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THE LATTICE-ISOMETRIC COPIES OF $\ell_\infty(\Gamma)$ IN QUOTIENTS OF BANACH LATTICES

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Let E be a Banach lattice and let M be a norm-closed and Dedekind σ-complete ideal of E. If E contains a lattice-isometric copy of ℓ_∞, then E/M contains such a copy as well, or M contains a lattice copy of ℓ_∞. This is one of the consequences of more general results presented in this paper.

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1. Introduction. Let E be a locally solid linear lattice (Riesz space), for example, a Banach lattice, let M be a closed ideal of E, and let Γ be an infinite set. In [7, Theorem 1] it is proved that if E contains a lattice copy U of ℓ_∞ and M is Dedekind σ-complete, then E/M or M contains such a copy as well. (Here and in what follows the term *lattice copy* means *both lattice and topological copy, and lattice-isometric copy* means *both lattice and isometric copy.*) This is a lattice-topological version of the classical by now theorem of Drewnowski and Roberts which asserts that the noncontainment of ℓ_∞ is a three-space property in the class of Banach spaces (see [3, Theorem 3.2.f] and [4]). A natural question to ask is what happens if E is a Banach lattice and U is a *lattice-isometric* copy of ℓ_∞, that is, can we then expect that E/M or M contains a lattice-isometric copy of ℓ_∞? A partial positive answer to this question, even for $\ell_\infty(\Gamma)$ instead of ℓ_∞, is given in Theorem 1.1 and Corollary 1.2.

For the basic notions and results regarding Banach lattices we refer the reader to the monographs [2, 6]. For the convenience of the reader we recall some definitions. The lattice E is called Dedekind α-complete, where α is an infinite cardinal number, if every subset V of E with $\operatorname{card}(V) \le \alpha$ and bounded from above has a supremum in E; if $\alpha = \aleph_0$ then this notion coincides with the notion of Dedekind σ-completeness, and E is Dedekind complete provided that it is Dedekind α-complete for every α (cf. [1]). If $E = (E, \|\cdot\|)$ is a Banach lattice, then E_a denotes the largest ideal in E such that the norm restricted to E_a is order continuous: $E_a = \{x \in E : |x| \ge x_s \downarrow 0 \text{ implies } \|x_s\| \to 0\}$; for example, if $E = \ell_\infty(\Gamma)$, then $E_a = c_0(\Gamma)$. We have that E_a is both norm-closed in E and Dedekind complete, and it does not contain any of the lattice copy of ℓ_∞ (see, e.g., [6, Proposition 2.4.10 and Corollary 2.4.3]). If M is an ideal of E, then Q denotes the natural quotient mapping from E onto the lattice E/M; we have that Q is a lattice homomorphism, that is, $|Q(x)| = Q(|x|)$ for all $x \in E$.

Our main result reads as follows.

Theorem 1.1. *Let Γ denote a set with $\operatorname{card}(\Gamma) \ge \alpha \ge \aleph_0$. Let E be a Banach lattice and let M be a norm-closed and Dedekind α-complete ideal of E. If E contains a lattice-isometric copy of $\ell_\infty(\Gamma)$, then the following alternative holds:*

(i) *E/M contains such a copy as well, or*

(ii) *M contains a lattice copy of $\ell_\infty(A)$, where $\operatorname{card}(A) = \alpha$.*

Since Dedekind α-completeness is inherited by order ideals, the following corollary is an immediate consequence of the theorem.

Corollary 1.2. *Let M be a norm-closed ideal of a Dedekind complete (resp., Dedekind σ-complete) Banach lattice E. If E contains a lattice-isometric copy of $\ell_\infty(\Gamma)$, then E/M contains such a copy as well or M contains a lattice copy of $\ell_\infty(\Gamma)$ (resp., of ℓ_∞).*

Since the ideal $M = E_a$ contains no lattice copy of ℓ_∞, from Corollary 1.2 we immediately obtain the following corollary.

