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ON COMMON FIXED POINTS OF PAIRS OF A SINGLE AND A MULTIVALUED COINCIDENTALLY COMMUTING MAPPINGS IN D-METRIC SPACES

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The present paper studies some common fixed-point theorems for pairs of a single-valued and a multivalued coincidentally commuting mappings in D-metric spaces satisfying a certain generalized contraction condition. Our result generalizes more than a dozen known fixed-point theorems in D-metric spaces including those of Dhage (2000) and Rhoades (1996).

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1. Introduction. The concept of a D-metric space introduced by the first author in [1] is as follows. A nonempty set, together with a function $\rho : X\times X\times X \to [0,\infty)$, is called a *$D$-metric space* and denoted by (X,ρ) if the function ρ, called a *D-metric* on X, satisfies the following properties:

(i) $\rho(x,y,z) = 0 \Leftrightarrow x = y = z$ (coincidence),

(ii) $\rho(x,y,z) = 0 = \rho(p\{x,y,z\})$ (symmetry), where p is a permutation,

(iii) $\rho(x,y,z) \le \rho(x,y,a)+\rho(x,a,z)+\rho(a,y,z)$ for all $x,y,z,a\in X$ (tetrahedral inequality).

It is known that the D-metric ρ in a continuous function on X^3 in the topology of D-metric convergence is Hausdorff. The details of a D-metric space and its topological properties appear in Dhage [8]. Some specific examples of a D-metric space are presented in Dhage [2].

A sequence $\{x_n\} \subset X$ is called *convergent* and *converges* to a point x if $\lim_{m,n}\rho(x_m,x_n,x) = 0$. Again a sequence $\{x_n\} \subset X$ is called *D-Cauchy* if $\lim_{m,n,p}\rho(x_m,x_n,x_p) = 0$. A complete D-metric space X is one in which every D-Cauchy sequence converges to a point in X. A subset S of a D-metric space X is called *bounded* if there exists a constant $k > 0$ such that $\rho(x,y,z) \le k$ for all $x,y,z \in X$ and the constant k is called a *D-bound* of S. The smallest among all such D-bounds k of S is called the *diameter* of X and it is denoted by $\delta(S)$.

Let 2^X and $\mathrm{CB}(X)$ denote the classes of nonempty closed and nonempty, closed, bounded subsets of X, respectively. A correspondence $F : X \to 2^X$ is called a *multivalued mapping* on a D-metric space X, and a point $u \in X$ is called a *fixed point* of F if $u \in Fu$.

In [3], the first author has defined a notion of the generalized or Kasusai D-metric on X. Let $\kappa : (\mathrm{CB}(X))^3 \to [0,\infty)$ be a function defined by

$$\kappa(A,B,C) = \inf\{\epsilon > 0 \mid A\cup B \subset N(c,\epsilon),\ B\cup C \subset N(A,\epsilon),\ C\cup A \subset N(B,\epsilon)\}, \tag{1.1}$$

where $N(A,\epsilon) = \cup_{a\in A} N(a,\epsilon)$, $N(a,\epsilon) = \{x \in N^*(a,\epsilon) \mid \rho(a,x,y) < \epsilon$ for all $y \in N^*(a,\epsilon)\}$, and $N^*(a,\epsilon) = \{x \in X \mid \rho(a,x,x) < \epsilon\}$.

The definition (1.1) is equivalent to

$$\kappa(A,B,C) = \max\Big\{\sup_{a\in A,\, b\in B} D(a,b,c),\ \sup_{b\in B,\, c\in C} D(b,c,A),\ \sup_{c\in C,\, a\in A} D(c,a,B)\Big\}, \tag{1.2}$$

where $D(a,b,c) = \inf\{\rho(a,b,c) \mid c \in C\}$.

