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T-FUZZY MULTIPLY POSITIVE IMPLICATIVE BCC-IDEALS OF BCC-ALGEBRAS

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The concept of fuzzy multiply positive BCC-ideals of BCC-algebras is introduced, and then some related results are obtained. Moreover, we introduce the concept of T-fuzzy multiply positive implicative BCC-ideals of BCC-algebras and investigate T-product of T-fuzzy multiply positive implicative BCC-ideals of BCC-algebras, examining its properties. Using a t-norm T, the direct product and T-product of T-fuzzy multiply positive implicative BCC-ideals of BCC-algebras are discussed and their properties are investigated.

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1. Introduction and preliminaries. A BCK-algebra is an important class of logical algebras introduced by K. Iséki in 1966. After that, Iséki posed an interesting problem (solved by Wroński [8]) of whether the class of BCK-algebra is a variety. In connection with this problem, Komori [6] introduced a notion of BCC-algebras and Dudek [5] redefined it by using a dual form of the ordinary definition in the sense of Komori. In 1965, Zadeh introduced the notion of fuzzy sets [9]. At present, this concept has been applied to many mathematical branches such as group, functional analysis, probality theory and topology, and so on. In 1991, Ougen applied this concept to BCK-algebras [7], and also many fuzzy structures in BCC-algebras are considered. In this paper, the concept of fuzzy multiply positive implicative BCC-ideals of BCC-algebras is introduced, and some related results are obtained. Moreover, we introduce the concept of T-fuzzy multiply positive implicative BCC-ideals of BCC-algebras, investigating its properties. Using a t-norm T, the direct product and T-product of T-fuzzy multiply positive implicative BCC-ideals of BCC-algebras are discussed, and their properties are investigated.

By a BCC-algebra, we mean a nonempty set G with a constant 0 and a binary operation $*$ satisfying the following conditions:

(I) $((x*y)*(z*y))*(x*z)=0$,

(II) $x*x=0$,

(III) $0*x=0$,

(IV) $x*0=x$,

(V) $x*y=0$ and $y*x=0$ imply $x=y$ for all $x,y,z\in G$.

On any BCC-algebra, one can define the partial ordering "$\le$" by putting $x \le y$ if and only if $x * y = 0$.

A BCK-algebra is a BCC-algebra, but there are not BCC-algebra which are not BCK-algebras (cf. [5]). Note that a BCC-algebra X is a BCK-algebra if and only if it satisfies $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$.

A nonempty subset A of a BCC-algebra G is called a BCC-ideal if (i) $0 \in A$ and (ii) $(x * y) * z \in A$ and $y \in A$ imply $x * z \in A$. For any elements x and y of a BCC-algebra, $x * y^n$ denotes $(\cdots((x * y) * y) * \cdots) * y$ in which y occurs n times. A nonempty subset A of a BCC-algebra G is called an n-fold BCC-ideal of G if (i) $0 \in A$ and (ii) for every $x, y, z \in G$, there exists a natural number n such that $x * z^n \in A$ whenever $(x * y) * z^n \in A$ and $y \in A$.

We now review some fuzzy logical concepts. A fuzzy set in set G is a function $\mu : G \to [0,1]$. For a fuzzy set μ in G and $\alpha \in [0,1]$, define $\mu_\alpha = \{x \in G \mid \mu(x) \ge \alpha\}$ which is called a level set of G. A fuzzy set μ in a BCC-algebra G is called a fuzzy BCC-ideal of G if (i) $\mu(0) \ge \mu(x)$ and (ii) $\mu(x * z) \ge \min\{\mu((x * y) * z), \mu(y)\}$ for all $x, y, z \in G$. A fuzzy set μ in a BCC-algebra G is called an n-fold fuzzy BCC-ideal of G if (i) $\mu(0) \ge \mu(x)$ for all $x \in G$ and (ii) for every $x, y, z \in G$, there exists a natural number n such that $\mu(x * z^n) \ge \min\{\mu((x * y) * z^n), \mu(y)\}$.

2. Fuzzy multiply positive implicative BCC-ideals

Definition 2.1. A nonempty subset A of a BCC-algebra G is called a multiply positive implicative BCC-ideal of G if

(i) $0 \in A$,

(ii) for every $x, y, z \in X$, there exists a natural number $k = k(x, y, z)$ such that $x * z^k \in A$ whenever $(x * y) * z^n \in A$ and $y * z^m \in A$ for any natural numbers m and n.

Example 2.2. (i) Consider a BCC-algebra $G = \{0, 1, 2, 3, 4, 5\}$ with the Cayley table as follows:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Then G is a proper BCC-algebra since $(4 * 5) * 2 \ne (4 * 2) * 5$. It is routine to check that $A = \{0, 1, 2, 3, 4\}$ is a multiply positive implicative BCC-ideal of G.

(ii) Consider a BCC-algebra $G = \{0, a, b, c, d\}$ with the Cayley table as follows:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	0
c	c	b	a	0	a
d	d	d	d	d	0

Then G is a proper BCC-algebra since $(c * a) * d \neq (c * d) * a$. It is routine to check that $A = \{0, a, b, c\}$ is a multiply positive implicative BCC-ideal of G.

