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ON THE p-ADIC SPECTRAL ANALYSIS AND MULTIWAVELET ON $L^2(\mathbb{Q}_p^n)$

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To Dong-Sik Kim on his 62nd birthday

We try to generalize an explicit construction for an orthonormal system of eigenfunctions of the Vladimirov operator on $L^2(\mathbb{Q}_p^n)$ which was originally given by Vladimirov (1988). Also the multiwavelet analysis can be considered as the p-adic spectral analysis in $L^2(\mathbb{Q}_p^n)$, using in part the recent work of Kozyrev (2002).

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1. Introduction. Let $\mathbb{Z}$, $\mathbb{C}, \mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of integers, the field of complex numbers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let ord_p denote the unique p-adic ordinal over $\overline{\mathbb{Q}}_p$ such that $\operatorname{ord}_p(p) = 1$. The corresponding non-Archimedean absolute value is $|x|_p = p^{-\operatorname{ord}_p(x)}$.

Any p-adic number $x \neq 0$ is uniquely represented in the canonical form $x = \sum_{k=\gamma(x)}^{\infty} x_k p^k$, where $\gamma = \gamma(x) \in \mathbb{Z}$ and x_j are integers such that $0 \leq x_k \leq p-1$, $x_0 > 0$, $k = 0, 1, 2, \ldots$. The fractional part $\{x\}_p$ of a number $x \in \mathbb{Q}_p$ is defined by

$$\{x\}_p = \begin{cases} 0, & \text{if } \gamma \geq 0 \text{ or } x = 0, \\ \displaystyle\sum_{k=\gamma}^{-1} x_k p^k, & \text{if } \gamma < 0. \end{cases} \tag{1.1}$$

It is easy to see from this definition that $p^{\gamma} \leq \{x\}_p \leq 1 - p^{\gamma}$ if $\gamma < 0$. The function $\chi_p(\xi x) = \exp(2\pi i \{\xi x\}_p)$ for every fixed $\xi \in \mathbb{Q}_p$ is an additive character of the field $\mathbb{Q}_p$ and the group B_γ (see [10]). From the relation for fractional parts we have $\{x + y\}_p = \{x\}_p + \{y\}_p - N$, $N = 0, 1$. For later use, we define the step function $\Omega(t)$ by

$$\Omega(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq 1, \\ 0, & \text{if } t > 1. \end{cases} \tag{1.2}$$

The space $\mathbb{Q}_p^n$ consists of points $x = (x_1,\dots,x_n)$, $x_j \in \mathbb{Q}_p$, $j = 1,2,\dots,n$. The norm on $\mathbb{Q}_p^n$ is $|x|_p = \max_{1\le j\le n}|x_j|_p$, $x_j \in \mathbb{Q}_p$. This is a non-Archimedean norm since $|x+y|_p \le \max(|x|_p,|y|_p)$, $x,y \in \mathbb{Q}_p^n$. The space $\mathbb{Q}_p^n$ is clearly complete metric, locally compact, and totally disconnected space. We introduce the inner product by $\langle x,y\rangle = x_1y_1+\cdots+x_ny_n$, $x,y \in \mathbb{Q}_p^n$.

We denote by $B_\gamma^n(a)$ the ball of radius p^γ with center at the point $a \in \mathbb{Q}_p^n$ and by $S_\gamma^n(a)$ its boundary (sphere), that is,

$$\begin{aligned} B_\gamma^n(a) &= \{x \in \mathbb{Q}_p^n : |x-a|_p \le p^\gamma\}, \\ S_\gamma^n(a) &= \{x \in \mathbb{Q}_p^n : |x-a|_p = p^\gamma\}. \end{aligned} \tag{1.3}$$

For the notational convenience, let $B_\gamma^n(0) = B_\gamma^n$ and $S_\gamma^n(0) = S_\gamma^n, \gamma \in \mathbb{Z}$. If $a = (a_1,\dots,a_n) \in \mathbb{Q}_p^n$, then $B_\gamma^n(a) = B_\gamma(a_1)\times\cdots\times B_\gamma(a_n)$ in $\mathbb{Q}_p^n$. Clearly, $B_\gamma^n(a)$ and $S_\gamma^n(a)$ are closed-open sets.

Recently, very interesting properties of spectral theory in the p-adic number field were studied (cf. [5, 6, 8, 10]). The important basic operator in the analysis on complex-valued functions over non-Archimedean local fields $K = \mathbb{Q}_p$ is the fractional differential operator D^α (see Section 1) introduced and studied by Vladimirov in [8], and for a general local field K by Kochubei [5]. This operator considered on $L_2(\mathbb{Q}_p)$ has a pure point spectrum with eigenvalues of the infinite multiplicity. An explicit construction of an eigenbasis was first given by Vladimirov [8]. Due to the infinite multiplicity, it is possible to construct eigenbases with different properties, and a new simpler construction was proposed recently by Kozyrev (see [6]) and applied by him to the 2-adic interpretation of wavelets. His result was a motive for our study (see Section 3). A detailed analysis of the spectrum and eigenfunctions of the Schrödinger-type operator $D^\alpha + V(|x|_p)$ over $\mathbb{Q}_p$, with $V(r)\to\infty$ as $r\to\infty$, is given in [10].

In Section 2, we try to generalize an explicit construction for an orthonormal system of eigenfunctions of the Vladimirov operator on $L^2(\mathbb{Q}_p^n)$ which was originally given by Vladimirov [8], and their properties of dimension 1 are also given in [6]. In Section 3, we give a generalization of Kozyrev's results (see Theorem 3.4). Those of multidimensional case with some conditions are proved. Also the multiwavelet analysis can be considered as the p-adic spectral analysis in $L^2(\mathbb{Q}_p^n)$.

