

CHARACTERIZATIONS OF VECTOR-VALUED WEAKLY ALMOST PERIODIC FUNCTIONS

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We characterize the weak almost periodicity of a vector-valued, bounded, continuous function. We show that if the range of the function is relatively weakly compact, then the relative weak compactness of its right orbit is equivalent to that of its left orbit. At the same time, we give the function some other equivalent properties.

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1. Introduction. Let S be a semitopological semigroup, let \mathcal{X} be a Banach space, and let $\mathcal{C}(S, \mathcal{X})$ be the space of bounded continuous functions from S to \mathcal{X} with supremum norm. Let $f \in \mathcal{C}(S, \mathcal{X})$. The right (left) translate of f by $s \in S$ is the function $R_s f$ ($L_s f$) such that $R_s f(t) = f(ts)$ and $L_s f(t) = f(st)$ for all $t \in S$. The function f is said to be weakly almost periodic if its right orbit $R_s f = \{R_s f : s \in S\}$ is relatively weakly compact in $\mathcal{C}(S, \mathcal{X})$. We denote by $\mathcal{WAP}(S, \mathcal{X})$ all such functions.

In the case that $\mathcal{X} = \mathbb{C}$, the complex number field, we will omit \mathcal{X} from our notations and write, for example, $\mathcal{C}(S)$ for $\mathcal{C}(S, \mathbb{C})$.

Recently, some authors have investigated $\mathcal{WAP}(S, \mathcal{X})$ and exploited its applications in many areas [1, 2, 4, 5, 6, 7, 8, 9]. However, some questions remain unsolved. For example, [2, Theorems 4.2.3 and 4.2.6] give a number of equivalent properties for a function $f \in \mathcal{C}(S)$ to be weakly almost periodic. It is natural to ask if the similar equivalent properties are true for a function in $\mathcal{C}(S, \mathcal{X})$. In this paper, we investigate these problems and give positive answers.

It is shown in [4, Proposition 2.8] that the equivalence of relative weak compactness for $R_s f$ and the left orbit $L_s f = \{L_s f : s \in S\}$ holds if S admits an identity and the range $f(S)$ is relative compact in \mathcal{X} . We will give an example at the end of the paper to show that the assumptions both on S and on $f(S)$ are not essential to get the equivalence. We will show the equivalence under the assumption that $f(S)$ is relatively weakly compact. At the same time, we characterize a vector-valued weakly almost periodic function by giving it as many equivalent properties as a scalar-valued weakly almost periodic function has. We will not assume that a semitopological semigroup S admits an identity. In fact, if S has an identity, we can drop the condition $f(S)$, being either relative norm compact or relatively weakly compact (see [Remark 3.6\(b\)](#)).

2. Vector-valued means. To show the main results of the paper in Section 3, we need some facts of vector-valued means. Unless otherwise mentioned, all the results of this section come from [8, Sections 2 and 3].

Let \mathcal{X}, \mathcal{Y} be two normed linear spaces and let \mathcal{Y}^* be the dual space of \mathcal{Y} . Let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ be the space of bounded linear operators from \mathcal{X} to \mathcal{Y} . With the norm topology, $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a Banach space. We can also furnish $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ with the following two topologies, both of them make $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ a locally convex topological space [3, VI.1.2, VI.1.3]:

- (1) the strong operator topology τ_s , which is the weakest topology of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ relative to which the mapping $\mu \rightarrow \mu(z) : \mathcal{L}(\mathcal{X}, \mathcal{Y}) \rightarrow \mathcal{Y}$, is continuous for each $z \in \mathcal{X}$;
- (2) the weak operator topology τ_w , which is the weakest topology of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ relative to which the mapping $\mu \rightarrow \mathcal{Y}^*[\mu(z)] : \mathcal{L}(\mathcal{X}, \mathcal{Y}) \rightarrow \mathbb{C}$, is continuous for each $z \in \mathcal{X}$ and $\mathcal{Y}^* \in \mathcal{Y}^*$.

For $\mathcal{L}(\mathcal{X}, \mathcal{Y}^*)$, we have the following topology that also makes $\mathcal{L}(\mathcal{X}, \mathcal{Y}^*)$ a locally convex topological space [3, page 476]:

- (3) the weak* operator topology τ_{w^*} , which is the weakest topology of $\mathcal{L}(\mathcal{X}, \mathcal{Y}^*)$ relative to which the mapping $\mu \rightarrow [\mu(z)](\mathcal{Y}) : \mathcal{L}(\mathcal{X}, \mathcal{Y}^*) \rightarrow \mathbb{C}$, is continuous for each $z \in \mathcal{X}$ and $\mathcal{Y} \in \mathcal{Y}$.