(iii) Consider a BCC-algebra $G = \{0, a, b, c, 1\}$ with the Cayley table as follows:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	0
c	c	b	a	0	a
1	1	c	c	c	0

Then G is a proper BCC-algebra since $(1 * b) * a \neq (1 * a) * b$. Let $A = \{0, b, c\}$, then A is not a multiply positive implicative BCC-ideals of G because $(1 * c) * 0^n = c * 0^m = c \in A$ while $1 * 0^k = 1 \notin A$.

Definition 2.3. A fuzzy set μ in a BCC-algebra G is called a fuzzy multiply positive implicative BCC-ideal of G if

(i) $\mu(0) \geq \mu(x)$ for all $x \in G$,

(ii) for any $n, m \in \mathbb{N}$, there exists a natural number $k = k(x, y, z)$ such that $\mu(x * z^k) \geq \min\{\mu((x * y) * z^n), \mu(y * z^m)\}$ for all $x, y, z \in G$.

Example 2.4. (i) Consider a BCC-algebra $G = \{0, 1, 2, 3, 4\}$ with the Cayley table as follows:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

It is a proper BCC-algebra since $(3 * 1) * 2 \neq (3 * 2) * 1$. Define a fuzzy set μ in G by $\mu(4) = 0.3$ and $\mu(x) = 0.8$ for all $x \neq 4$. Then μ is a fuzzy multiply positive implicative BCC-ideal of G.

(ii) Let G be a proper BCC-algebra as (i) and let μ be a fuzzy set in G defined by

$$\mu(x) = \begin{cases} \alpha_1 & \text{if } x \in \{0,2,3\}, \\ \alpha_2 & \text{otherwise}, \end{cases} \tag{2.1}$$

where $\alpha_1 > \alpha_2$ in $[0,1]$. It is easy to check that μ is not a fuzzy multiply positive implicative BCC-ideal of G because $\mu(4 * 0^k) = \mu(4) = \alpha_2 \le \min\{\mu((4*3)*0^n), \mu(3*0^m)\}$ for any positive integer numbers m, n, and k.

Theorem 2.5. *Let μ be a fuzzy set in a BCC-algebra G, then μ is a fuzzy multiply positive implicative BCC-ideal of G if and only if the nonempty level set $\mu_\alpha = \{x \in G \mid \mu(x) \ge \alpha\}$ of μ is a multiply positive implicative BCC-ideal of G.*

Proof. Suppose that μ is a fuzzy multiply positive implicative BCC-ideal of G and $\mu_\alpha \ne \emptyset$ for any $\alpha \in [0,1]$. Then there exists $x \in \mu_\alpha$ and so $\mu(x) \ge \alpha$. It follows that $\mu(0) \ge \mu(x) \ge \alpha$ so that $0 \in \mu_\alpha$. Let $x, y, z \in G$ be such that $(x*y)*z^n \in \mu_\alpha$ and $y * z^m \in \mu_\alpha$. By Definition 2.3, there exists a natural number k such that $\mu(x*z^k) \ge \min\{\mu((x*y)*z^n), \mu(y*z^m)\} \ge \min\{\alpha,\alpha\} = \alpha$ and that $x * z^k \in \mu_\alpha$. Hence μ_α is a multiply positive implicative BCC-ideal of G. Conversely, assume that μ_α is a multiply positive implicative BCC-ideal of G for every $\alpha \in [0,1]$. For any $x \in G$, let $\mu(x) = \alpha$. Then $x \in \mu_\alpha$. Since $0 \in \mu_\alpha$, it follows that $\mu(0) \ge \alpha = \mu(x)$ so that $\mu(0) \ge \mu(x)$ for all $x \in G$. Now suppose that there exist $x_0, y_0, z_0 \in G$ such that $\mu(x_0 * z_0^k) < \min\{\mu((x_0*y_0)*z_0), \mu(y_0 * z_0^m)\}$. Let $\lambda_0 = (\mu(x_0*z_0^k) + \min\{\mu((x_0*y_0)*z_0), \mu(y_0*z_0^m)\})/2$, then $\lambda_0 > \mu(x_0*z_0^k)$ and $0 \le \lambda_0 < \min\{\mu((x_0*y_0)*z_k^n), \mu(y_0*z_0^m)\} \le 1$, so we have $\mu((x_0*y_0)*z_0^n) \ge \lambda_0$ and $\mu(y_0*z_0^m) \ge \lambda_0$, then $(x_0*y_0)*z_0^n \in \mu_{\lambda_0}$ and $y_0 * z_0^n \in \mu_{\lambda_0}$. As μ_{λ_0} is a multiply positive BCC-ideal of G, it implies $x_0 * z_0^k \in \mu_{\lambda_0}$ and $\mu(x_0*z_0^k) \ge \lambda_0$. This is a contradiction. Hence μ is a fuzzy multiply positive implicative BCC-ideal of G. □

Theorem 2.6. *Let A be a nonempty subset of a BCC-algebra G, and μ a fuzzy set in G defined by*

$$\mu(x) = \begin{cases} \alpha_1 & \textit{if } x \in A, \\ \alpha_2 & \textit{otherwise}, \end{cases} \tag{2.2}$$

where $\alpha_1 > \alpha_2$ in $[0,1]$. Then μ is a fuzzy multiply positive implicative BCC-ideal of G if and only if A is a multiply positive implicative BCC-ideal of G.