Define

$$\begin{aligned} D(A,B,C) &= \inf\{\rho(a,b,c) \mid a \in A,\ b \in B,\ c \in C\}, \\ \delta(A,B,C) &= \sup\{\rho(a,b,c) \mid a \in A,\ b \in B,\ c \in C\}. \end{aligned} \tag{1.3}$$

Notice that D and δ are continuous functions on $(\mathrm{CB}(X))^3$ and satisfy

$$D(A,B,C) \le \kappa(A,B,C) \le \delta(A,B,C). \tag{1.4}$$

A multivalued mapping $F : X \to \mathrm{CB}(X)$ is called *continuous* if

$$\lim_{m,n} \rho(x_m,x_n,x) = 0 \Longrightarrow \kappa(Fx_m,Fx_n,Fx) = 0. \tag{1.5}$$

In [3], the first author has proved some fixed-point theorem for multivalued contraction mappings in D-metric spaces, and in [5] he has proved some common fixed-point theorems for coincidentally commuting single-valued mappings in D-metric spaces satisfying a condition of generalized contraction.

In this paper, we prove some common fixed-point theorems for a pair of singlevalued and multivalued mappings in a D-metric space satisfying a contraction condition more general than that given in Dhage [1, 2, 3, 4, 5, 7] and Rhoades [12]. The results of this paper are new to the fixed-point theory in D-metric spaces and include nearly a dozen of known fixed-point theorems as special cases (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12]).

2. Preliminaries. Before going to the main results of this paper, we give some preliminaries needed in the sequel.

Let $F : X \to 2^X$. Then by an orbit of F at a point $x \in X$ we mean a set $O_F(x)$ in X defined by

$$O_F(x) = \{x_0 = x,\ x_{n+1} \in Fx_n,\ n \ge 0\}. \tag{2.1}$$

An orbit $O_F(x)$ is called *bounded* if $\delta(O_F(x)) < \infty$, and a D-metric space X is called *F-orbitally bounded* if $O_F(x)$ is bounded for each $x \in X$. Again an F-orbit $O_F(x)$ is called *complete* if every D-Cauchy sequence in $O_F(x)$ converges to a point in X. A D-metric space X is said to be *F-orbitally complete* if $O_F(x)$ is complete for each $x \in X$. Finally, F is called *F-orbitally continuous* if for any sequence $\{x_n\} \subseteq O_F(x)$, we have

$$\lim_{m,n} \rho(x_m, x_n, x^*) = 0 \Longrightarrow \lim_{m,n} \kappa(Fx_m, Fx_n, Fx^*) = 0 \tag{2.2}$$

for each $x \in X$.

Let Φ denote the class of all functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following properties:

(i) ϕ is continuous,
(ii) ϕ is nondecreasing,
(iii) $\phi(t) < t$, $t > 0$,
(iv) $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t \in [0,\infty)$.

The function ϕ is called a *Lipschitz control function* or *Lipschitz growth function* and the usual growth function is $\phi(t) = \alpha t$, $0 \le t < 1$. The following lemma concerning the function ϕ appears in [7].

Lemma 2.1. *If $\phi \in \Phi$, then $\phi^n(t) = 0$ for each $n \in \mathbb{N}$ and $\lim_n \phi^n(t) = 0$ for each $t \in [0,\infty)$.*

We need the following D-Cauchy principle of Dhage [7] in the sequel.

Lemma 2.2 (D-Cauchy principle)**.** *Let $\{x_n\}$ be a bounded sequence in a D-metric space X with D-bound k satisfying, for some positive real number r,*

$$\rho(x_n, x_{n+1}, x_m) \le [\phi^n(k^r)]^{1/r} \tag{2.3}$$

for all $m > n \in \mathbb{N}$, where $\phi : [0,\infty) \to [0,\infty)$ satisfies $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t \in [0,\infty)$. Then $\{x_n\}$ is a D-Cauchy sequence in X.

Proof. The proof appears in [7], but for the sake of completeness we give the details. Let $p, t \in \mathbb{N}$ be arbitrary but fixed. Then from (2.3) it follows that

$$\begin{aligned} \rho(x_n, x_{n+1}, x_{n+p}) &\le [\phi^n(k^r)]^{1/r}, \\ \rho(x_n, x_{n+1}, x_{n+p+t}) &\le [\phi^n(k^r)]^{1/r}, \end{aligned} \tag{2.4}$$

for all $n \in \mathbb{N}$.