Let S be a nonempty set, let \mathcal{X} be a Banach space, and let $\mathcal{B}(S, \mathcal{X})$ be the space of bounded functions from S to \mathcal{X} with supremum norm. Let \mathcal{A} be a subspace of $\mathcal{B}(S, \mathcal{X})$ containing the constant functions.

DEFINITION 2.1. A mean μ on \mathcal{A} is a linear operator from \mathcal{A} to \mathcal{X} such that $\mu(f) \in \overline{\text{co}}f(S)$ for all $f \in \mathcal{A}$, denoted by $M(\mathcal{A})$ of all means on \mathcal{A} .

We define the evaluation mapping $\epsilon : S \rightarrow M(\mathcal{A})$ as follows: for $s \in S$, $\epsilon(s)f = f(s)$, $f \in \mathcal{A}$. The following proposition comes from [9, Propositions 1.5 and 1.6].

PROPOSITION 2.2. *Let \mathcal{A} be a subspace of $\mathcal{B}(S, \mathcal{X})$ containing the constant functions. Then for both τ_s and τ_w , $M(\mathcal{A})$ is convex and closed in $\mathcal{L}(\mathcal{A}, \mathcal{X})$, and $\text{co}(\epsilon(S))$ is dense in $M(\mathcal{A})$. Furthermore, if \mathcal{A} is such that $f(S)$ is relatively weakly compact in \mathcal{X} for all $f \in \mathcal{A}$, then $M(\mathcal{A})$ is τ_w -compact.*

We embed \mathcal{X} into its double dual space \mathcal{X}^{**} canonically and let $\iota(\mathcal{X})$ denote its canonical image in \mathcal{X}^{**} ; similarly, we embed $f(S)$ into \mathcal{X}^{**} for every $f \in \mathcal{A}$ and get a subspace $\iota(\mathcal{A})$ of $\mathcal{B}(S, \iota(\mathcal{X}))$. A function of \mathcal{A} may be regarded as a function of $\iota(\mathcal{A})$, and vice versa. Replacing \mathcal{A} and \mathcal{X} by $\iota(\mathcal{A})$ and \mathcal{X}^{**} in Definition 2.1, respectively, we get $M(\iota(\mathcal{A}))$. A mean of $M(\mathcal{A})$ may be regarded as a mean of $M(\iota(\mathcal{A}))$, and vice versa. This leads to the following more general definition of means.

DEFINITION 2.3. Let \mathcal{A} be a subspace of $\mathcal{B}(S, \mathcal{X}^{**})$. A linear map $\mu : \mathcal{A} \rightarrow \mathcal{X}^{**}$ is called a w^* mean on \mathcal{A} provided $\mu(f) \in \overline{\text{co}}^{w^*} f(S)$, for all $f \in \mathcal{A}$, where w^* stands for the weak* topology $\sigma(\mathcal{X}^{**}, \mathcal{X}^*)$. Denote by $w^*M(\mathcal{A})$ the set of

all w^* means on \mathcal{A} . In the case that $\mathcal{A} \subset \mathcal{B}(S, \mathcal{X})$, we define $w^*M(\mathcal{A})$ to be $w^*M(\iota(\mathcal{A}))$.

Both Definitions 2.1 and 2.3 will reduce to the definition of a scalar-valued mean when $\mathcal{X} = \mathbb{C}$ [1, 2.1.2].

PROPOSITION 2.4. *Let \mathcal{A} be a linear subspace of $\mathcal{B}(S, \mathcal{X}^{**})$. Then for τ_{w^*}*

- (1) $w^*M(\mathcal{A})$ is convex and compact;
- (2) let ϵ be the evaluation map $S \rightarrow w^*M(\mathcal{A})$, then $\text{co}(\epsilon(S))$ is dense in $w^*M(\mathcal{A})$.

PROPOSITION 2.5. *Every member of $w^*M(\mathcal{A})$ can be extended to a member of $w^*M(\mathcal{B}(S, \mathcal{X}^{**}))$.*

We call the scalar-valued function space

$$\mathcal{F} = \text{sp} \{ [f(\cdot)](x^*) : x^* \in \mathcal{X}^*, f \in \mathcal{A} \} \tag{2.1}$$

generated space of \mathcal{A} .