Proof. Assume that μ is a fuzzy multiply positive implicative BCC-ideal of G. Since $\mu(0) \ge \mu(x)$ for all $x \in G$, we have $\mu(0) = \alpha_1$ and so $0 \in A$. Let $x, y, z \in G$ be such that $(x*y)*z^n \in A$ and $y*z^m \in A$. By Definition 2.3, there exists a natural number $k = k(x,y,z)$ such that $\mu(x*z^k) \ge \min\{\mu((x*y)*z^n), \mu(y*z^m)\} = \alpha_1$ and that $x*z^k \in A$. Hence A is a multiply positive implicative BCC-ideal of G.

Conversely, suppose that A is a multiply positive implicative BCC-ideal of G. Since $0 \in A$, it follows that $\mu(0) = \alpha_1 \geq \mu(x)$ for all $x \in G$. Let $x, y, z \in G$. If $y * z^m \notin A$ and $(x * y) * z^n \in A$, then clearly $\mu(x * z^k) \geq \min\{\mu((x * y) * z^n), \mu(y * z^m)\}$. Assume that $y * z^m \in A$ and $(x * y) * z^n \notin A$, we have $(x * y) * z^k \notin A$. Therefore $\mu(x * z^k) = \alpha_2 = \min\{\mu((x * y) * z^n), \mu(y * z^m)\}$. Hence, μ is a fuzzy multiply positive implicative BCC-ideal of G. □

A fuzzy relation on any set S is a fuzzy subset $\mu : S \times S \to [0,1]$. If μ is a fuzzy relation on a set S and ν is a fuzzy subset of S, then μ is a fuzzy relation on ν if $\mu(x, y) \leq \min\{\nu(x), \nu(y)\}$ for all $x, y \in S$. Let μ and ν on S be defined as $(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\}$. One can prove that $\mu \times \nu$ is a fuzzy relation on S and $(\mu \times \nu)_t = \mu_t \times \nu_t$ for all $t \in [0,1]$. If μ is a fuzzy subset of a set S, the strongest fuzzy relation on S that is a fuzzy relation on ν is μ_ν, given by $\mu_\nu(x, y) = \min\{\mu(x), \nu(y)\}$ for all $x, y \in S$. In this case we have $(\mu_\nu)_t = \nu_t \times \nu_t$ for all $t \in [0,1]$ (see [2]).

Theorem 2.7. *For a given fuzzy subset ν of a BCC-algebra G, let μ_ν be the strongest fuzzy relation on G. If μ_ν is a fuzzy multiply positive implicative BCC-ideal of $G \times G$, then $\nu(0) \geq \nu(x)$ for all $x \in G$.*

Proof. Since μ_ν is a fuzzy multiply positive implicative BCC-ideal of $G \times G$, it follows that $\mu_\nu(0,0) \geq \mu_\nu(x,x)$ for all $x \in G$. This means that $\min\{\nu(0), \nu(0)\} \geq \min\{\nu(x), \nu(x)\}$, which implies that $\nu(0) \geq \nu(x)$. □

Theorem 2.8. *If ν is a fuzzy multiply positive implicative BCC-ideal of a BCC-algebra G, then the level multiply positive implicative BCC-ideals of $(\mu_\nu)_t$ are given by*

$$(\mu_\nu)_t = \mu_t \times \nu_t \quad \forall t \in [0,1]. \tag{2.3}$$

The proof is obvious.

Theorem 2.9. *If μ and ν are fuzzy multiply positive implicative BCC-ideals of a BCC-algebra G, then $\mu \times \nu$ is a fuzzy multiply positive implicative BCC-ideal of $G \times G$.*

Proof. For any $(x, y) \in G \times G$,

$$\begin{aligned}(\mu \times \nu)(0,0) &= \min\{\mu(0), \nu(0)\} \geq \min\{\mu(x), \nu(x)\} \\ &= (\mu \times \nu)(x, y).\end{aligned} \tag{2.4}$$

Now, let $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $z = (z_1, z_2) \in G \times G$. For any $n, m \in \mathbb{N}$, there exists a natural number k such that

$$\begin{aligned}(\mu \times \nu)(x * z^k) &= (\mu \times \nu)\Big((x_1, x_2) * (z_1, z_2)^k\Big) \\ &= (\mu \times \nu)(x_1 * z_1^k, x_2 * z_2^k) \\ &= \min\{\mu(x_1 * z_1^k), \nu(x_2 * z_2^k)\}\end{aligned}$$

$$\begin{aligned}
&\geq \min\{\min\{\mu((x_1*y_1)*z_1^n),\mu(y_1*z_1^m)\},\\
&\qquad \min\{\nu((x_1*y_2)*z_2^n),\nu(y_2*z_2^m)\}\}\\
&=\min\{\min\{\mu((x_1*y_1)*z_1^n),\nu((x_2*y_2)*z_2^n)\},\\
&\qquad \min\{\mu(y_1*z_1^m),\nu(y_2*z_2^m)\}\}\\
&=\min\Big\{(\mu\times\nu)\Big(((x_1,x_2)*(y_1,y_2))*(z_1,z_2)^n\Big),\\
&\qquad (\mu\times\nu)((y_1,y_2)*(z_1,z_2)^m)\Big\}\\
&=\min\{(\mu\times\nu)((x*y)*z^n),(\mu\times\nu)(y*z^m)\}.
\end{aligned} \tag{2.5}$$