Now by repeated application of the tetrahedral inequality, we obtain

$$\begin{aligned} &\rho(x_n, x_{n+p}, x_{n+p+t}) \\ &\quad \le \rho(x_n, x_{n+1}, x_{n+p}) + \rho(x_n, x_{n+1}, x_{n+p+t}) + \rho(x_{n+1}, x_{n+p}, x_{n+p+t}) \\ &\quad \le \rho(x_n, x_{n+1}, x_{n+p}) + \rho(x_n, x_{n+1}, x_{n+p+t}) + \rho(x_{n+1}, x_{n+2}, x_{n+p}) \\ &\qquad + \rho(x_{n+1}, x_{n+2}, x_{n+p+t}) + \rho(x_{n+2}, x_{n+p}, x_{n+p+t}) \end{aligned}$$

$$\begin{aligned}&\le 2[\phi^n(k^r)]^{1/r}+2[\phi^{n+1}(k^r)]^{1/r}+\rho(x_{n+2},x_{n+p},x_{n+p+t})\\&\le 2\Big\{[\phi^n(k^r)]^{1/r}+\cdots+[\phi^{n+p-2}(k^r)]^{1/r}\Big\}+\rho(x_{n+p-1},x_{n+p},x_{n+p+t})\\&\le 2\sum_{j=n}^{n+p-1}[\phi^j(k^r)]^{1/r}.\end{aligned}\tag{2.5}$$

Since $\sum_{n=1}^{\infty}\phi^n(t)<\infty$ for each $t\in[0,\infty)$, we have $\sum_{j=1}^{\infty}[\phi^j(k^r)]^{1/r}<\infty$ and so $\lim_n\sum_{j=n}^{n+p-1}[\phi^j(k^r)]^{1/r}=0$. Now from (2.5) it follows that

$$\lim_{n\to\infty}\rho(x_n,x_{n+p},x_{n+p+t})=0.\tag{2.6}$$

This proves that $\{x_n\}$ is a D-Cauchy sequence in X and the proof of the lemma is complete. □

As a direct application of Lemma 2.2, we obtain the following result proved in [5].

Lemma 2.3. *Let $\{x_n\}$ be a bounded sequence in a D-metric space X with D-bound k satisfying*

$$\rho(x_n,x_{n+1},x_m)\le\lambda^n k\tag{2.7}$$

for all $m>n\in\mathbb{N}$, where $0\le\lambda<1$. Then $\{x_n\}$ is D-Cauchy.

We use contractive conditions of the form

$$a^r\le\phi(b^r)\tag{2.8}$$

for some positive real number r, where a and b are nonnegative real numbers and $\phi\in\Phi$, because sometimes inequality (2.8) holds, but for the same real numbers a and b, the inequality

$$a\le\phi(b)\tag{2.9}$$

does not hold. To see this, let $\phi:\mathbb{R}^+\to\mathbb{R}^+$ be a function defined by

$$\phi(t)=\frac{\alpha t}{1+t},\quad 0\le\alpha<1.\tag{2.10}$$

Obviously the function ϕ is continuous, nondecreasing and satisfies $\phi(t)=\alpha t/(1+t)<t$ for $t>0$. Again since

$$\sum_{n=1}^{\infty}\phi^n(t)=\sum_{n=1}^{\infty}\frac{\alpha^n t}{1+t+\cdots+\alpha^{n-1}t}<\sum_{n=1}^{\infty}\alpha^n<\infty,\tag{2.11}$$

we have that $\phi\in\Phi$.

Now for $a = 1/2$ and $b = 2/3$, we have, by (2.9),

$$\frac{1}{2} \le \phi\left(\frac{2}{3}\right) = \frac{(2/3)\alpha}{1+2/3} = \frac{2}{5}\alpha, \tag{2.12}$$

which is not true since $0 \le \alpha < 1$. But for the same values of a and b, we have a positive real number $r = 2$ such that

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} \le \frac{4\alpha}{13} = \phi\left(\left(\frac{2}{3}\right)^2\right) \tag{2.13}$$

for $13/16 \le \alpha < 1$. Hence inequality (2.8) holds. Thus inequality (2.9) does not imply inequality (2.8). Actually, inequalities (2.8) and (2.9) are independent. To show that inequality (2.8) does not imply inequality (2.9), let $a = 1/4$, $b = 4/9$, and $r = 1/2$. Clearly, inequality (2.8) does not hold, but for the same values of a, b, and r, one has

$$\frac{1}{4} \le 4\frac{\alpha}{13} = \frac{\alpha(4/9)}{1+4/9} = \phi\left(\frac{4}{9}\right) \tag{2.14}$$

for $\alpha \ge 13/16$, and so inequality (2.9) holds. Thus inequalities (2.8) and (2.9) are independent.