PROPOSITION 2.6. *Let \mathcal{A} be a linear subspace of $\mathcal{B}(S, \mathcal{X}^{**})$ containing the constant functions, and let \mathcal{F} be its generated space. Then, there is an isometric $\tau_{w^*} \cdot \sigma(\mathcal{F}^*, \mathcal{F})$ homeomorphism $\mu \rightarrow \varphi_\mu : w^*M(\mathcal{A}) \rightarrow M(\mathcal{F})$ such that*

$$[\mu(f)](x^*) = \varphi_\mu[f(\cdot)](x^*) \quad (f \in \mathcal{A}, x^* \in \mathcal{X}^*). \tag{2.2}$$

3. Main results. A nonempty set S that is a semigroup and also a topological space is called a semitopological semigroup provided that the maps $s \rightarrow ts$ and $s \rightarrow st$ from S to S are continuous for all $t \in S$. Let S be such a set, and let \mathcal{A} be a subspace of $\mathcal{C}(S, \mathcal{X})$. We say \mathcal{A} is right (resp., left) translation invariant if $R_S \mathcal{A} = \{R_S f : s \in S, f \in \mathcal{A}\} \subset \mathcal{A}$ (resp., $L_S \mathcal{A} = \{L_S f : s \in S, f \in \mathcal{A}\} \subset \mathcal{A}$). We say \mathcal{A} is translation invariant if it is both right and left translation invariant.

Let \mathcal{A} be a translation invariant subspace of $\mathcal{C}(S, \mathcal{X})$. For $\mu \in M(\mathcal{A})$, define $T_\mu : \mathcal{A} \rightarrow \mathcal{B}(S, \mathcal{X})$ by

$$T_\mu f(s) = \mu(L_s f) \quad (f \in \mathcal{A}, s \in S) \tag{3.1}$$

and $U_\mu : \mathcal{A} \rightarrow \mathcal{B}(S, \mathcal{X})$ by

$$U_\mu f(s) = \mu(R_s f) \quad (f \in \mathcal{A}, s \in S). \tag{3.2}$$

We call T_μ (U_μ) left (right) introversion operator determined by μ . We will say that \mathcal{A} is left (right) introverted if $T_\mu \mathcal{A} \subset \mathcal{A}$ ($U_\mu \mathcal{A} \subset \mathcal{A}$) for all $\mu \in M(\mathcal{A})$. We will say that \mathcal{A} is introverted if it is both left and right introverted.

Similarly, we define an introversion operator from \mathcal{A} to $\mathcal{B}(S, \mathcal{X}^{**})$ if \mathcal{A} is a translation invariant subspace of $\mathcal{B}(S, \mathcal{X}^{**})$ and $\mu \in w^*M(\mathcal{A})$.

To show Theorem 3.2, we need the following proposition that characterizes weak almost periodicity of a function in $\mathcal{C}(S)$.

PROPOSITION 3.1 [2, 4.2.6]. *Let S be a semitopological semigroup, let $f \in \mathcal{C}(S)$, and let $\mathcal{F} = \mathcal{WAP}(S)$. Then, the following statements are equivalent:*

- (1) $f \in \mathcal{WAP}(S)$;
- (2) $L_S f$ is relatively weakly compact in $\mathcal{C}(S)$;
- (3) the mapping $\varphi \rightarrow T_\varphi f : M(\mathcal{F}) \rightarrow \mathcal{B}(S)$ is $\sigma(\mathcal{F}^*, \mathcal{F}) \rightarrow \sigma(\mathcal{B}(S), \mathcal{B}(S)^*)$ continuous;
- (4) the mapping $\varphi \rightarrow U_\varphi f : M(\mathcal{F}) \rightarrow \mathcal{B}(S)$ is $\sigma(\mathcal{F}^*, \mathcal{F}) \rightarrow \sigma(\mathcal{B}(S), \mathcal{B}(S)^*)$ continuous;
- (5) for all $\varphi, \psi \in M(\mathcal{F})$, $\varphi(T_\psi f) = \psi(U_\varphi f)$.