Corollary 1.3. *Let E be a Banach lattice. If E contains a lattice-isometric copy of $\ell_\infty(\Gamma)$, then E/E_a contains such a copy as well.*

In particular, the quotient Banach lattice $\ell_\infty(\Gamma)/c_0(\Gamma)$ contains a lattice-isometric copy of $\ell_\infty(\Gamma)$.

Corollaries 1.2 and 1.3 apply for Orlicz spaces endowed with the Luxemburg norm (which form a nontrivial sample class of Dedekind complete Banach lattices, and which contain lattice-isometric copies of ℓ_∞ whenever their norms are not order continuous); one can obtain similar results for Musielak-Orlicz spaces, Lorentz-Orlicz spaces, and Calderón-Lozanovsky spaces (see [5, page 526]).

2. Proof of Theorem 1.1. The symbol e_γ denotes the γth unit vector of $\ell_\infty(\Gamma)$, and if $B \subset \Gamma$ then e_B denotes the element $\sup_{\gamma \in B} e_\gamma$. The proof of the theorem depends essentially on the following lemma.

Lemma 2.1. *Let A be a set with $\operatorname{card} A = \alpha \ge \aleph_0$, and let M be a Dedekind α-complete and norm-closed ideal of a normed lattice E. If there exist $u \in E^+$ and a set $\{u_a : a \in A\}$ of pairwise disjoint elements of E^+ such that*

(a) *$u_a \le u$ for all $a \in A$,*

(b) *$b_A := \inf_{a \in A} \|u_a\| > \|Qu\|_{E/M}$,*

then M contains a lattice copy of $\ell_\infty(A)$.

Proof. We partially follow an idea of the proof of [7, Proposition 1(b)]. By (b), there exists $v \in M$ such that

$$\|u - v\| < b_A. \tag{2.1}$$

From the inequality $|u-v| \geq |u-|v||$ we may assume that $v \geq 0$, and from (a) and (2.1) we obtain that $v \neq 0$. For every $a \in A$, we define $v_a := v \wedge u_a$ (notice that $v_a \in M$ for all $a \in A$ since M is an ideal). From the equality $u_a = u_a \wedge u$ for all $a \in A$, the triangle inequality, and from the inequality $|x_1 \wedge y - x_2 \wedge y| \leq |x_1 - x_2|$, which holds in every linear lattice (see [2, Theorem 1.6]), we obtain

$$\|v_a\| \geq \|u_a\| - \|u-v\| \geq b_A - \|u-v\|. \tag{2.2}$$

By (2.1), the number $c := b_A - \|u-v\|$ is positive, thus from (2.2) we obtain that $\|v_a\| \geq c$ for all $a \in A$. Since the elements $(v_a)_{a\in A}$ are pairwise disjoint and dominated by v, and since M is Dedekind α-complete, we can define an additive function R_0 from the cone $\ell_\infty(A)^+$ into M by the rule $R_0(\xi) := \sup_{a\in A} \xi_a v_a$, where $\xi = (\xi_a)_{a\in A}$, with

$$\xi_a v_a \leq R_0(\xi) \leq \|\xi\|_{\ell_\infty(A)} v \quad \forall a \in A. \tag{2.3}$$

By [2, Theorem 1], the formula $R(\xi) := R_0(\xi^+) - R_0(\xi^-)$, $\xi \in \ell_\infty(A)$, defines a linear (positive) mapping from $\ell_\infty(A)$ into M. Moreover, since for every $\xi \in \ell_\infty(A)$ the elements $R_0(\xi^+)$ and $R_0(\xi^-)$ are disjoint, we have $|R(\xi)| = R(|\xi|)$, that is, R is a lattice homomorphism; in particular, the range of R is a linear sublattice of M (see [2, page 88]). Finally, from (2.3) we obtain that $c\|\xi\|_{\ell_\infty(A)} \leq \|R(\xi)\| \leq \|\xi\|_{\ell_\infty(A)}\|v\|$ for all $\xi \in \ell_\infty(A)$, and thus (see [2, page 89]) R is a lattice-topological isomorphism. □