In the following sections, we will prove the main results of this paper.

3. Weak commuting mappings in D-metric spaces. Let $F : X \to 2^X$ and $g : X \to X$. Then the pair $\{F,g\}$ of maps is called *limit coincident* if $\lim_n Fx_n = \{\lim_n gx_n\}$ for some sequence $\{x_n\}$ in X, and *coincident* if there exists a point $u \in X$ such that $Fu = \{gu\}$. Again two maps F and g are called *limit commuting* if $\lim_n Fgx_n = \{\lim_n gFx_n\}$, where $\{x_n\}$ is a sequence in X, and *commuting* if $Fgx = \{gFx\}$ for all $x \in X$. Two maps F and g are called *limit coincidentally commuting* if their limit coincidence implies the limit commutativity on X. Similarly, they are called *coincidentally commuting* if they are commuting at the coincidence points. Again two maps F and g are said to be *limit pseudocommuting* if $\lim_n Fgx_n \cap \lim_n gFx_n \ne \phi$, that is, $\lim_n D(Fgx_n, gFx_n, gFx_n) = 0$, where $\{x_n\}$ is a sequence in X, and *pseudocommuting* if $Fgx \cap gFx \ne \varnothing$ for each $x \in X$. Finally, the pair $\{F,g\}$ is called *limit coincidentally pseudocommuting* if its limit coincidence implies the limit pseudocommutativity on X, and *coincidentally pseudocommuting* if it is pseudocommuting at the coincidence points. It is known that a coincidentally commuting pair is limit coincidentally commuting and a coincidentally pseudocommuting pair is limit coincidentally pseudocommuting, but the converse implications need not hold. A pair of maps $\{F,g\}$ is *weak commuting* if it is either limit commuting, coincidentally commuting, limit pseudocommuting, or coincidentally pseudocommuting on X. Below, we will prove some common fixed-point theorems for each of these weak commuting mappings on D-metric spaces.

3.1. Limit coincidentally commuting maps in D-metric spaces. Let $F : X \to 2^X$ and $g : X \to X$. By an (F/g)-orbit of the pair $\{F, g\}$ of maps at a point $x \in X$, we mean a set $O_F(gx)$ in X defined by

$$O_F(gx) = \{y_n \mid y_0 = gx_0,\ y_n = gx_n \in Fx_{n-1},\ n \in \mathbb{N},\ \text{where } x_0 = x\} \tag{3.1}$$

for some sequence $\{x_n\}$ in X. The orbit $O_F(gx)$ is well defined for each $x \in X$ if $F(X) \subseteq g(X)$. By $\overline{O_F(gx)}$ we denote the closure of the set $O_F(gx)$ in X.

A D-metric space X is called (F/g)-*orbitally bounded* if $\delta(O_F(gx)) < \infty$ for each $x \in X$. Further X is called (F/g)-*orbitally complete* if every D-Cauchy sequence $\{x_n\} \subset O_F(gx)$ converges to a point in X for each $x \in X$. Finally, a mapping $T : X \to \mathrm{CB}(X)$ is called (F/g)-*orbitally continuous* if for any $\{x_n\} \subset O_F(gx)$, $x_n \to x^*$ implies that $Tx_n \to Tx^*$ for each $x \in X$.