We will generalize **Proposition 3.1** from scalar-valued function to vector-valued function in the next theorem. We will use some results of the previous section to show the theorem. To make notations short, we let

$$\iota(\mathcal{C}) = \mathcal{C}(S, \iota(\mathcal{X})), \quad \iota(\mathcal{B}) = \mathcal{B}(S, \iota(\mathcal{X})), \quad \mathcal{B} = \mathcal{B}(S, \mathcal{X}^{**}). \tag{3.3}$$

As in the paragraph before **Definition 2.3**, $\iota(\mathcal{X})$ is the canonical image of \mathcal{X} in \mathcal{X}^{**} ; an $f \in \mathcal{B}(S, \mathcal{X})$ and its corresponding function in $\iota(\mathcal{B})$ will be regarded as same function.

Note that both \mathcal{B} and $\iota(\mathcal{B})$ have the same generated space $\mathcal{B}(S)$, the space of bounded scalar-valued functions on S .

THEOREM 3.2. *Let S be a semitopological semigroup and let \mathcal{X} be a Banach space. Let $\mathcal{A} = \mathcal{WAP}(S, \mathcal{X})$ and let $f \in \mathcal{C}(S, \mathcal{X})$ be such that $f(S)$ is relatively weakly compact in \mathcal{X} . Then the following statements are equivalent:*

- (1) $f \in \mathcal{A}$, that is, $R_S f$ is relatively weakly compact in $\mathcal{C}(S, \mathcal{X})$;
- (2) $L_S f$ is relatively weakly compact in $\mathcal{C}(S, \mathcal{X})$;
- (3) the mapping $\mu \rightarrow T_\mu f : w^*M(\mathcal{B}) \rightarrow \iota(\mathcal{B})$ is $\tau_{w^*} \text{-}\sigma(\iota(\mathcal{B}), \iota(\mathcal{B})^*)$ continuous;
- (4) the mapping $\mu \rightarrow U_\mu f : w^*M(\mathcal{B}) \rightarrow \iota(\mathcal{B})$ is $\tau_{w^*} \text{-}\sigma(\iota(\mathcal{B}), \iota(\mathcal{B})^*)$ continuous;
- (5) for all $\mu, \nu \in w^*M(\mathcal{B})$,

$$\mu(T_\nu f) = \nu(U_\mu f). \tag{3.4}$$

PROOF. Since $f(S)$ is relatively weakly compact in \mathcal{X} , the functions $T_\mu f$ and $U_\mu f$ are in $\iota(\mathcal{B})$ for all $\mu \in w^*M(\mathcal{B})$.

Let ϵ and ϵ' be the evaluation mappings on \mathcal{B} and $\mathcal{B}(S)$, respectively. Let $B = \overline{\text{co}}^{\tau_{w^*}} \epsilon(S)$, and let $B_{\mathcal{B}(S)^*}$ be the unit ball of $\mathcal{B}(S)^*$. By [2, 2.1.14], $B_{\mathcal{B}(S)^*} = \overline{\text{co}}^{w^*} \epsilon'(S)$, where w^* stands for $\sigma(\mathcal{B}(S)^*, \mathcal{B}(S))$. It follows from Propositions 2.4(2) and 2.6 that B and $B_{\mathcal{B}(S)^*}$ are $\tau_{w^*} \text{-}\sigma(\mathcal{B}(S)^*, \mathcal{B}(S))$ homeomorphic.

Define $V : B \rightarrow \mathcal{B}$ by

$$V(\mu) = T_\mu f. \tag{3.5}$$

V is continuous from τ_{w^*} to the weak* pointwise convergence topology P . For, if $\{\mu_\alpha\} \subset B$ and $\mu \in B$ are such that $\mu_\alpha \rightarrow \mu$ in τ_{w^*} , then

$$\begin{aligned} [T_{\mu_\alpha}f(s)](x^*) &= [\mu_\alpha(L_S f)](x^*) \rightarrow [\mu(L_S f)](x^*) \\ &= [T_\mu f(s)](x^*) \quad (x^* \in \mathcal{X}^*, s \in S). \end{aligned} \tag{3.6}$$

Since B is τ_{w^*} -compact (Proposition 2.4(1)), we have

$$V(B) = \overline{\text{cco}}(R_S f), \tag{3.7}$$

where closure in $\iota(\mathcal{B})$ is taken in the weak* pointwise convergence topology P .

Now, we show that (1) implies (3).