2. The Vladimirov operator D^α on $L^2(\mathbb{Q}_p^n)$.

We now recall the definitions and results from the p-adic spectral theory on the p-adic space $\mathbb{Q}_p^n$ (see [2, 4, 5, 6, 8, 9, 10]). The Haar measure dx_i $(i = 1,2,\dots,n)$ is the essentially invariant measure on the additive group $\mathbb{Q}_p : d(x_i+a) = dx_i$ for any $a \in \mathbb{Q}_p$ and so it is extended up to an invariant measure $dx = dx_1dx_2\cdots dx_n$ on $\mathbb{Q}_p^n$ in a standard way, and all integration theories are carried over to $\mathbb{Q}_p^n$. Normalization is fixed by taking the measure of $\mathbb{Z}_p^n$, the n-times Cartesian product of p-adic integers, as being equal to 1:

$$\mu(\mathbb{Z}_p^n) = \int_{\mathbb{Z}_p^n} dx = \int_{|x|_p \le 1} dx = 1. \tag{2.1}$$

It is now straightforward to calculate the measure of any n-ball and also of n-sphere, that is, for $\gamma \in \mathbb{Z}$,

$$\mu(B_\gamma^n) = \int_{B_\gamma^n} dx = p^{n\gamma}, \tag{2.2}$$

$$\mu(S_\gamma^n) = \int_{S_\gamma^n} dx = \int_{B_\gamma^n} dx - \int_{B_{\gamma-1}^n} dx = p^{n\gamma}\left(1 - \frac{1}{p^n}\right). \tag{2.3}$$

A complex-valued function $f(x)$ defined on $\mathbb{Q}_p^n$ is called *locally constant* if for any point $x \in \mathbb{Q}_p^n$ there exists an integer $l(x) \in \mathbb{Z}$ such that

$$f(x+x') = f(x), \qquad |x'|_p \le p^{l(x)}. \tag{2.4}$$

For the set of locally constant functions on $\mathbb{Q}_p^n$ we denote $\mathcal{E} = \mathcal{E}(\mathbb{Q}_p^n)$. We call a *test function* for each function in $\mathcal{E}$ compact support. The set of test functions which are linear is denoted by $\mathcal{D} = \mathcal{D}(\mathbb{Q}_p^n)$. Let $\varphi \in \mathcal{D}$. Then there exists $l \in \mathbb{Z}$ such that

$$\varphi(x+x') = \varphi(x), \quad x' \in B_l^n,\ x \in \mathbb{Q}_p^n. \tag{2.5}$$

Such largest number l we call it *the parameter of constancy* of a function φ, $l = l(\varphi)$. We denote by $\mathcal{D}_\gamma^l = \mathcal{D}_\gamma^l(\mathbb{Q}_p^n)$ the set of test functions with support in the disc B_γ^n and with parameter of constancy greater than or equal to l. Let $\varphi \in \mathcal{D}$. Its *Fourier-transform* $F[\varphi] = \tilde{\varphi}$ is defined by the formula

$$\tilde{\varphi}(\xi) = F[\varphi](\xi) = \int_{\mathbb{Q}_p^n} \chi_p(\langle \xi, x\rangle)\varphi(x)dx, \quad \xi \in \mathbb{Q}_p^n. \tag{2.6}$$

The Fourier transform $\varphi \to \tilde{\varphi}$ is the linear isomorphism from $\mathcal{D}$ onto $\mathcal{D}$, and also the inversion formula is valid:

$$\varphi(x) = F^{-1}[\tilde{\varphi}](x) = \int_{\mathbb{Q}_p^n} \chi_p(-\langle x, \xi\rangle)\tilde{\varphi}(\xi)d\xi, \quad \tilde{\varphi}, \varphi \in \mathcal{D}. \tag{2.7}$$

Lemma 2.1. *Let $\mathcal{D}'$ be the set of linear continuous functionals on $\mathcal{D}$. Every function $f \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^n)$ defines a generalized function $f \in \mathcal{D}'$ by*

$$\langle f, \psi\rangle = \int_{\mathbb{Q}_p^n} f(x)\psi(x)dx, \quad \varphi \in \mathcal{D}. \tag{2.8}$$

The Vladimirov operator $D^\alpha : \psi \to D^\alpha\psi$ is defined by the convolution of generalized function $f_{-\alpha}$ and ψ:

$$\begin{aligned}(D^\alpha\psi)(x) &= (f_{-\alpha} * \psi)(x) \\ &= \langle f_{-\alpha}(y), \psi(x-y)\rangle, \quad \alpha \ne -n \in \mathbb{C},\ x, y \in \mathbb{Q}_p^n,\end{aligned} \tag{2.9}$$

where $f_{-\alpha}(x) = (1-p^{\alpha})/(1-p^{-\alpha-n})|x|_p^{-\alpha-n}$. *Then*

$$(D^{\alpha}\psi)(x) = \left(F^{-1}\circ|\xi|_p^{\alpha}\cdot F[\psi]\right)(x) = \frac{p^{\alpha}-1}{1-p^{-\alpha-n}}\int_{\mathbb{Q}_p^n}\frac{\psi(x)-\psi(y)}{|x-y|_p^{\alpha+n}}dy. \tag{2.10}$$

Proof. By (2.2) and (2.3) we calculate the following integrals:

$$\begin{aligned}\int_{B_0^n}|x|_p^{\alpha-n}dx &= \sum_{-\infty<\gamma\le 0}p^{\gamma(\alpha-n)}\int_{S_\gamma^n}dx\\ &= \left(1-\frac{1}{p^n}\right)\sum_{-\infty<\gamma\le 0}p^{\gamma\alpha} = \frac{1-p^{-n}}{1-p^{-\alpha}}\quad(\operatorname{Re}\alpha>0),\\ \int_{\mathbb{Q}_p^n\setminus B_0^n}|x|_p^{\alpha-n}dx &= \sum_{1\le\gamma<\infty}p^{\gamma(\alpha-n)}\int_{S_\gamma^n}dx\\ &= \left(1-\frac{1}{p^n}\right)\sum_{1\le\gamma<\infty}p^{\gamma\alpha} = -\frac{1-p^{-n}}{1-p^{-\alpha}}\quad(\operatorname{Re}\alpha<0),\end{aligned} \tag{2.11}$$