Theorem 3.1. *Let $F : X \to \mathrm{CB}(X)$ and $g : X \to X$ be two mappings satisfying, for some positive real number r,*

$$\begin{aligned}\delta^r&(Fx, Fy, Fz)\\ &\le \phi(\max\{\rho^r(gx, gy, gz), \delta^r(Fx, Fy, gz), \delta^r(gx, Fx, gz),\\ &\qquad \delta^r(gy, Fy, gz), \delta^r(gx, Fy, gz), \delta^r(gy, Fx, gz)\})\end{aligned} \tag{3.2}$$

for all $x, y, z \in X$, where $\phi \in \Phi$. Suppose that

(a) *$F(X) \subseteq g(X)$ and $g(X)$ is bounded,*
(b) *$\{F, g\}$ is limit coincidentally commuting,*
(c) *F or g is (F/g)-orbitally continuous.*

Further if X is (F/g)-orbitally complete D-metric space, then F and g have a unique common fixed point $u \in X$ such that $Fu = \{u\} = gu$. Moreover, if g is continuous at u, then F is also continuous at u in the Kasubai D-metric on X.

Proof. Let $x \in X$ be arbitrary and define a sequence $\{y_n\}$ in X as follows. Take $x_0 = x$ and $y_0 = gx_0$. Choose a point $y_1 \in Fx_0 = X_1$. Since $F(X) \subseteq g(X)$, there is a point $x_1 \in X$ such that $y_1 = gx_1$. Again choose a point $y_2 \in Fx_1 = X_2$. By hypothesis (a), there is a point $x_2 \in X$ such that $y_2 = gx_2$. Proceeding in this way, by induction there is a sequence $\{x_n\}$ of points in X such that

$$y_0 = gx_0, \qquad y_{n+1} = gx_{n+1} \in X_{n+1} = Fx_n, \quad n = 0, 1, 2, \ldots. \tag{3.3}$$

From hypothesis (a), it follows that

$$\delta(X_m, X_n, X_p) \le \delta(g(X)) = k < \infty \tag{3.4}$$

for all $m, n, p \in \mathbb{N}$.

Now there are two cases.

Case 1. Suppose that $y_r = y_{r+1}$ for some $r \in \mathbb{N}$. Then we have $gx_r = gx_{r+1} = u$ for some $u \in X$.

We will show that $Fx_r = \{u\}$. Suppose not. Then by (3.2),

$$\begin{aligned}
&\delta^r(Fx_r, Fx_r, u) \\
&\quad = \delta^r(Fx_r, Fx_r, gx_{r+1}) \\
&\quad \le \delta^r(Fx_r, Fx_r, Fx_r) \\
&\quad \le \phi(\max\{\rho^r(gx_r, gx_r, gx_r), \delta^r(gx_r, Fx_r, gx_r), \delta^r(Fx_r, Fx_r, gx_r)\}) \\
&\quad \le \phi(\max\{0, \delta^r(gx_r, Fx_r, gx_r), \delta^r(Fx_r, Fx_r, u)\}) \\
&\quad = \phi(\max\{\delta^r(u, Fx_r, u), \delta^r(Fx_r, Fx_r, u)\}) \\
&\quad = \phi(\delta^r(u, Fx_r, u))
\end{aligned} \tag{3.5}$$

because $\delta^r(Fx_r, Fx_r, u) \le \phi(\delta^r(Fx_r, Fx_r, u))$ is not possible in view of $\phi \in \Phi$.

Again by (3.2),

$$\begin{aligned}
\delta^r(Fx_r, u, u) &= \delta^r(Fx_r, gx_{r+1}, gx_{r+1}) \\
&\le \delta^r(Fx_r, Fx_r, Fx_r) \\
&\le \phi(\max\{\delta^r(u, Fx_r, u), \delta^r(Fx_r, Fx_r, u)\}) \\
&= \phi(\delta^r(Fx_r, Fx_r, u)).
\end{aligned} \tag{3.6}$$

Substituting (3.6) in (3.5), we obtain

$$\delta^r(Fx_r, Fx_r, u) \le \phi^2(\delta^r(Fx_r, Fx_r, u)), \tag{3.7}$$

which is a contradiction since $\phi \in \Phi$. Hence $Fx_r = u$. Since F and g are limit coincidentally commuting, one has $Fgx_r = \{gFx_r\}$.

We will show that u is a common fixed point of F and g such that $Fu = \{u\} = gu$.