By the Krein-Smulian theorem [2, Theorem A.10], $\text{cco}(R_S f)$ is relatively weakly compact in $\iota(\mathcal{C})$, which in view of (3.7) implies that $V(B)$ is $\sigma(\iota(\mathcal{C}), \iota(\mathcal{C})^*)$ -closure of $\text{cco}(R_S f)$ in $\iota(\mathcal{C})$ and that $V(B)$ is $\sigma(\iota(\mathcal{C}), \iota(\mathcal{C})^*)$ -compact. Therefore, the weak topology $\sigma(\iota(\mathcal{C}), \iota(\mathcal{C})^*)$ and the topology P coincide on $V(B)$. So, V is τ_{w^*} - $\sigma(\iota(\mathcal{B}), \iota(\mathcal{B})^*)$ continuous on B .

To show (3), we define, for $\Phi \in \iota(\mathcal{B})^*$, the linear functional $\Phi \circ T : \mathcal{B}(S)^* \rightarrow \mathbb{C}$ by

$$\Phi \circ T(\varphi_\mu) = \Phi(T_\mu f), \tag{3.8}$$

where μ and φ_μ are as in (2.2). It follows from the τ_{w^*} - $\sigma(\iota(\mathcal{B}), \iota(\mathcal{B})^*)$ continuity on B of V that $\Phi \circ T$ is $\sigma(\mathcal{B}(S)^*, \mathcal{B}(S))$ continuous on $B_{\mathcal{B}(S)^*}$. By Grothendieck's completeness theorem [2, Proposition A.8], $\Phi \circ T$ is $\sigma(\mathcal{B}(S)^*, \mathcal{B}(S))$ continuous on $\mathcal{B}(S)^*$. Now, we claim that (3) holds. For, if μ_α and μ of $w^*M(\mathcal{B})$ are such that $\mu_\alpha \rightarrow \mu$ in τ_{w^*} , then $\varphi_{\mu_\alpha} \rightarrow \varphi_\mu$ in $\sigma(\mathcal{B}(S)^*, \mathcal{B}(S))$ (Proposition 2.6), and therefore

$$\Phi(T_{\mu_\alpha} f) = \Phi \circ T(\varphi_{\mu_\alpha}) \rightarrow \Phi \circ T(\varphi_\mu) = \Phi(T_\mu f). \tag{3.9}$$

Since Φ is arbitrary in $\iota(\mathcal{B})^*$, we have $T_{\mu_\alpha} f \rightarrow T_\mu f$ in $\sigma(\iota(\mathcal{B}), \iota(\mathcal{B})^*)$. Thus (3) holds.

By Proposition 2.4(1), B is τ_{w^*} -compact. If (3) holds, then it follows from (3.7) that $R_S f$ is relatively weakly compact in $\iota(\mathcal{B})$. Thus (1) holds. So, (1) and (3) are equivalent.

Similarly, we show that (2) and (4) are equivalent. Next, we show that (1) and (3) imply (5).

Let $\mu, \nu \in w^*M(\mathcal{B})$ and $\varphi_\nu, \varphi_\mu \in M(\mathcal{B}(S))$ be as in (2.2). Since f is in $\mathcal{WAP}(S, \mathcal{X})$, $f(\cdot)(x^*)$ is in $\mathcal{WAP}(S)$ for all $x^* \in \mathcal{X}^*$. By Proposition 3.1(5),

$$\varphi_\nu\{T_{\varphi_\mu}[f(\cdot)(x^*)]\} = \varphi_\mu\{U_{\varphi_\nu}[f(\cdot)(x^*)]\}. \tag{3.10}$$

It follows from (2.2) and (3.10) that

$$\begin{aligned} [\nu(T_\mu f)](x^*) &= \varphi_\nu \{T_{\varphi_\mu} [f(\cdot)(x^*)]\} \\ &= \varphi_\mu \{U_{\varphi_\nu} [f(\cdot)(x^*)]\} \\ &= [\mu(U_\nu f)](x^*). \end{aligned} \quad (3.11)$$

Since $x^* \in \mathcal{X}^*$ is arbitrary, we have

$$\nu(T_\mu f) = \mu(U_\nu f) \quad (\mu, \nu \in w^*M(\mathcal{B})). \quad (3.12)$$

Thus (5) holds.

Similarly, we show that (2) and (4) imply (5).