and we have an analytical continuation for $\operatorname{Re}\alpha \ge 0$, $\alpha \ne \alpha_k = 2k\pi i/\ln p$, $k \in \mathbb{Z}$. Therefore,

$$\int_{\mathbb{Q}_p^n}|x|_p^{\alpha-n}dx = \int_{B_0^n}|x|_p^{\alpha-n}dx + \int_{\mathbb{Q}_p^n\setminus B_0^n}|x|_p^{\alpha-n}dx = 0, \tag{2.12}$$

where $\alpha \ne \alpha_k$, $k \in \mathbb{Z}$. Then we see that

$$\langle |x|_p^{\alpha-n},\psi\rangle = \int_{\mathbb{Q}_p^n}|x|_p^{\alpha-n}\psi(x)dx = \int_{\mathbb{Q}_p^n}|x|_p^{\alpha-n}[\psi(x)-\psi(0)]dx \tag{2.13}$$

for $\alpha \ne \alpha_k$, $k \in \mathbb{Z}$. Set

$$f_{-\alpha}(x) = \frac{1-p^{\alpha}}{1-p^{-\alpha-n}}|x|_p^{-\alpha-n}. \tag{2.14}$$

We obtain

$$\begin{aligned}(D^{\alpha}\psi)(x) = (f_{-\alpha}*\psi)(x) &= \frac{1-p^{\alpha}}{1-p^{-\alpha-n}}\langle |y|_p^{-\alpha-n},\psi(x-y)\rangle\\ &= \frac{1-p^{\alpha}}{1-p^{-\alpha-n}}\int_{\mathbb{Q}_p^n}|y|_p^{-\alpha-n}[\psi(x-y)-\psi(x)]dy\\ &= \frac{p^{\alpha}-1}{1-p^{-\alpha-n}}\int_{\mathbb{Q}_p^n}\frac{\psi(x)-\psi(x-y)}{|y|_p^{\alpha+n}}dy\\ &= \frac{p^{\alpha}-1}{1-p^{-\alpha-n}}\int_{\mathbb{Q}_p^n}\frac{\psi(x)-\psi(\tilde{y})}{|x-\tilde{y}|_p^{\alpha+n}}dy.\end{aligned} \tag{2.15}$$

On the other hand, by the formula (2.3) we obtain

$$F[f_{-\alpha}(x)] = F\left[\frac{1-p^{\alpha}}{1-p^{-\alpha-n}}|x|_p^{-\alpha-n}\right] = |\xi|_p^{\alpha}, \tag{2.16}$$

for $\alpha \neq \alpha_k$, $k \in \mathbb{Z}$. Since $D^{\alpha}\psi = f_{-\alpha} * \psi$, we have

$$\begin{aligned} F[D^{\alpha}\psi] &= F[f_{-\alpha} * \psi] = F[f_{-\alpha}] \cdot F[\psi] \quad (\text{see [5, 10]}) \\ &\Longleftrightarrow F[D^{\alpha}\psi] = |\xi|_p^{\alpha} \cdot F[\psi] \\ &\Longleftrightarrow D^{\alpha}\psi = F^{-1} \circ |\xi|_p^{\alpha} \cdot F[\psi]. \end{aligned} \tag{2.17}$$

This completes the proof. □

Lemma 2.2 (see [2, 5, 10]). *Let χ_p be the additive character of the field $\mathbb{Q}_p$. Then*

$$\int_{B_\gamma^n} \chi_p(\langle \xi, x\rangle)dx = p^{n\gamma}\Omega(|\xi p^{-\gamma}|_p); \tag{2.18}$$

$$\int_{S_\gamma^n} \chi_p(\langle \xi, x\rangle)dx = \begin{cases} p^{n\gamma}(1-p^{-n}), & |\xi|_p \le p^{-\gamma}, \\ -p^{n(\gamma-1)}, & |\xi|_p = p^{-\gamma+1}, \\ 0, & |\xi|_p \ge p^{-\gamma+2}; \end{cases} \tag{2.19}$$

$$\int_{S_\gamma^n} [\chi_p(\langle \xi, x\rangle) - 1]dx = \begin{cases} 0, & |\xi|_p \le p^{-\gamma}, \\ -p^{n\gamma}, & |\xi|_p = p^{-\gamma+1}, \\ -p^{n\gamma}(1-p^{-n}), & |\xi|_p \ge p^{-\gamma+2}. \end{cases} \tag{2.20}$$

Corollary 2.3 (cf. [5, page 37, Proposition 2.3]). *Let $\xi \in \mathbb{Q}_p^n$ with $|\xi|_p \le 1$ and let $\operatorname{Re}\alpha < 0$. Then*

$$|\xi|_p^{-\alpha} = \frac{1-p^{-\alpha}}{1-p^{\alpha-n}} \int_{\mathbb{Q}_p^n} |\xi|_p^{\alpha-n}[\chi_p(\langle \xi, x\rangle) - 1]dx. \tag{2.21}$$

Proof. Since $\mathbb{Q}_p^n = \cup_{\gamma\in\mathbb{Z}} S_\gamma^n$,

$$\begin{aligned} &\int_{\mathbb{Q}_p^n} |\xi|_p^{\alpha-n}[\chi_p(\langle \xi, x\rangle) - 1]dx \\ &\quad = \sum_{-\infty<\gamma<\infty} p^{\gamma(\alpha-n)} \int_{S_\gamma^n} [\chi_p(\langle \xi, x\rangle) - 1]dx. \end{aligned} \tag{2.22}$$

Let $|\xi|_p = p^{-k}$, $k \ge 0$. By Lemma 2.2, we see that

$$\begin{aligned} \sum_{-\infty<\gamma<\infty} p^{\gamma(\alpha-n)} \int_{S_\gamma^n} [\chi_p(\langle \xi, x\rangle) - 1]dx &= -p^{(k+1)\alpha} - \sum_{k+2\le\gamma<\infty} p^{\gamma\alpha}(1-p^{-n}) \\ &= p^{k\alpha}\frac{1-p^{\alpha-n}}{1-p^{-\alpha}}. \end{aligned} \tag{2.23}$$