Hence $\mu\times\nu$ is a fuzzy multiply positive implicative BCC-ideals of $G\times G$. □

Theorem 2.10. *Let μ and ν be fuzzy subsets of a BCC-algebra G such that $\mu\times\nu$ is a fuzzy multiply positive implicative BCC-ideal of $G\times G$. Then*

(i) *either $\mu(x)\leq\mu(0)$ or $\nu(x)\leq\nu(0)$ for all $x\in G$,*

(ii) *if $\mu(x)\leq\mu(0)$ for all $x\in G$, then either $\mu(x)\leq\nu(0)$ or $\nu(x)\leq\nu(0)$,*

(iii) *if $\nu(x)\leq\nu(0)$ for all $x\in G$, then either $\mu(x)\leq\mu(0)$ or $\nu(x)\leq\mu(0)$,*

(iv) *either μ or ν is a fuzzy multiply positive implicative BCC-ideal of G.*

Proof. (i) Suppose that $\mu(x)>\mu(0)$ and $\nu(x)>\nu(0)$ for some $x,y\in G$. Then $(\mu\times\nu)(x,y)=\min\{\mu(x),\nu(y)\}>\min\{\mu(0),\nu(0)\}=(\mu\times\nu)(0,0)$. This is a contradiction and we obtain (i).

(ii) Assume that there exist $x,y\in G$ such that $\mu(x)>\nu(0)$ and $\nu(y)>\nu(0)$. Then $(\mu\times\nu)(0,0)=\min\{\mu(0),\nu(0)\}=\nu(0)$. It follows that $(\mu\times\nu)(x,y)=\min\{\mu(x),\nu(y)\}>\nu(0)=(\mu\times\nu)(0,0)$. This is a contradiction. Hence (ii) holds.

(iii) Item (iii) is proved by similar method to part (ii).

(iv) Since by (i), either $\mu(x)\leq\mu(0)$ or $\nu(x)\leq\nu(0)$ for all $x\in G$, without loss of generality, we may assume that $\nu(x)\leq\nu(0)$ for all $x\in G$. Form (iii), it follows that either $\mu(x)\leq\mu(0)$ or $\nu(x)\leq\mu(0)$. If $\nu(x)\leq\mu(0)$ for all $x\in G$, then $(\mu\times\nu)(0,x)=\min\{\mu(0),\nu(x)\}=\nu(x)$. Let $(x_1,x_2),(y_1,y_2),(z_1,z_2)\in G\times G$. Since $\mu\times\nu$ is a fuzzy multiply positive implicative BCC-ideal of $G\times G$, then for any $n,m\in\mathbb{N}$, there exists a natural number k such that

$$\begin{aligned}
&(\mu\times\nu)\Big((x_1,x_2)*(z_1,z_2)^k\Big)\\
&\quad\geq\min\Big\{(\mu\times\nu)\Big(((x_1,x_2)*(y_1,y_2))*(z_1,z_2)^n\Big),\\
&\qquad (\mu\times\nu)\Big((y_1,y_2)*(z_1,z_2)^m\Big)\Big\} \qquad (2.6)\\
&\quad=\min\{(\mu\times\nu)(((x_1*y_1)*z_1^n),((x_2*y_2)*z_2^n)),\\
&\qquad (\mu\times\nu)(y_1*z_1^m,y_2*z_2^m)\}.
\end{aligned}$$

If we take $x_1 = y_1 = z_1 = 0$, then

$$\begin{aligned}
\nu(x_2 * z_2^k) &= (\mu\times\nu)(0, x_2 * z_2^k) \\
&= (\mu\times\nu)\big((0,x_2) * (0,z_2)^k\big) \\
&\geq \min\{(\mu\times\nu)(0,(x_2 * y_2) * z_2^n), (\mu\times\nu)(0, y_2 * z_2^m)\} \\
&= \min\{\min\{\mu(0), \nu((x_2 * y_2) * z_2^n)\}, \min\{\nu(0), \nu(y_2 * z_2^m)\}\} \\
&= \min\{\nu((x_2 * y_2) * z_2^n), \nu(y_2 * z_2^m)\}.
\end{aligned} \tag{2.7}$$