For $x^* \in \mathcal{X}^*$ and $\mu \in w^*M(\iota(\mathcal{B}))$, define $\mu \circ x^* : \iota(\mathcal{B}) \rightarrow \mathbb{C}$ by

$$\mu \circ x^*(h) = [\mu(h)](x^*) \quad (h \in \iota(\mathcal{B})). \quad (3.13)$$

Then, $\mu \circ x^* \in \iota(\mathcal{B})^*$. Let $D = \{\mu \circ x^* : x^* \in \mathcal{X}^*, \|x^*\| = 1, \mu \in w^*M(\iota(\mathcal{B}))\}$. By the separation theorem, we show that $\overline{\text{cco}}^{w^*} D = B_{\iota(\mathcal{B})^*}$, the unit ball of $\iota(\mathcal{B})^*$, where w^* stands for $\sigma(\iota(\mathcal{B})^*, \iota(\mathcal{B}))$.

To show that (5) implies (4), we need to show that the mapping $\mu \rightarrow U_\mu f : B = \overline{\text{cco}}^{\tau_{w^*}} \epsilon(S) \rightarrow \iota(\mathcal{B})$ is τ_{w^*} - $\sigma(\iota(\mathcal{B}), \iota(\mathcal{B})^*)$ continuous. That is, we need to show that if $\mu_\alpha \subset B$ and $\mu \in B$ are such that $\mu_\alpha \rightarrow \mu$ in τ_{w^*} , and if $F \in B_{\iota(\mathcal{B})^*}$, then

$$F(U_{\mu_\alpha} f) \rightarrow F(U_\mu f). \quad (3.14)$$

Note that $\overline{\text{cco}}^{w^*} D$ is the unit ball of $\iota(\mathcal{B})^*$. For $\nu \circ x^* \in \text{cco} D$, we have

$$\nu \circ x^* [U_{\mu_\alpha} f] \rightarrow \nu \circ x^* [U_\mu f], \quad (3.15)$$

because it follows from (3.4) and (3.13) that

$$\begin{aligned} \nu \circ x^* [U_{\mu_\alpha} f] &= [\nu(U_{\mu_\alpha} f)](x^*) \\ &= [\mu_\alpha(T_\nu f)](x^*) \rightarrow [\mu(T_\nu f)](x^*) \\ &= [\nu(U_\mu f)](x^*) \\ &= \nu \circ x^* (U_\mu f). \end{aligned} \quad (3.16)$$

Let $A = \{U_\mu f : \mu \in B\}$. Then, A is a bounded subset of $\iota(\mathcal{B})$. As in the discussion for Note (ii) and (iii) before Theorem 3 of [5], we regard $\text{cco} D$ as a set of bounded function on A , that is, a subset of $\mathcal{B}(A)$. Then, the weak* closure

$\overline{\text{cco}}^{w^*} D$ in $\iota(\mathcal{B})^*$ is contained in the weak closure $\overline{\text{cco}}^w D$ in $\mathcal{B}(A)$. Since $\overline{\text{cco}}^w D$ is also norm closed in $\mathcal{B}(A)$, for $F \in \overline{\text{cco}}^{w^*} D \subset \overline{\text{cco}}^w D$, we have a sequence of $\{\nu_n \circ \mathcal{X}_n^*\}$ of $\text{cco}D$ such that

$$|F(h) - \nu_n \circ \mathcal{X}_n^*(h)| \rightarrow 0 \tag{3.17}$$

uniformly in $h \in A$. Since

$$\begin{aligned} |F(U_{\mu\alpha}f) - F(U_{\mu}f)| &\leq |F(U_{\mu\alpha}f) - \nu_n \circ \mathcal{X}_n^*(U_{\mu\alpha}f)| \\ &\quad + |\nu_n \circ \mathcal{X}_n^*(U_{\mu\alpha}f) - \nu_n \circ \mathcal{X}_n^*(U_{\mu}f)| \\ &\quad + |\nu_n \circ \mathcal{X}_n^*(U_{\mu}f) - F(U_{\mu}f)|, \end{aligned} \tag{3.18}$$

now (3.14) is a consequence of (3.15) and (3.17).