Proof of Theorem 1.1. Let $\Gamma = \bigcup_{\omega\in\Omega}\Gamma_\omega$, where $\operatorname{card}(\Gamma_\omega) = \alpha$ for all $\omega \in \Omega$, $\operatorname{card}(\Omega) = \operatorname{card}(\Gamma)$, and $\Gamma_{\omega_1} \cap \Gamma_{\omega_2} = \varnothing$ for distinct $\omega_1, \omega_2 \in \Omega$. Let T be a lattice-isometric embedding of $\ell_\infty(\Gamma)$ into E. We will show that the opposite of (ii) implies (i).

Put $u^\omega := Te_{\Gamma_\omega}$ and $u^\omega_\gamma := Te_\gamma$, where $\gamma \in \Gamma_\omega$. We have that, for every $\omega \in \Omega$, the element u^ω and the set $\{u^\omega_\gamma : \gamma \in \Gamma_\omega\}$ fulfil Lemma 2.1(a), and, by hypothesis, the ideal M contains no copy of $\ell_\infty(\Gamma_\omega)$. Thus, from the lemma we obtain that

$$\|Qu^\omega\| = 1 \quad \forall \omega \in \Omega. \tag{2.4}$$

Put $W := \{x \in \ell_\infty(\Gamma) : x_{|\Gamma_\omega} = \text{const}\}$, and let H denote the linear-lattice isometry from $\ell_\infty(\Omega)$ onto W of the form $H(f_\omega) = e_{\Gamma_\omega}$, $\omega \in \Omega$, where f_ω is the ωth unit vector of $\ell_\infty(\Omega)$. We claim that the quotient mapping $Q : E \to E/M$ restricted to $T(W)$ is an isometry (and hence, since Q is a lattice homomorphism, it is a lattice isometry). To this end, let $v = \sup_{\omega\in\Omega} \lambda_\omega u^\omega$, where $(\lambda_\omega)_{\omega\in\Omega} \in \ell_\infty(\Omega)^+$ and the sup is taken in the lattice $T(W)$. We obviously have $v = T(\sup_{\omega\in\Omega} \lambda_\omega e_{\Gamma_\omega})$, whence $\|Q(v)\| \leq \|v\| = \|\sup_{\omega\in\Omega} \lambda_\omega e_{\Gamma_\omega}\| = \sup_{\omega\in\Omega} \lambda_\omega$. On the other hand, we have that $v \geq \lambda_\omega u^\omega$ for all $\omega \in \Omega$, and by (2.4), we obtain

$\|Q(v)\| \geq \sup_{\omega\in\Omega}\lambda_\omega$. Finally, $\|Q(v)\| = \|v\|$ for all positive v, and hence (since Q is a lattice homomorphism) we obtain $\|Q(v)\| = \||Q(v)|\| = \|Q(|v|)\| = \||v|\| = \|v\|$, as claimed. Since T, H, and $Q_{|T(W)}$ are lattice isometries, the operator QTH is a lattice isometry from $\ell_\infty(\Omega)$ into E/M, and since $\operatorname{card}(\Omega) = \operatorname{card}(\Gamma)$, the proof is complete. □

We want to point out that Lemma 2.1 can also be used to prove the following generalization, to the Banach lattice case, of the main result of [7] quoted in the introduction. Under the same assumptions on E, M, Γ, and α as in the theorem, let $T : \ell_\infty(\Gamma) \to E$ be a lattice-topological isomorphism. We define the number $b := \inf\{\|Q(Te_A)\|_{E/M} : A \subset \Gamma \text{ and } \operatorname{card} A = \alpha\}$. Then E/M contains a lattice copy of $\ell_\infty(\Gamma)$ (whenever $b > 0$; then we mimic the above part of the proof of the theorem with inequality $\|Qu^\omega\| \geq b$ instead of (2.4)), or M contains a lattice copy of $\ell_\infty(A)$ (whenever $b = 0$; then we directly apply the lemma).

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