This proves that ν is a fuzzy multiply positive BCC-ideal of G. Now we consider the case $\mu(x) \leq \mu(0)$ for all $x \in G$. Suppose that $\nu(y) > \mu(0)$ for some $y \in G$. Then $\nu(0) \geq \nu(y) > \mu(0)$. Since $\mu(0) \geq \mu(x)$ for all $x \in G$, it follows that $\nu(0) > \mu(x)$ for any $x \in G$. Hence $(\mu\times\nu)(x,0) = \min\{\mu(x), \nu(0)\} = \mu(x)$. Taking $x_2 = y_2 = z_2 = 0$ in (2.6), then

$$\begin{aligned}
\mu(x_1 * z_1^k) &= (\mu\times\nu)(x_1 * z_1^k, 0) \\
&= (\mu\times\nu)\big((x_1,0) * (z_1,0)^k\big) \\
&\geq \min\{(\mu\times\nu)((x_1 * y_1) * z_1^n, 0), (\mu\times\nu)(y_1 * z_1^m, 0)\} \\
&= \min\{\min\{\mu((x_1 * y_1) * z_1^n), \nu(0)\}, \min\{\mu(y_1 * z_1^m), \nu(0)\}\} \\
&= \min\{\mu((x_1 * y_1) * z_1^n), \mu(y_1 * z_1^m)\}
\end{aligned} \tag{2.8}$$

which proves that μ is a fuzzy multiply positive implicative BCC-ideal of G. □

Theorem 2.11. *Let ν be a fuzzy subset of a BCC-algebra G and let μ_ν be the strongest fuzzy relation on G. Then ν is a fuzzy multiply positive implicative BCC-ideal of G if and only if μ_ν is a fuzzy multiply positive implicative BCC-ideal of $G\times G$.*

Proof. Assume that ν is a fuzzy multiply positive implicative BCC-ideal of X, then

$$\mu_\nu(0,0) = \min\{\nu(0), \nu(0)\} \geq \min\{\nu(x), \nu(y)\} = \mu_\nu(x,y) \tag{2.9}$$

for any $(x,y) \in G\times G$. Moreover, for any $n, m \in \mathbb{N}$, there exists a natural number k such that

$$\begin{aligned}
\mu_\nu\big((x_1,x_2) * (z_1,z_2)^k\big) &= \mu_\nu(x_1 * z_1^k, x_2 * z_2^k) \\
&= \min\{\nu(x_1 * z_1^k), \nu(x_2 * z_2^k)\} \\
&\geq \min\{\min\{\nu((x_1 * y_1) * z_1^n), \nu(y_1 * z_1^m)\}, \\
&\qquad \min\{\nu((x_2 * y_2) * z_2^n), \nu(y_2 * z_2^m)\}\}
\end{aligned}$$

$$\begin{aligned}
&= \min\{\min\{\nu((x_1*y_1)*z_1^n),\nu((x_2*y_2)*z_2^n)\},\\
&\qquad \min\{\nu(y_1*z_1^m),\nu(y_2*z_2^m)\}\}\\
&= \min\{\mu_\nu(((x_1*y_1)*z_1^n),(x_2*y_2)*z_2^n),\\
&\qquad \mu_\nu(y_1*z_1^m,y_2*z_2^m)\}\\
&= \min\Big\{\mu_\nu\Big(((x_1,x_2)*(y_1,y_2))*(z_1,z_2)^n\Big),\\
&\qquad \mu_\nu\Big((y_1,y_2)*(z_1,z_2)^m\Big)\Big\}
\end{aligned} \tag{2.10}$$

for any $(x_1,x_2),(y_1,y_2),(z_1,z_2)\in G\times G$.

Hence μ_ν is a fuzzy multiply positive implicative BCC-ideal of $G\times G$.

Conversely, suppose that μ_ν is a fuzzy multiply positive implicative BCC-ideal of $G\times G$. Then for all $(x_1,x_2)\in G\times G$,

$$\min\{\nu(0),\nu(0)\}=\mu_\nu(0,0)\geq\mu_\nu(x_1,x_2)=\min\{\nu(x_1),\nu(x_2)\}. \tag{2.11}$$

It follows that $\nu(0)\geq\nu(x)$ for all $x\in G$. Now, for any $n,m\in\mathbb{N}$, there exists a natural number k such that

$$\begin{aligned}
&\min\{\nu(x_1*z_1^k),\nu(x_2*z_2^k)\}\\
&\quad= \mu_\nu(x_1*z_1^k,x_2*z_2^k)=\mu_\nu\Big((x_1,x_2)*(z_1,z_2)^k\Big)\\
&\quad\geq \min\Big\{\mu_\nu\Big(((x_1,x_2)*(y_1,y_2))*(z_1,z_2)^n\Big),\mu_\nu\Big((y_1,y_2)*(z_1,z_2)^m\Big)\Big\}\\
&\quad= \min\{\mu_\nu((x_1*y_1)*z_1^n,(x_2*y_2)*z_2^n),\mu_\nu(y_1*z_1^m,y_2*z_2^m)\}\\
&\quad= \min\{\min\{\nu((x_1*y_1)*z_1^n),\nu((x_2*y_2)*z_2^n)\},\\
&\qquad\quad \min\{\nu(y_1*z_1^m),\nu(y_2*z_2^m)\}\}\\
&\quad= \min\{\min\{\nu((x_1*y_1)*z_1^n),\nu(y_1*z_1^m)\},\\
&\qquad\quad \min\{\nu((x_2*y_2)*z_2^n),\nu(y_2*z_2^m)\}\}.
\end{aligned} \tag{2.12}$$

If we take $x_2=y_2=z_2=0$ (resp., $x_1=y_1=z_1=0$), then $\nu(x_1*z_1^k)\geq\min\{\nu((x_1*y_1)*z_1^n),\nu(y_2*z_2^m)\}$. Hence ν is a fuzzy multiply positive implicative BCC-ideal of G. □

3. *T*-fuzzy multiply positive implicative BCC-ideals

Definition 3.1 [1]. By a t-norm T, we mean a function $T:[0,1]\times[0,1]\to[0,1]$ satisfying the following conditions:

(I) $T(x,1)=x$,

(II) $T(x,y)\leq T(x,z)$ if $y\leq z$,

(III) $T(x,y)=T(y,x)$,

(IV) $T(x,T(y,z))=T(T(x,y),z)$ for all $x,y,z\in[0,1]$.