We therefore obtain the corollary. □

Lemma 2.4. *For $a \in \mathbb{Q}_p^n$ with $|a|_p > 1$, the Vladimirov function*

$$\psi(x) = \chi_p(\langle a, x\rangle)\Omega(|x|_p) \tag{2.24}$$

is an eigenfunction of the Vladimirov operator

$$D^{\alpha}\psi(x) = |a|_p^{\alpha}\psi(x). \tag{2.25}$$

Proof. Note that, by Lemma 2.1,

$$\begin{aligned}(D^{\alpha}\psi)(x) &= \left(F^{-1}\circ|\xi|_p^{\alpha}\cdot F[\psi]\right)(x)\\ &= \frac{(p^{\alpha}-1)\chi_p(\langle a,x\rangle)}{1-p^{-\alpha-n}}\int_{\mathbb{Q}_p^n}\frac{\Omega(|x|_p)-\chi_p(\langle a,y-x\rangle)\Omega(|y|_p)}{|x-y|_p^{\alpha+n}}dy,\\ d(cx) &= |c|_p dx \quad \text{for } c\in\mathbb{Q}_p^*.\end{aligned} \tag{2.26}$$

Let $|x|_p \le 1$. Then using the fact that every point of a disk is the center of this disk, we get

$$\begin{aligned}\int_{\mathbb{Q}_p^n}\frac{\Omega(|x|_p)-\chi_p(\langle a,y-x\rangle)\Omega(|y|_p)}{|x-y|_p^{\alpha+n}}dy &= \int_{\mathbb{Q}_p^n}\frac{1-\chi_p(\langle a,y-x\rangle)\Omega(|y|_p)}{|x-y|_p^{\alpha+n}}dy\\ &= \int_{\mathbb{Q}_p^n}\frac{1-\chi_p(\langle a,z\rangle)\Omega(|z|_p)}{|z|_p^{\alpha+n}}dz.\end{aligned} \tag{2.27}$$

Let $|x|_p > 1$. We see that

$$\begin{aligned}\int_{\mathbb{Q}_p^n}\frac{\chi_p(\langle a,y-x\rangle)\Omega(|y|_p)}{|x-y|_p^{\alpha+n}}dy &= \frac{1}{|x|_p^{\alpha+n}}\int_{|y|_p\le 1}\chi_p(\langle a,y-x\rangle)dy\\ &= \frac{\chi_p^{-1}(\langle a,x\rangle)}{|x|_p^{\alpha+n}}\int_{|y|_p\le 1}\chi_p(\langle a,y\rangle)dy.\end{aligned} \tag{2.28}$$

By (2.18) of Lemma 2.2 with $|a|_p > 1$, we have

$$\int_{B_0^n}\chi_p(\langle a,x\rangle)dx = \begin{cases}0, & \text{if } |a|_p > 1,\\ 1, & \text{if } |a|_p \le 1.\end{cases} \tag{2.29}$$

Therefore, for $a, x \in \mathbb{Q}_p^n$ with $|x|_p > 1$ and $|a|_p > 1$,

$$\begin{aligned}&\int_{\mathbb{Q}_p^n}\frac{\Omega(|x|_p)-\chi_p(\langle a,y-x\rangle)\Omega(|y|_p)}{|x-y|_p^{\alpha+n}}dy\\ &\quad = -\int_{\mathbb{Q}_p^n}\frac{\chi_p(\langle a,y-x\rangle)\Omega(|y|_p)}{|x-y|_p^{\alpha+n}}dy = 0.\end{aligned} \tag{2.30}$$

If $|z|_p > 1$, that is, $|z|_p = p^1$, $|z|_p = p^2, \dots$, then $\Omega(|z|_p) = 0$. From (2.19) of Lemma 2.2 we obtain

$$\begin{aligned}\int_{\mathbb{Q}_p^n} \frac{1-\chi_p(\langle a,z\rangle)\Omega(|z|_p)}{|z|_p^{\alpha+n}}dz &= \int_{|z|_p \geq p} \frac{1}{|z|_p^{\alpha+n}}dz = \sum_{1\leq \gamma<\infty} \int_{S_\gamma^n} |z|_p^{-\alpha-n}dz \\ &= \sum_{1\leq\gamma<\infty} p^{-\gamma(\alpha+n)}p^{\gamma n}\left(1-\frac{1}{p^n}\right) = \frac{1-p^{-n}}{p^\alpha-1}.\end{aligned} \tag{2.31}$$

Suppose that $|z|_p \leq 1$, that is, $\Omega(|z|_p) = 1$. Then by (2.19) of Lemma 2.2 and $\mu(S_\gamma^n) = p^{n\gamma}(1-p^{-n})$, we have

$$\begin{aligned}&\int_{\mathbb{Q}_p^n} \frac{1-\chi_p(\langle a,z\rangle)\Omega(|z|_p)}{|z|_p^{\alpha+n}}dz \\ &\quad = \sum_{-\infty<\gamma\leq 0} p^{-\gamma(\alpha+n)}\left(\int_{S_\gamma^n}(1-\chi_p(\langle a,z\rangle))dz\right) \\ &\quad = \frac{1-p^{-n}}{1-p^\alpha} - \sum_{0\leq\gamma<\infty} p^{\gamma(\alpha+n)} \cdot \begin{cases} p^{-n\gamma}\left(1-\dfrac{1}{p^n}\right), & |a|_p \leq p^\gamma, \\ -p^{n(-\gamma-1)}, & |a|_p = p^{\gamma+1}, \\ 0, & |a|_p \geq p^{\gamma+2}.\end{cases}\end{aligned} \tag{2.32}$$