Now,

$$\begin{aligned}
\delta^r(Fu, gu, u) &= \delta^r(FFx_r, Fgx_r, Fx_r) \\
&\le \phi(\max\{\rho^r(gFx_r, ggx_r, gx_r), \delta^r(FFx_r, Fgx_r, gx_r), \\
&\qquad \delta^r(gFx_r, FFx_r, gx_r), \delta^r(ggx_r, Fgx_r, gx_r), \\
&\qquad \delta^r(gFx_r, Fgx_r, gx_r), \delta^r(ggx_r, FFx_r, gx_r)\}) \\
&= \phi(\max\{\rho^r(gFx_r, ggx_r, gx_r), \delta^r(ggx_r, FFx_r, gx_r)\}) \\
&= \phi(\delta^r(Fu, gu, u)),
\end{aligned} \tag{3.8}$$

which is possible only when $Fu = \{u\} = gu$ since $\phi \in \Phi$.

Case 2. Assume that $y_n \ne y_{n+1}$ for each $n \in \mathbb{N}$. We will show that $\{y_n\}$ is a D-Cauchy sequence in X. Let $x = x_0$, $y = x_1$, and $z = x_{m-1}$, $m \ge 1$. Then by (3.2),

$$\begin{aligned}
&\rho^r(y_1,y_2,y_m)\\
&\quad\le \delta^r(Fx_0,Fx_1,Fx_{m-1})\\
&\quad\le \phi(\max\{\rho^r(gx_0,gx_1,gx_{m-1}),\delta^r(Fx_0,Fx_1,gx_{m-1}),\delta^r(gx_0,Fx_0,gx_{m-1}),\\
&\qquad\qquad \delta^r(gx_1,Fx_1,gx_{m-1}),\delta^r(gx_0,Fx_1,gx_{m-1}),\delta^r(gx_1,Fx_0,gx_{m-1})\})\\
&\quad\le \phi(\max\{\delta^r(X_0,X_1,X_{m-1}),\delta^r(X_1,X_2,X_{m-1}),\delta^r(X_0,X_1,X_{m-1}),\\
&\qquad\qquad \delta^r(X_1,X_2,X_{m-1}),\delta^r(X_0,X_2,X_{m-1}),\delta^r(X_1,X_1,X_{m-1})\})\\
&\quad\le \phi\Big(\max_{0\le a\le 1,\,1\le b\le 2}\delta^r(X_a,X_b,X_{m-1})\Big)\\
&\quad\le \phi(k^r),
\end{aligned} \tag{3.9}$$

that is,

$$\rho(y_1,y_2,y_m)\le[\phi(k^r)]^{1/r}. \tag{3.10}$$

Similarly, letting $x=x_1$, $y=x_2$, and $z=z_{m-1}$, $m\ge 2$ in (3.2), we obtain

$$\begin{aligned}
&\rho^r(y_2,y_3,y_m)\\
&\quad\le \delta^r(Fx_1,Fx_2,Fx_{m-1})\\
&\quad\le \phi(\max\{\rho^r(gx_1,gx_2,gx_{m-1}),\delta^r(Fx_1,Fx_2,gx_{m-1}),\\
&\qquad\qquad \delta^r(gx_1,Fx_1,gx_{m-1}),\delta^r(gx_2,Fx_2,gx_{m-1}),\\
&\qquad\qquad \delta^r(gx_1,Fx_2,gx_{m-1}),\delta^r(gx_2,Fx_1,gx_{m-1})\})\\
&\quad\le \phi(\max\{\delta^r(Fx_0,Fx_1,Fx_{m-2}),\delta^r(Fx_1,Fx_2,Fx_{m-2}),\\
&\qquad\qquad \delta^r(Fx_0,Fx_1,Fx_{m-2}),\delta^r(Fx_1,Fx_2,Fx_{m-2}),\\
&\qquad\qquad \delta^r(Fx_0,Fx_2,Fx_{m-2}),\delta^r(Fx_1,Fx_1,Fx_{m-2})\})\\
&\quad\le \phi\Big(\phi\Big(\max_{0\le a\le 2,\,1\le b\le 3}\delta^r(X_a,X_b,X_{m-2})\Big)\Big)\\
&\quad\le \phi(\phi(k^r))\\
&\quad= \phi^2(k^r),
\end{aligned} \tag{3.11}$$

that is,

$$\rho(y_2,y_3,y_m)\le[\phi^2(k^r)]^{1/r}. \tag{3.12}$$

In general, by induction,

$$\rho(y_n,y_{n+1},y_m)\le[\phi^n(k^r)]^{1/r} \tag{3.13}$$

for all $m>n\in\mathbb{N}$.