Similarly, we show that (5) implies (3). The proof is complete. □

COROLLARY 3.3. *Let S, \mathcal{X} , and \mathcal{A} be as in Theorem 3.2. Let $f \in \mathcal{A}$. Then, $T_{\mu}f \in \mathcal{A}$ for all $\mu \in M(\mathcal{A})$. Furthermore, if $f(S)$ is relatively weakly compact in \mathcal{X} , then $U_{\mu}f \in \mathcal{A}$ for all $\mu \in M(\mathcal{A})$.*

PROOF. As in the proof of (1) implying (3) of Theorem 3.2, we show the first statement. If $f(S)$ is relatively weakly compact in \mathcal{X} , then by the theorem, $L_S f$ is relatively weakly compact in $\mathcal{C}(S, \mathcal{X})$. Note that this time f is in \mathcal{A} . We show the second statement as in the proof of (2) implying (4). □

For every $x^* \in \mathcal{X}^*$, $\|x^*\| = 1$, and $s \in S$, define $x^* \circ s : \mathcal{C}(S, \mathcal{X}) \rightarrow \mathbb{C}$, by

$$x^* \circ s(f) = x^*[f(s)] \quad (f \in \mathcal{C}(S, \mathcal{X})). \tag{3.19}$$

Then, $x^* \circ s \in \mathcal{C}(S, \mathcal{X})^*$, the dual space of $\mathcal{C}(S, \mathcal{X})$. Set $E = \{x^* \circ s : x^* \in \mathcal{X}^*, \|x^*\| = 1, s \in S\}$. Let $B = \overline{E}^{w^*}$, where w^* stands for the weak* topology $\sigma(\mathcal{C}(S, \mathcal{X})^*, \mathcal{C}(S, \mathcal{X}))$. Then, B is weak* compact.

For every $f \in \mathcal{C}(S, \mathcal{X})$, define $\hat{f} : E \rightarrow \mathbb{C}$, by

$$\hat{f}(x^* \circ s) = x^*[f(s)] \quad (x^* \circ s \in E). \tag{3.20}$$

We extend \hat{f} from E to B continuously. So we have $\hat{f} \in \mathcal{C}(B)$.

Obviously, we have $\widehat{R_t f}$ for $t \in S$ and $R_t f$, that is, $\widehat{R_t f}(x^* \circ s) = x^*[R_t f(s)]$. The mapping $f \rightarrow \hat{f} : \mathcal{C}(S, \mathcal{X}) \rightarrow \mathcal{C}(B)$ is one to one and linear isometric. The space $\mathcal{C}(B)$ is a Banach space with norm topology. In the next theorem, we will also equip $\mathcal{C}(B)$ with the P -topology, the pointwise convergence topology.

THEOREM 3.4. *Let S be a semitopological semigroup, and let $f \in \mathcal{C}(S, \mathcal{X})$ such that $f(S)$ is relatively weakly compact in \mathcal{X} . Then, the following statements are equivalent:*

- (1) $f \in \mathcal{WAP}(S, \mathcal{X})$;
- (2) $L_S f$ is relatively weakly compact in $\mathcal{C}(S, \mathcal{X})$;
- (3) $\widehat{R_S f}$ is relatively compact in $\mathcal{C}(B)$ in the P -topology;
- (4) $\widehat{L_S f}$ is relatively compact in $\mathcal{C}(B)$ in the P -topology;
- (5) $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} x_m^* f(s_m t_n) = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} x_m^* f(s_m t_n)$, whenever $\{x_m^*\} \subset \mathcal{X}^*$, $\|x^*\| = 1$, $\{t_n\}, \{s_m\} \subset S$ and all the limits exist;
- (6) $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} x_m^* f(t_n s_m) = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} x_m^* f(t_n s_m)$, whenever $\{x_m^*\} \subset \mathcal{X}^*$, $\|x^*\| = 1$, $\{t_n\}, \{s_m\} \subset S$ and all the limits exist.

PROOF. The equivalence of (1) and (2) comes from [Theorem 3.2](#).

If (1) holds, then $R_S f$ is relatively weakly compact in $\mathcal{C}(S, \mathcal{X})$. Since the mapping $f \rightarrow \hat{f}$ is one to one and linear isometric, $\widehat{R_S f}$ is relatively weakly compact in $\mathcal{C}(B)$. So, $\widehat{R_S f}$ is relatively compact in $\mathcal{C}(B)$ in P -topology. Thus (3) holds.

