_1}\lambda_y = \lambda_{x_2}\lambda_y$ for every $\lambda_y \in \Sigma_X$, it follows that $\lambda_{x_1} = \lambda_{x_2}$. So Σ_X is right reductive. □

Corollary 2.4. *Let sS (or Ss) be dense in S, for some $s \in S$, then for every compactification (ψ, X) of S, it follows that $X^2 = X$ and so (σ_X, Σ_X) is reductive.*

Now, we are going to construct the universal $E\mathcal{F}$-compactification of S. For this end we need the following lemma.

Lemma 2.5. *Let $\mathcal{F}$ be an m-admissible subalgebra of $\mathcal{C}(S)$. Then $T_\nu f \in \sigma_{S^{\mathcal{F}}}^*(\mathcal{C}(\Sigma_{S^{\mathcal{F}}}))$ for all $f \in \mathcal{F}$ and $\nu \in S^{\mathcal{LMC}}$.*

Proof. Since $\Sigma_{S^{\mathcal{F}}} = \{\lambda_\mu : \mu \in S^{\mathcal{F}}\}$, we can define $g : \Sigma_{S^{\mathcal{F}}} \to \mathbb{C}$ by $g(\lambda_\mu) = \mu(T_\nu f)$, where $\mathbb{C}$ denotes the complex numbers. Since the mapping $\lambda_\mu \to \mu\nu : \Sigma_{S^{\mathcal{F}}} \to S^{\mathcal{F}}$ is p-weak* continuous, g is a bounded continuous function and it is easy to see that $\sigma_{S^{\mathcal{F}}}^*(g) = T_\nu(f)$. Therefore, $T_\nu f \in \sigma_{S^{\mathcal{F}}}^*(\mathcal{C}(\Sigma_{S^{\mathcal{F}}}))$ for all $\nu \in S^{\mathcal{F}}$. If $\tilde{\nu} \in S^{\mathcal{LMC}}$ and ν is the restriction of $\tilde{\nu}$ to $\mathcal{F}$, then $T_{\tilde{\nu}} f = T_\nu f$ for all $f \in \mathcal{F}$. So the conclusion follows. □

Proposition 2.6. *Let $\mathcal{F}$ be an E-algebra. Then*

$$G_{\mathcal{F}} := \{f \in \mathcal{LMC} : T_\nu f \in \mathcal{F} \ \forall \nu \in S^{\mathcal{LMC}}\} \tag{2.1}$$

is an m-admissible subalgebra of $\mathcal{C}(S)$ and $(\epsilon, S^{G_{\mathcal{F}}})$ is the universal $E\mathcal{F}$-compactification of S.

Proof. It is easy to verify that $G_{\mathcal{F}}$ is an m-admissible subalgebra of $\mathcal{C}(S)$ containing $\mathcal{F}$. By definition of $G_{\mathcal{F}}$ we can define the mapping $\theta : S^{\mathcal{F}} \to \Sigma_{S^{G_{\mathcal{F}}}}$ by $\theta(\mu) = \lambda_{\tilde{\mu}}$, where $\tilde{\mu}$ is an extension of μ to $S^{G_{\mathcal{F}}}$. Clearly, θ is continuous and $\theta \circ \epsilon = \sigma_{S^{G_{\mathcal{F}}}}$. Thus $(\epsilon, S^{\mathcal{F}}) \geq (\sigma_{S^{G_{\mathcal{F}}}}, \Sigma_{S^{G_{\mathcal{F}}}})$. On the other hand, since $\mathcal{F}$ is an E-algebra, there exists a compactification (ϕ, Y) of S such that $(\sigma_Y, \Sigma_Y) \cong (\epsilon, S^{\mathcal{F}})$ and $\mathcal{F} = \sigma_Y^*(\mathcal{C}(\Sigma_Y))$. By Lemma 2.5, we have $T_\nu f \in \sigma_Y^*(\mathcal{C}(\Sigma_Y))$, for each $\nu \in S^{\mathcal{LMC}}$ and each $f \in \phi^*(\mathcal{C}(Y))$. This means that $\phi^*(\mathcal{C}(Y)) \subset G_{\mathcal{F}}$ and so, by [1, Propositiion 1.6.7], $(\sigma_Y, \Sigma_Y) \leq (\sigma_{S^{G_{\mathcal{F}}}}, \Sigma_{S^{G_{\mathcal{F}}}})$. Therefore, $(\epsilon, S^{\mathcal{F}}) \cong (\sigma_{S^{G_{\mathcal{F}}}}, \Sigma_{S^{G_{\mathcal{F}}}})$ and $(\epsilon, S^{G_{\mathcal{F}}})$ is an $E\mathcal{F}$-compactification of S. Finally, if (ψ, X) is an $E\mathcal{F}$-compactification of S and $f \in \psi^*(\mathcal{C}(X))$, then by Lemma 2.5, $T_\mu f \in \sigma_X^*(\mathcal{C}(\Sigma_X)) = \mathcal{F}$ for all $\mu \in S^{\mathcal{LMC}}$. So $\psi^*(\mathcal{C}(X)) \subset G_{\mathcal{F}}$ and $(\psi, X) \leq (\epsilon, S^{G_{\mathcal{F}}})$. □

Examples 2.7. (a) We have $G_{\mathcal{MD}} = \mathcal{D}$. To see this, if $f \in G_{\mathcal{MD}}$, then for all $\mu, \nu, \eta \in S^{\mathcal{LMC}}$ with $\eta^2 = \eta$, we have $\mu\eta\nu(f) = \mu\eta(T_\nu f) = \mu(T_\nu f) = \mu\nu(f)$. So $f \in \mathcal{D}$. Also if $f \in \mathcal{D}$, then for all $\mu, \nu, \eta \in S^{\mathcal{LMC}}$ with $\eta^2 = \eta$, we have $\mu\eta(T_\nu f) = \mu\eta\nu(f) = \mu\nu(f) = \mu(T_\nu f)$. That is, $T_\nu f \in \mathcal{MD}$ for all $\nu \in S^{\mathcal{LMC}}$ and so $f \in G_{\mathcal{MD}}$ (see also [4, Lemma 2.2]).

(b) By a similar proof, we can show that $G_{\mathcal{SP}} = \mathcal{SD}$ (see [4, Lemma 2.2 and Theorem 2.6]).

(c) Let $\mathcal{R} := \{f \in \mathcal{LMC}(S) : f(rst) = f(rt) \text{ for } r, s, t \in S\}$. Clearly, $\mathcal{R}$ is an m-admissible subalgebra of $\mathcal{C}(S)$. If $f \in \mathcal{R}$ and $\nu \in S^{\mathcal{LMC}}$, then for each $r, s, t \in S$ we have $L_{rt} f(s) = f(rts) = f(rs) = L_r f(s)$. So $T_\nu f(rt) = \nu(L_{rt} f) = \nu(L_r f) = T_\nu f(r)$. That is, $T_\nu f \in \mathcal{LZ}$. On the other hand, if $f \in G_{\mathcal{LZ}}$, then $f(rst) = (T_{\epsilon(t)} f)(rs) = (T_{\epsilon(t)} f)(r) = f(rt)$ and so $f \in \mathcal{R}$. Therefore, $G_{\mathcal{LZ}} = \mathcal{R}$.

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