Hence, the application of Lemma 2.2 yields that $\{y_n\}$ is a D-Cauchy sequence in X. The D-metric space X being complete, there is a point $u\in X$ such that $\lim_n y_n=u$. The definition of $\{y_n\}$ implies that $\lim_n gx_n=u$. We will show that $\lim_n Fx_n=\{u\}$.

Now,

$$\begin{aligned}
&\lim_n \delta^r(Fx_n, Fx_n, u)\\
&\quad= \lim_n \delta^r(Fx_n, Fx_n, y_{n+1})\\
&\quad\le \lim_n \delta^r(Fx_n, Fx_n, Fx_n)\\
&\quad\le \lim_n \phi(\max\{\rho^r(gx_n, gx_n, gx_n), \delta^r(Fx_n, Fx_n, gx_n), \delta^r(gx_n, Fx_n, gx_n)\})\\
&\quad= \lim_n \phi(\max\{\delta^r(Fx_n, Fx_n, u), 0\})\\
&\quad= \phi\Big(\lim_n \delta^r(Fx_n, Fx_n, u)\Big),
\end{aligned} \tag{3.14}$$

which implies that $\lim_n Fx_n = u$. Thus we have

$$\lim_n Fx_n = \{u\} = \lim_n gx_n. \tag{3.15}$$

Since F and g are limit coincidentally commuting, one has

$$\lim_n Fgx_n = \Big\{\lim_n gFx_n\Big\}. \tag{3.16}$$

Suppose that g is (F/g)-orbitally continuous on X. Then we have

$$\lim_n Fgx_n = \lim_n gFx_n = \lim_n ggx_n = gu. \tag{3.17}$$

First, we will show that u is a common fixed point of F and g. Suppose not. Then we have

$$\begin{aligned}
\delta^r(u, u, gu) &= \lim_n \delta^r(Fx_n, Fx_n, gFx_n)\\
&= \lim_n \delta^r(Fx_n, Fx_n, Fgx_n)\\
&\le \lim_n \phi(\max\{\rho^r(gx_n, gx_n, ggx_n),\\
&\qquad\qquad \delta^r(Fx_n, Fx_n, ggx_n), \delta^r(gx_n, Fx_n, ggx_n)\})\\
&= \phi\Big(\max\Big\{\lim_n \delta^r(gx_n, gx_n, ggx_n), \lim_n \delta^r(Fx_n, Fx_n, ggx_n)\Big\}\Big)\\
&= \phi(\delta^r(u, u, gu)),
\end{aligned} \tag{3.18}$$

which is a contradiction and hence $gu = u$.

Again

$$\begin{aligned}
&\delta^r(Fu, gu, u)\\
&\quad= \lim_n \delta^r(Fu, Fx_n, Fgx_n)\\
&\quad\le \lim_n \phi(\max\{\rho^r(gu, gx_n, ggx_n), \delta^r(Fu, Fx_n, ggx_n), \delta^r(gu, Fu, ggx_n),\\
&\qquad\qquad \delta^r(gx_n, Fx_n, ggx_n), \delta^r(gu, Fx_n, ggx_n), \delta^r(gx_n, Fu, ggx_n)\})
\end{aligned}$$

$$\begin{aligned}&= \phi(\max\{\rho^r(gu,u,gu),\delta^r(Fu,u,gu),\delta^r(gu,Fu,gu),\\&\qquad\delta^r(u,u,gu),\delta^r(gu,u,gu),\delta^r(u,Fu,gu)\})\\&= \phi(\delta^r(Fu,gu,u)),\end{aligned} \tag{3.19}$$

which is possible only when $Fu = \{u\} = gu$ since $\phi \in \Phi$. Thus u is a common fixed point of F and g.