Now, we show (3) implies (5). Let $\{x_m^*\} \subset \mathcal{X}^*$, $\|x_m^*\| = 1$, and $\{t_n\}, \{s_m\} \subset S$ be sequences such that the iterated limits

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} x_m^* f(s_m t_n), \quad \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} x_m^* f(s_m t_n) \tag{3.21}$$

exist. Thus, we have $\{\widehat{R_{t_n} f}\} \subset \mathcal{C}(B)$ and $\{x_m^* \circ s_m\} \subset E$. Let $\hat{g} \in \mathcal{C}(B)$ be P -topological cluster point of $\{\widehat{R_{t_n} f}\}$, and let $\gamma \in B$ be cluster point of $\{x_m^* \circ s_m\}$. Then,

$$\begin{aligned} \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \widehat{R_{t_n} f}(x_m^* \circ s_m) &= \lim_{m \rightarrow \infty} \hat{g}(x_m^* \circ s_m) \\ &= \hat{g}(\gamma) = \lim_{n \rightarrow \infty} \widehat{R_{t_n} f}(\gamma) \\ &= \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \widehat{R_{t_n} f}(x_m^* \circ s_m). \end{aligned} \tag{3.22}$$

Note that

$$\begin{aligned} \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \widehat{R_{t_n} f}(x_m^* \circ s_m) &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} x_m^* f(s_m t_n), \\ \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \widehat{R_{t_n} f}(x_m^* \circ s_m) &= \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} x_m^* f(s_m t_n). \end{aligned} \tag{3.23}$$

Therefore, (5) holds.

That (5) implies (1) is a consequence of Grothendieck’s double theorem [2, A.5]. Similarly, we show the equivalence among (2), (4), and (6). The proof is complete. □

EXAMPLE 3.5. Let $S = \mathbb{N} = \{1, 2, \dots\}$, the semigroup of natural numbers. Let l_p , $1 < p < \infty$ be the usual sequence spaces with basis $\{e_n\}_{n=1}^\infty$, where $e_n = \{x_k\}_{k=1}^\infty$ with $x_k = 1$ if $k = n$ and $x_k = 0$ otherwise. Define $f : \mathbb{N} \rightarrow l_p$ by

$$f(n) = e_n \quad (n \in \mathbb{N}). \tag{3.24}$$

Since \mathbb{N} is abelian, $R_S f = L_S f$. For any subsequences $\{m_i\}$, $\{n_k\}$ of S , and $\{\varphi_i\}$ of l_q with $\|\varphi\| \neq 1$, where $1/p + 1/q = 1$ and the norm being taken in the space l_q , we have that

$$\begin{aligned} \limlim_i^k \varphi_i(f(m_i + n_k)) &= \limlim_i^k \varphi_i(e_{m_i+n_k}) = \limlim_i^k \varphi_{i,m_i+n_k} = 0, \\ \limlim_k^i \varphi_i(f(m_i + n_k)) &= \limlim_k^i \varphi_i(e_{m_i+n_k}) = \limlim_k^i \varphi_{i,m_i+n_k} = 0, \end{aligned} \quad (3.25)$$

where φ_{i,m_i+n_k} is the $(m_i + n_k)$ th component of φ_i . Therefore, by [Theorem 3.4](#) we have $f \in \mathcal{WAP}(\mathbb{N}, l_p)$. However, $f(S)$ is not relatively norm compact but relatively weakly compact. Since $R_S f = L_S f$, the two orbits of f are all relatively weakly compact. We note that \mathbb{N} does not admit an identity.

REMARK 3.6. (a) The equivalence of (1) and (5) of [Theorem 2.4](#) appeared in [\[5, Theorem 6\]](#); though it was assumed that S admits an identity, the proof of [Theorem 6](#) did not use the identity. (b) In both [Theorems 3.2](#) and [3.4](#), we assume that the range of $f(S)$ is relatively weakly compact to show that the relative weak compactness of $L_S f$ is equivalent to that of $R_S f$. We do not know if the condition of $f(S)$ is essential. We showed in [\[8, Corollary 8.4\]](#) that if S admits an identity, then $f(S)$ is relatively weakly compact in \mathcal{X} for all $f \in \mathcal{WAP}(S, \mathcal{X})$. Of course, if \mathcal{X} is reflexive, then any bounded function has a relatively weakly compact range. (c) From the proof of [Theorem 3.4](#), we see that to get the equivalence among (1), (3), and (5), it does not need the condition $f(S)$ being relatively weakly compact, neither does the equivalence among (2), (4), and (6).

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