We now set $|a|_p = p^k$, $k \geq 1$. Then

$$\begin{aligned}&\int_{\mathbb{Q}_p^n} \frac{1-\chi_p(\langle a,z\rangle)\Omega(|z|_p)}{|z|_p^{\alpha+n}}dz \\ &\quad = \frac{1-p^{-n}}{1-p^\alpha} - \sum_{0\leq\gamma<\infty} p^{\gamma(\alpha+n)} \cdot \begin{cases} p^{-n\gamma}\left(1-\dfrac{1}{p^n}\right), & k \leq \gamma, \\ -p^{-n(\gamma+1)}, & k-1 = \gamma, \\ 0, & k-2 \geq \gamma\end{cases} \\ &\quad = \frac{1-p^{-n}}{1-p^\alpha} - \left(\sum_{k\leq\gamma<\infty} p^{\gamma(\alpha+n)}p^{-n\gamma}\left(1-\frac{1}{p^n}\right) - p^{(k-1)(\alpha+n)}p^{-nk}\right).\end{aligned} \tag{2.33}$$

Hence we have

$$\begin{aligned}D^\alpha\psi(x) &= \frac{(p^\alpha-1)\chi_p(\langle a,x\rangle)}{1-p^{-\alpha-n}} \int_{\mathbb{Q}_p^n} \frac{1-\chi_p(\langle a,z\rangle)\Omega(|z|_p)}{|z|_p^{\alpha+n}}dz \\ &= \psi(x)\frac{(p^\alpha-1)}{1-p^{-\alpha-n}}\left(\frac{1-p^{-n}}{p^\alpha-1} + \frac{1-p^{-n}}{1-p^\alpha} - \frac{1-p^{-n}p^{k\alpha}}{1-p^\alpha} + p^{(k-1)\alpha-n}\right) \\ &= \psi(x)\frac{(p^\alpha-1)}{1-p^{-\alpha-n}}\,\frac{p^{k\alpha}(1-p^{-\alpha-n})}{p^\alpha-1} \\ &= \psi(x)p^{k\alpha} = |a|_p^\alpha\psi(x)\end{aligned} \tag{2.34}$$

that finishes the proof of the lemma. ∎

We will consider a direct product group

$$G_{(n)} = \underbrace{\mathbb{Q}_p/\mathbb{Z}_p \times \cdots \times \mathbb{Q}_p/\mathbb{Z}_p}_{n \text{ times}}. \tag{2.35}$$

Then any nonzero element $N \in G_{(n)}$ is representable in the form

$$N = (N_1, \ldots, N_n) = \left(\sum_{i=1}^{m_1} N_{1i} p^{-i}, \ldots, \sum_{i=1}^{m_n} N_{ni} p^{-i} \right), \tag{2.36}$$

where $N_{ki} \in \{0, 1, \ldots, p-1\}$, $k = 1, \ldots, n$.

Theorem 2.5. *Suppose that Vladimirov functions are as follows:*

$$\psi^{(n)}_{\gamma,j,N}(x) = \frac{p^{-\gamma n/2}}{\sqrt{c(|a|_p, n)}} \chi_p(\langle aj, p^\gamma x \rangle) \Omega\Big(|p^\gamma x - N|_p \Big), \quad |a|_p > 1, \tag{2.37}$$

where $\gamma \in \mathbb{Z}$, $N \in G_{(n)}$, $j = 1, \ldots, p-1$, and $a, x \in \mathbb{Q}_p^n$ such that

$$\sqrt{c(|a|_p, n)} = \frac{p^n(p-1)}{|a|_p^n (p^n - 1)}. \tag{2.38}$$

Denote by $(\cdot, \cdot)$ the inner product in $L^2(\mathbb{Q}_p^n)$. Set the inner product

$$\begin{aligned} \Big(\psi^{(n)}_{\gamma,j,N}, \psi^{(n)}_{\gamma',j',N'} \Big) &= \frac{p^{-(\gamma n + \gamma' n)/2}}{c(|a|_p, n)} \int_{\mathbb{Q}_p^n} \chi_p(-\langle aj, p^\gamma x \rangle) \Omega\Big(|p^\gamma x - N|_p \Big) \\ &\quad \times \chi_p(\langle aj', p^{\gamma'} x \rangle) \Omega\Big(|p^{\gamma'} x - N'|_p \Big) dx. \end{aligned} \tag{2.39}$$

Then

$$\Big(\psi^{(n)}_{\gamma,j,N}, \psi^{(n)}_{\gamma',j',N'} \Big) = \frac{1}{c(|a|_p, n)} \delta_{\gamma\gamma'} \delta_{NN'} \delta_{jj'}, \tag{2.40}$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol.

Proof. Without loss of generality, we now assume that $\gamma \le \gamma'$. If $|p^\gamma x|_p \ne |N|_p > 1$, then by the isosceles triangle principle of the non-Archimedian norm,

$$\Omega\Big(|p^\gamma x - N|_p \Big) = 0. \tag{2.41}$$

This means that we can set $|p^\gamma x|_p = |N|_p$ and also denote

$$\Omega\Big(|p^\gamma x - N|_p \Big) \Omega\Big(|p^{\gamma'} x - N'|_p \Big) = \Omega\Big(|p^\gamma x - N|_p \Big) \Omega\Big(|p^{\gamma' - \gamma} N - N'|_p \Big). \tag{2.42}$$

Therefore we have

$$\begin{aligned}
&\left(\psi^{(n)}_{\gamma,j,N},\psi^{(n)}_{\gamma',j',N'}\right)\\
&\quad=\frac{p^{-(\gamma n+\gamma' n)/2}}{c(|a|_p,n)}\int_{\mathbb{Q}_p^n}\chi_p(\langle a(p^{\gamma'}j'-p^{\gamma}j),x\rangle)\\
&\qquad\qquad\qquad\times\Omega\left(|p^{\gamma}x-N|_p\right)\Omega(|p^{\gamma'-\gamma}N-N'|_p)dx\\
&\quad=\begin{cases}0, & \text{if } \gamma<\gamma',\\ \dfrac{p^{-\gamma n}}{c(|a|_p,n)}\displaystyle\int_{\mathbb{Q}_p^n}\chi_p(\langle ap^{\gamma}(j'-j),x\rangle)\\ \qquad\times\Omega\left(|p^{\gamma}x-N|_p\right)\Omega(|N-N'|_p)dx, & \text{if } \gamma=\gamma'.\end{cases}
\end{aligned}\tag{2.43}$$