Every t-norm T has a useful property $T(\alpha,\beta)\leq\min\{\alpha,\beta\}$ for all $\alpha,\beta\in[0,1]$.

Lemma 3.2 [1]. *Let T be a t-norm. Then $T(T(\alpha,\beta),T(\nu,\delta)) = T(T(\alpha,\nu),T(\beta,\delta))$ for all $\alpha,\beta,\nu,\delta \in [0,1]$.*

Definition 3.3. A fuzzy subset $\mu : G \to [0,1]$ in a BCC-algebra G is called a fuzzy multiply positive implicative BCC-ideal of G with respect to a t-norm T (briefly, T-fuzzy multiply positive implicative BCC-ideal of G) if

(i) $\mu(0) \geq \mu(x)$ for all $x \in G$,

(ii) for any $n,m \in \mathbb{N}$, there exists a natural number $k = k(x,y,z)$ such that $\mu(x * z^k) \geq T(\mu((x*y)*z^n),\mu(y*z^m))$ for any $x,y,z \in G$.

Example 3.4. Consider a BCC-algebra $G = \{0,1,2,3,4\}$ with the Cayley table as follows:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	3	4	4	3	0

By routine calculation, G is a proper BCC-algebra (cf. [5]). Define a fuzzy set μ by $\mu(0) = \mu(1) = \mu(2) = \mu(3) = 0.8$ and $\mu(4) = 0.3$. Let $T(\alpha,\beta) = \max\{\alpha+\beta-1,0\}$ for all $\alpha,\beta \in [0,1]$. Then T is a t-norm. It is easy to check that μ is a T-fuzzy multiply positive implicative BCC-ideal of G.

Theorem 3.5. *Let μ be a T-fuzzy multiply positive implicative BCC-ideal of a BCC-algebra G and let $\alpha \in [0,1]$ if $\alpha = 1$, then the nonempty subset μ_α is a multiply positive implicative BCC-ideal of G.*

Proof. Assume that $\alpha = 1$ and $x \in \mu_\alpha$, then $\mu(x) \geq 1$. Thus $\mu(0) \geq \mu(x) \geq 1$ and $0 \in \mu_\alpha$.

Moreover, suppose that $(x*y)*z^n \in \mu_\alpha$ and $y*z^m \in \mu_\alpha$, then $\mu((x*y)*z^n) \geq 1$ and $\mu(y*z^m) \geq 1$. By Definition 3.3, there exists a natural number k such that $\mu(x*z^k) \geq T(\mu((x*y)*z^n),\mu(y*z^m)) \geq T(1,1) = 1$ and that $x*z^k \in \mu_\alpha$. Hence μ_α is a multiply positive implicative BCC-ideal of G. □

For a fuzzy set μ on a BCC-algebra G and a map $\theta : G \to G$, we define a mapping $\mu[\theta] : G \to [0,1]$ by $\mu[\theta](x) = \mu(\theta(x))$ for all $x \in G$.

Theorem 3.6. *If μ is a T-fuzzy multiply positive implicative BCC-ideal of a BCC-algebra G and θ is an epimorphism of G, then $\mu[\theta]$ is a T-fuzzy multiply positive implicative BCC-ideal of G.*

Proof. Let $\mu[\theta](0) = \mu(\theta(0)) = \mu(0) \geq \mu(y)$ for any $y \in G$. Since θ is an epimorphism of G, then there exists $x \in G$ such that $\theta(x) = y$. Thus $\mu[\theta](0) \geq \mu(\theta(x)) = \mu[\theta](x)$. As y is an arbitrary element of G, the above result is true for any $x \in G$.

Moreover, for any $n, m \in \mathbb{N}$, there exists a natural number k such that

$$\begin{aligned} \mu[\theta](x * z^k) &= \mu(\theta(x * z^k)) = \mu(\theta(x) * \theta(z)^k) \\ &\geq T(\mu((\theta(x) * \theta(y)) * \theta(z)^n), \mu(\theta(y) * \theta(z)^m)) \\ &= T(\mu(\theta((x * y) * z^n)), \mu(\theta(y * z^m))) \\ &= T(\mu[\theta]((x * y) * z^n), \mu[\theta](y * z^m)). \end{aligned} \tag{3.1}$$

Hence $\mu[\theta]$ is a T-fuzzy multiply positive implicative BCC-ideal of G. □

Let f be a mapping defined on a BCC-algebra G. If ν is a fuzzy set in $f(G)$, then the fuzzy set μ_ν of G defined by $\mu(x) = \nu(f(x))$ is called the preimage of ν under f.