Since $N,N'\in G_{(n)}$, $\Omega(|N-N'|_p)=\delta_{NN'}$. Let $|a|_p>1$. We note that

$$\begin{aligned}
&p^{-\gamma n}\int_{\mathbb{Q}_p^n}\chi_p(\langle ap^{\gamma}(j'-j),x\rangle)\Omega\left(|p^{\gamma}x-N|_p\right)dx\\
&\quad=p^{-\gamma n}\int_{|p^{\gamma}x-N|_p\le 1}\chi_p(\langle ap^{\gamma}(j'-j),x\rangle)dx\\
&\quad=\chi_p(\langle a(j'-j),N\rangle)\int_{|\tilde{x}|_p\le 1}\chi_p(\langle a(j'-j),\tilde{x}\rangle)d\tilde{x}\\
&\quad=\begin{cases}0, & j\ne j',\\ 1, & j=j',\end{cases}
\end{aligned}\tag{2.44}$$

using (2.18) of Lemma 2.2. From above it follows the formula

$$\left(\psi^{(n)}_{\gamma,j,N},\psi^{(n)}_{\gamma',j',N'}\right)=\frac{1}{c(|a|_p,n)}\delta_{\gamma\gamma'}\delta_{NN'}\delta_{jj'}.\tag{2.45}$$

This completes the proof. □

Theorem 2.6. *Let the inner product $(\cdot,\cdot)$, and so forth, be as in Theorem 2.5. Then*

$$\left(\Omega(|x|_p),\psi^{(n)}_{\gamma,j,N}\right)=\frac{p^{-\gamma n/2}}{\sqrt{c(|a|_p,n)}}\theta(\gamma,a)\delta_{N0},\tag{2.46}$$

where $\theta(\gamma,a)=0$ if $|a|_p\ge p^{\gamma+1}$ and $\theta(\gamma,a)=1$ if $|a|_p\le p^{\gamma}$.

Proof. First, let $|p^{\gamma}x|_p\ne|N|_p$. Then $|p^{\gamma}x-N|_p=\max(|p^{\gamma}x|_p,|N|_p)$. Since $N\in G_{(n)}$, $|N|_p\ge p$. Hence we see that

$$\Omega\left(|p^{\gamma}x-N|_p\right)=0\quad\text{if } |p^{\gamma}x|_p\ne|N|_p.\tag{2.47}$$

Next, let $|p^{\gamma}x|_p = |N|_p$, that is, $\text{ord}_p(p^{\gamma}x) = \text{ord}_p(N)$. Set $p^{\gamma}x = (p^{\gamma}x_1,\ldots,p^{\gamma}x_n)$ and $N = (N_1,\ldots,N_n)$. Then

$$\min_{1\le i\le n}\{\text{ord}_p(p^{\gamma}x_i)\} = \min_{1\le i\le n}\{\text{ord}_p(N_i)\}. \tag{2.48}$$

If $|x|_p > 1$, $\Omega(|x|_p) = 0$, and we assume that $|x|_p \le 1$, then it follows the canonical form

$$\begin{aligned} x_i &= x_{i0} + x_{i1}p + \cdots + x_{i(|\gamma|-1)}p^{|\gamma|-1} + x_{i|\gamma|}p^{|\gamma|} + \cdots, \\ N_i &= p^{\gamma}N_{i|\gamma|} + \cdots + p^{-1}N_{i1}, \quad i = 1,\ldots,n, \end{aligned} \tag{2.49}$$

where $x = (x_1,\ldots,x_n) \in \mathbb{Z}_p^n$ and $0 \le x_{ij} \le p-1$, $j = 0,1,\ldots$. Then we get

$$\begin{aligned} p^{\gamma}x - N &= (p^{\gamma}x_1,\ldots,p^{\gamma}x_n) - (N_1,\ldots,N_n) \\ &= \left(p^{\gamma}(x_{10} + \cdots + x_{1|\gamma|}p^{|\gamma|} + \cdots),\ldots,p^{\gamma}(x_{n0} + \cdots + x_{n|\gamma|}p^{|\gamma|} + \cdots)\right) \\ &\quad - \left((p^{\gamma}N_{1|\gamma|} + \cdots + p^{-1}N_{11}),\ldots,(p^{\gamma}N_{n|\gamma|} + \cdots + p^{-1}N_{n1})\right) \\ &= \left((\tilde{x}_{10}p^{\gamma} + \cdots + \tilde{x}_{1(|\gamma|-1)}p^{-1} + x_{1|\gamma|} + x_{1(|\gamma|+1)}p + \cdots),\ldots,\right. \\ &\qquad \left.(\tilde{x}_{n0}p^{\gamma} + \cdots + \tilde{x}_{n(|\gamma|-1)}p^{-1} + x_{n|\gamma|} + x_{n(|\gamma|+1)}p + \cdots)\right). \end{aligned} \tag{2.50}$$

If $\tilde{x}_{i0} \ne 0,\ldots,$ or $\tilde{x}_{i(|\gamma|-1)} \ne 0$ for $i = 1,\ldots,n$, then

$$\Omega\left(|p^{\gamma}x - N|_p\right) = 0 \quad \text{since } |p^{\gamma}x - N|_p \ge p. \tag{2.51}$$

Let $\tilde{x}_{i0} = \cdots = \tilde{x}_{i(|\gamma|-1)} = 0$ for $i = 1,\ldots,n$; we have

$$\Omega\left(|p^{\gamma}x - N|_p\right) \ne 0, \tag{2.52}$$

that is, $p^{\gamma}x - N = ((x_{1|\gamma|} + x_{1(|\gamma|+1)}p + \cdots),\ldots,(x_{n|\gamma|} + x_{n(|\gamma|+1)}p + \cdots)) \in B_0^n$. Hence

$$\begin{aligned} \left(\Omega(|x|_p),\psi_{\gamma,j,N}^{(n)}\right) &= \frac{p^{-\gamma n/2}}{\sqrt{c(|a|_p,n)}}\int_{\mathbb{Q}_p^n}\Omega(|x|_p)\chi_p(\langle aj,p^{\gamma}x\rangle)\Omega\left(|p^{\gamma}x - N|_p\right)dx \\ &= \frac{p^{-\gamma n/2}}{\sqrt{c(|a|_p,n)}}\int_{B_0^n}\chi_p(\langle ajp^{\gamma},x\rangle)dx = 0 \end{aligned} \tag{2.53}$$

by (2.18) of Lemma 2.2 with $\gamma < 0$. Under the assumption that $N \neq 0$, we always obtain

$$\left(\Omega(|x|_p), \psi^{(n)}_{\gamma,j,N}\right) \equiv 0. \tag{2.54}$$

On the other hand, now set $N = 0$, we see that

$$\begin{aligned}\left(\Omega(|x|_p), \psi^{(n)}_{\gamma,j,N}\right) &= \frac{p^{-\gamma n/2}}{\sqrt{c(|a|_p, n)}} \int_{x \in B_0^n,\, x \in B_\gamma^n} \chi_p(\langle aj, p^\gamma x\rangle)\,dx \\ &= \begin{cases} 0, & \text{if } B_\gamma^n \subseteqq B_0^n \\ \dfrac{p^{-\gamma n/2}}{\sqrt{c(|a|_p, n)}} \displaystyle\int_{x \in B_0^n} \chi_p(\langle aj, p^\gamma x\rangle)\,dx, & \text{if } B_0^n \subsetneqq B_\gamma^n \end{cases} \\ &= \begin{cases} \dfrac{p^{-\gamma n/2}}{\sqrt{c(|a|_p, n)}}, & |a|_p \leq p^\gamma, \\ 0, & |a|_p \geq p^{\gamma+1}. \end{cases}\end{aligned} \tag{2.55}$$

Therefore, we have

$$\left(\Omega(|x|_p), \psi^{(n)}_{\gamma,j,N}\right) = \frac{p^{-\gamma n/2}}{\sqrt{c(|a|_p, n)}} \theta(\gamma, a)\delta_{N0} \tag{2.56}$$

for $\theta(\gamma, a) = 0$ if $|a|_p \geq p^{\gamma+1}$ and $\theta(\gamma, a) = 1$ if $|a|_p \leq p^\gamma$. This completes the proof. □

Theorem 2.7. *The system of Vladimirov function in Theorem 2.5 with $|a|_p > 1$ is a complete orthonormal system of eigenfunctions of Vladimirov operator D^α in $L^2(\mathbb{Q}_p^n)$.*

Proof. To prove that $\{\psi^{(n)}_{\gamma,j,N}\}$ is a complete system, it is enough to check that the Parseval identity for the function $\Omega(|x|_p)$ forms an orthonormal basis in $L^2(\mathbb{Q}_p^n)$ (see [3, 4]). Set $|a|_p = p^k$ for $k = 1, 2, \dots$. By Theorem 2.6, it is obvious that

$$\begin{aligned}\sum_{\gamma,j,N} \left(\left\langle \Omega(|x|_p), \psi^{(n)}_{\gamma,j,N}\right\rangle\right)^2 &= \sum_{\gamma=k}^{\infty} \sum_{j=1}^{p-1} \frac{p^{-\gamma n}}{c(|a|_p, n)} \\ &= \frac{p-1}{c(|a|_p, n)} \frac{p^{-kn}}{1-p^{-n}} = 1,\end{aligned} \tag{2.57}$$

where $\sqrt{c(|a|_p, n)} = p^n(p-1)/|a|_p^n(p^n - 1)$. On the other hand, by (2.39),

$$(\Omega(|x|_p), \Omega(|x|_p)) = 1, \tag{2.58}$$

which implies our result. □

Corollary 2.8 (see [6]). *The system of Vladimirov function in Theorem 2.7 with $a = p^{-1}$ and $n = 1$ is a complete orthonormal system of eigenfunctions of Vladimirov operator D^{α} in $L^2(\mathbb{Q}_p)$.*

3. Interpretation of multiwavelets. Multiwavelets have been used in the data compression, noise reduction, and solution of integral equations. Because multiwavelets are able to offer a combination of orthogonality, symmetry, higher order of approximation, and short support, methods using multiwavelets frequently outperform those using the comparable scale wavelets. Multiresolution produces an orthonormal basis of wavelets at all scales $\gamma \in \mathbb{Z}$ (cf. [1]).

The wavelet basis in $L^2(\mathbb{R}^n_+)$ is a basis given by shifts and dilations of the so-called mother wavelet function. For $x \in \mathbb{R}^n_+$, we define the Haar wavelet $\Psi^{(n)}(x)$ by

$$\Psi^{(n)}(x) \equiv \begin{cases} 1, & x \in \left[0, \frac{1}{2}\right]^n, \\ -1, & x \in \left[\frac{1}{2}, 1\right]^n, \\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

where

$$[a,b]^n = \underbrace{[a,b] \times \cdots \times [a,b]}_{n \text{ times}} \quad \text{for } a, b \in \mathbb{R}. \tag{3.2}$$

A wavelet basis is a function $\Psi^{(n)} \in L^2(\mathbb{R}^n_+)$ such that

$$\left\{\Psi^{(n)}_{\gamma,N}(x) \equiv 2^{-\gamma n/2}\Psi^{(n)}(2^{-\gamma}x - N) : \gamma \in \mathbb{Z},\ N \in \mathbb{Z}^n\right\} \tag{3.3}$$

is a basis for $L^2(\mathbb{R}^n_+)$ (see [7]).