Theorem 3.7. *An onto homomorphic preimage of a T-fuzzy multiply positive implicative BCC-ideal is a T-fuzzy multiply positive implicative BCC-ideal.*

Proof. Let $f : G \to G'$ be an onto homomorphism of BCC-algebra, ν a T-fuzzy multiply positive implicative BCC-ideal of G', and μ the preimage of ν under f. Then $\mu(0) = \nu(f(0)) = \nu(0') \geq \nu(f(x)) = \mu(x)$ for all $x \in G$. Moreover, for any $n, m \in \mathbb{N}$, there exists a natural number k such that

$$\begin{aligned} \mu(x * z^k) &= \nu(f(x * z^k)) = \nu(f(x) * f(z)^k) \\ &\geq T(\nu((f(x) * f(y)) * f(z)^n), \nu(f(y) * f(z)^m)) \\ &= T(\nu(f((x * y) * z^n)), \nu(f(y * z^m))) \\ &= T(\mu((x * y) * z^n), \mu(y * z^m)) \end{aligned} \tag{3.2}$$

for any $x, y, z \in G$. Hence μ is a T-fuzzy multiply positive implicative BCC-ideal of G. □

If μ is a fuzzy set in a BCC-algebra G and f is a mapping defined on G, then the fuzzy set μ^f in $f(G)$ defined by $\mu^f(y) = \sup_{x \in f^{-1}(y)} \mu(x)$ for all $y \in G$ is called the image of μ under f. A fuzzy set μ in G is said to have sup property if, for every subset $T \subseteq G$, there exists $t_0 \in T$ such that $\mu(t_0) = \sup_{t \in T} \mu(t)$.

Theorem 3.8. *An onto homomorphic image of a T-fuzzy multiply positive implicative BCC-ideal with sup property is a T-fuzzy multiply positive implicative BCC-ideal.*

Proof. Let $f : G \to G'$ be an onto homomorphism of BCC-algebras and let μ be a T-fuzzy multiply positive implicative BCC-ideal of G with sup property. Then $\mu^f(0) = \sup_{f \in f^{-1}(0)} \mu(t) = \mu(0) \geq \mu(x)$ for any $x \in G$. Furthermore, we have $\mu^f(x_1) = \sup_{t\in f^{-1}(x_1)} \mu(t)$ for any $x_1 \in G'$. Thus $\mu^f(0) \geq \sup_{t\in f^{-1}(x_1)} \mu(t) = \mu^f(x_1)$ for any $x_1 \in G'$. Moreover, for any $x_1, y_1, z_1 \in G'$, let $x \in f^{-1}(x_1)$, $y \in f^{-1}(y_1)$, and $z \in f^{-1}(z_1)$ such that

$$\begin{aligned} \mu(x * z^k) &= \sup_{t\in f^{-1}(x_1 * z_1^k)} \mu(t), \\ \mu((x * y) * z^n) &= \sup_{t\in f^{-1}((x*y)*z^n)} \mu(t), \\ \mu(y * z^n) &= \sup_{t\in f^{-1}(y_1 * z_1^m)} \mu(t). \end{aligned} \tag{3.3}$$

Thus

$$\begin{aligned} \mu^f(x_1 * z_1^k) &= \sup_{t\in f^{-1}(x_1 * z_1^k)} \mu(t) = \mu(x * z^k) \\ &\geq T(\mu((x*y)*z^n), \mu(y * z^m)) \\ &= T\left(\sup_{t\in f^{-1}((x_1*y_1)*z_1^n)} \mu(t), \sup_{t\in f^{-1}(y_1*z_1^m)} \mu(t)\right) \\ &= T(\mu^f((x_1 * y_1) * z_1^n), \mu^f(y_1 * z_1^m)). \end{aligned} \tag{3.4}$$

Therefore, μ^f is a T-fuzzy multiply positive implicative BCC-ideal of G'. □

4. Fuzzy multiply positive implicative BCC-ideals induced by norms

Theorem 4.1. *Let T be a t-norm and $G = G_1 \times G_2$ the direct product BCC-algebra of BCC-algebras G_1 and G_2. If μ_1 (resp., μ_2) is a T-fuzzy multiply positive implicative BCC-ideal of G_1 (resp., G_2), then $\mu = \mu_1 \times \mu_2$ is a T-fuzzy multiply positive implicative BCC-ideal of G defined by $\mu(x_1, x_2) = (\mu_1 \times \mu_2)(x_1, x_2) = T(\mu_1(x_1), \mu_2(x_2))$ for all $(x_1, x_2) \in G_1 \times G_2$.*

The proof is identical with the corresponding proof from [3].

We will generalize the idea to the product of n T-fuzzy multiply positive implicative BCC-ideals. We first need to generalize the domain of t-norm T to $\Pi_{i=1}^n [0,1]$ as follows.