Remark 3.1. Spline bases have a maximal approximation order with respect to their length; however, spline uniwavelets are only semiorthogonal. A family of spline multiwavelets that are symmetric and orthogonal is developed (cf. [1]).

We may define a mapping $\rho : \mathbb{Q}^n_p \to \mathbb{R}^n_+$ by

$$\rho\left(\sum_{i_1=\gamma_1}^{\infty} a_{i_1} p^{i_1}, \ldots, \sum_{i_n=\gamma_n}^{\infty} a_{i_n} p^{i_n}\right) = \frac{1}{p}\left(\sum_{i_1=\gamma_1}^{\infty} a_{i_1} p^{-i_1}, \ldots, \sum_{i_n=\gamma_n}^{\infty} a_{i_n} p^{-i_n}\right), \tag{3.4}$$

where $\gamma_j \in \mathbb{Z}$ and $a_{i_j} \in \{0,1,\dots,p-1\}$ for $j = 1,\dots,n$. This map ρ is clearly not one to one. The following map is a one-to-one map:

$$\rho : G_{(n)} \longrightarrow \mathbb{Z}_+^n, \tag{3.5}$$

where $\mathbb{Z}_+^n = \mathbb{N}^n \cup \{0\}$, $\mathbb{N} = \{1,2,\dots\}$.

Lemma 3.2. (1) *The map ρ satisfies the estimate*

$$\left|\rho(x) - \rho(y)\right| \le \sqrt{n}|x-y|_p. \tag{3.6}$$

(2) *For $N \in G_{(n)}$ and $m,k \in \mathbb{Z}$, the map ρ satisfies the conditions*

$$\begin{aligned} \rho : p^m N + p^k \mathbb{Z}_p^n &\longrightarrow p^{-m}\rho(N) + [0,p^{-k}]^n, \\ \rho : \mathbb{Q}_p^n \backslash \{p^m N + p^k \mathbb{Z}_p^n\} &\longrightarrow \mathbb{R}_+^n \backslash \{p^{-m}\rho(N) + [0,p^{-k}]^n\}, \end{aligned} \tag{3.7}$$

up to a finite number of points.

(3) *The map ρ maps the Haar measure μ on $\mathbb{Q}_p^n$ onto the Lebesgue measure ν on $\mathbb{R}_+^n$: for measurable subspace $X \subset \mathbb{Q}_p^n$,*

$$\mu(X) = \nu(\rho(X)). \tag{3.8}$$

(4) *Suppose that*

$$\rho^* : L^2(\mathbb{R}_+^n) \longrightarrow L^2(\mathbb{Q}_p^n);\ f(x) \longmapsto f(\rho(x)) \tag{3.9}$$

is a unitary operator. Then the map ρ maps the Haar wavelet (3.1) onto the Vladimirov function $\psi(x)$ in Lemma 2.4 with $p = 2$ and $(2^{-1},\dots,2^{-1}) \in \mathbb{Q}_2^n$, that is,

$$\rho^* : \Psi^{(n)}(x) \longmapsto \psi(x). \tag{3.10}$$

Proof. The proof is essentially due to [6, Lemma 3-6]. □

Corollary 3.3. *Let $S_\gamma = \{x \in \mathbb{Q}_p^n \mid |x|_p = p^\gamma\}$. Then*

$$\begin{aligned} \rho(p^k \mathbb{Z}_p^n) &= \bigcup_{\gamma \le 0} \rho(p^k S_\gamma), \\ \rho(p^k S_\gamma) \cap \rho(p^k S_{\gamma'}) &= \begin{cases} \dfrac{1}{p^{\gamma+k}}, & \gamma' = \gamma + 1, \\ \varnothing, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.11}$$

Theorem 3.4. *For $p = 2$, the map ρ maps the orthonormal basis of wavelets in $L^2(\mathbb{R}_+^n)$ (see (3.3)) onto the basis $\psi_{\gamma,1,N}^{(n)}$ of the eigenfunctions for the Vladimirov* *operator in Theorem 2.5 with* $a = (2^{-1},\dots,2^{-1}) \in \mathbb{Q}_2^n$ *and* $|a|_2 > 1$*:*

$$\rho^* : \Psi^{(n)}_{\gamma,\rho(N)}(x) \longmapsto (-1)^{|N|}(2^n-1)\psi^{(n)}_{\gamma,1,N}(x), \tag{3.12}$$

where $|N| = N_1 + \cdots + N_n$.

Proof. From (3.3) and Part (4) of Lemma 3.2, we obtain

$$\begin{aligned} \Psi^{(n)}_{\gamma,\rho(N)}(\rho(x)) &= 2^{-\gamma n/2}\Psi^{(n)}(2^{-\gamma}\rho(x)-\rho(N)) \\ &= 2^{-\gamma n/2}\Psi^{(n)}(\rho(2^{\gamma}x-N)) \\ &= 2^{-\gamma n/2}\rho^*(\Psi^{(n)}(2^{\gamma}x-N)) \\ &= 2^{-\gamma n/2}\psi(2^{\gamma}x-N). \end{aligned} \tag{3.13}$$

Next, by the definition $\sqrt{c(|a|_2,n)}$ with $a = (2^{-1},\dots,2^{-1}) \in \mathbb{Q}_2^n$, we have

$$\sqrt{c(|a|_2,n)} = \frac{1}{(2^n-1)}. \tag{3.14}$$

Hence

$$\begin{aligned} \psi^{(n)}_{\gamma,1,N}(x) &= \frac{2^{-\gamma n/2}}{\sqrt{c(|a|_2,n)}}\chi_2(\langle a,2^{\gamma}x\rangle)\Omega(|2^{\gamma}x-N|_2) \\ &= (-1)^{|N|}2^{-\gamma n/2}(2^{n-1})\psi(2^{\gamma}x-N) \end{aligned} \tag{3.15}$$

since $\psi(2^{\gamma}x-N) = \chi_2(-\langle a,N\rangle)\chi_2(\langle a,2^{\gamma}x\rangle)\Omega(|2^{\gamma}x-N|_2)$. The proof now follows directly. □

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