The function $T_n : \Pi_{i=1}^n [0,1] \to [0,1]$ is defined by

$$T_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = T(\alpha_i, T_{n-1}(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n)) \tag{4.1}$$

for all $1 \leq i \leq n$, where $n \geq 2$, $T_2 = T$, and $T_1 = \text{id}$ (identity). For a t-norm T and every $\alpha_i, \beta_i \in [0,1]$, where $1 \leq i \leq n$ and $n \geq 2$, we have

$$\begin{aligned} &T_n(T(\alpha_1, \beta_1), T(\alpha_2, \beta_2), \ldots, T(\alpha_n, \beta_n)) \\ &\quad = T(T_n(\alpha_1, \alpha_2, \ldots, \alpha_n), T_n(\beta_1, \beta_2, \ldots, \beta_n)). \end{aligned} \tag{4.2}$$

Theorem 4.2. *Let T be a t-norm, $\{G_i\}_{i=1}^n$ the finite collection of BCC-algebras, and $G = \Pi_n^{i=1} G_i$ the direct product BCC-algebra of $\{G_i\}$. Let μ_i be a T-fuzzy multiply positive implicative BCC-ideal of $\{G_i\}$, where $1 \le i \le n$. Then $\mu = \Pi_{i=1}^n \mu_i$ defined by $\mu(x_1,x_2,\dots,x_n) = (\Pi_{i=1}^n \mu_i)(x_1,x_2,\dots,x_n) = T_n(\mu_1(x_1),\mu_2(x_2),\dots,\mu_n(x_n))$ is a T-fuzzy multiply positive implicative BCC-ideal of G.*

The proof is identical with the corresponding proof from [3].

Definition 4.3 [4]**.** Let T be a t-norm and let μ and ν be fuzzy sets in a BCC-algebra G. Then the T-product of μ and ν, written as $[\mu\cdot\nu]_T$, is defined by $[\mu\cdot\nu]_T(x) = T(\mu(x),\nu(x))$ for all $x \in G$.

Theorem 4.4. *Let T be a t-norm and let μ and ν be T-fuzzy multiply positive implicative BCC-ideals of a BCC-algebra G. If T^* is a t-norm which dominates T, that is, $T^*(T(\alpha,\beta),T(\nu,\delta)) \ge T(T^*(\nu,\delta),T^*(\beta,\delta))$ for all $\alpha,\beta,\nu,\delta \in [0,1]$, then the T^*-product of μ and ν, $[\mu\cdot\nu]_{T^*}$, is a T-fuzzy multiply positive implicative BCC-ideal of G.*

Proof. Let $[\mu\cdot\nu]_{T^*}(0) = T^*(\mu(0),\nu(0)) \ge T^*(\mu(x),\nu(x)) = [\mu\cdot\nu]_{T^*}(x)$ for any $x \in G$. Moreover, for any $n,m \in \mathbb{N}$, there exists a natural number k, such that

$$\begin{aligned}
&[\mu\cdot\nu]_{T^*}(x*z^k)\\
&\quad= T^*(\mu(x*z^k),\nu(x*z^k))\\
&\quad\ge T^*(T(\mu((x*y)*z^n),\mu(y*z^m)),T(\nu((x*y)*z^n),\nu(y*z^m)))\\
&\quad\ge T(T^*(\mu((x*y)*z^n),\nu((x*y)*z^n)),T^*(\mu(y*z^m),\nu(y*z^m)))\\
&\quad= T([\mu\cdot\nu]_{T^*}((x*y)*z^n),[\mu\cdot\nu]_{T^*}(y*z^m)).
\end{aligned} \tag{4.3}$$

Hence $[\mu\cdot\nu]_{T^*}$ is a T-fuzzy multiply positive implicative BCC-ideal of G. □

Let $f : G \to G'$ be an onto homomorphism of BCC-algebras. Let T and T^* be t-norms such that T^* dominates T. If μ and ν are T-fuzzy multiply positive implicative BCC-ideals of G', then the T^*-product of μ and ν, $[\mu\cdot\nu]_{T^*}$, is a T-fuzzy multiply positive implicative BCC-ideal of G'. Since every onto homomorphism preimage of a T-fuzzy multiply positive implicative BCC-ideal is a T-fuzzy multiply positive implicative BCC-ideal, the preimages $f^{-1}(\mu)$, $f^{-1}(\nu)$, and $f^{-1}([\mu\cdot\nu]_{T^*})$ are T-fuzzy multiply positive implicative BCC-ideals of G. The next theorem provides the relation between $f^{-1}([\mu\cdot\nu]_{T^*})$ and T^*-product $[f^{-1}(\mu)\cdot f^{-1}(\nu)]_{T^*}$ of $f^{-1}(\mu)$ and $f^{-1}(\nu)$.

Theorem 4.5. *Let $f : G \to G'$ be an onto homomorphism of BCC-algebras. Let T and T^* be t-norms such that T^* dominates T. Let μ and ν be T-fuzzy multiply positive implicative BCC-ideals of G'. If $[\mu\cdot\nu]_{T^*}$ is the T^*-product of μ and ν, and $[f^{-1}(\mu)\cdot f^{-1}(\nu)]_{T^*}$ is the T^*-product of $f^{-1}(\mu)$ and $f^{-1}(\nu)$, then $f^{-1}([\mu\cdot\nu]_{T^*}) = [f^{-1}(\mu)\cdot f^{-1}(\nu)]_{T^*}$.*

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