Next, suppose that F is (F/g)-orbitally continuous on X. Then we have

$$\lim_n Fgx_n = \lim_n gFx_n = \lim_n FFx_n = Fu = \{z\}. \tag{3.20}$$

We will show that z is a common fixed point of F and g. Since $F(X) \subseteq g(X)$, there is a point $v \in X$ such that $Fu = gv = z$. We will show that $Fv = gv = \{z\}$. By (3.2),

$$\begin{aligned}&\delta^r(Fv,gv,Fv)\\&\quad= \lim_n \delta^r(Fv,Fv,FFx_n)\\&\quad\le \lim_n \phi(\max\{\rho^r(gv,gv,gFx_n),\delta^r(Fv,gv,gFx_n),\delta^r(gv,Fv,gFx_n)\})\\&\quad= \phi(\max\{\delta^r(gv,gv,gv),\delta^r(Fv,gv,z)\}),\end{aligned} \tag{3.21}$$

that is,

$$\delta^r(Fv,gv,z) \le \phi(\delta^r(Fv,gv,z)), \tag{3.22}$$

which implies that $Fv = gv = \{z\}$ since $\phi \in \Phi$.

Since F and g are limit coincidentally commuting, they are coincidentally commuting on X. Therefore, we have $Fgv = gFv$. Now, proceeding with the arguments as in Case 1, it is proved that z is a common fixed point of F and g.

To prove the uniqueness, let z^* $(\neq z)$ be another common fixed point of F and g. Then by (3.2),

$$\begin{aligned}\rho^r(z,z,z^*) &= \delta^r(Fz,Fz,Fz^*)\\&\le \phi(\max\{\rho^r(gz,gz,gz^*),\delta^r(Fz,Fz,gz^*),\\&\qquad\delta^r(gz,Fz,gz^*),\delta^r(gz,Fz,gz^*)\})\\&= \phi(\rho^r(z,z,z^*)),\end{aligned} \tag{3.23}$$

which is a contradiction. Hence $z = z^*$. Then F and g have a unique common fixed point $z \in X$ with $Fz = \{z\} = gz$.

Finally, suppose that g is continuous at the common fixed point z of F and g. Then we will prove that F is also continuous at z. Let $\{z_n\}$ be any sequence in X converging to the common fixed point z. Since g is continuous on X, we have

$$\lim_{m,n}\rho(z_m,z_n,z)=0\Longrightarrow\lim_{m,n}\rho(gz_m,gz_n,gz)=0. \tag{3.24}$$

From (1.2), it follows that

$$\kappa(Fz_m,Fz_n,Fz)\le\delta(Fz_m,Fz_n,Fz). \tag{3.25}$$

Now,

$$\begin{aligned}\delta^r&(Fz_m,Fz_n,Fz)\\&\le\phi(\max\{\rho^r(gz_m,gz_n,gz),\delta^r(Fz_m,Fz_n,gz),\delta^r(gz_m,Fz_m,gz),\\&\qquad\delta^r(gz_n,Fz_n,gz),\delta^r(gz_m,Fz_n,gz),\delta^r(gz_n,Fz_m,gz)\}).\end{aligned} \tag{3.26}$$

Therefore,

$$\begin{aligned}\lim_{m,n}&\delta^r(Fz_m,Fz_n,Fz)\\&\le\lim_{m,n}\phi(\max\{\rho^r(gz_m,gz_n,gz),\delta^r(Fz_m,Fz_n,Fz),\delta^r(gz_m,Fz_m,z),\\&\qquad\delta^r(gz_n,Fz_n,z),\delta^r(gz_m,Fz_n,z),\delta^r(gz_n,Fz_m,z)\})\\&=\phi\Big(\max\Big\{0,\lim_{m,n}\delta^r(Fz_m,Fz_n,Fz),\lim_{m}\delta^r(z,Fz_m,z),\lim_{n}\delta^r(z,Fz_n,z)\Big\}\Big)\\&=\phi\Big(\max\Big\{\lim_{m}\delta^r(z,Fz_m,z),\lim_{n}\delta^r(z,Fz_n,z)\Big\}\Big).\end{aligned} \tag{3.27}$$

But

$$\begin{aligned}\lim_{m}&\delta^r(z,Fz_m,z)\\&=\lim_{m}\delta^r(Fz,Fz,Fz_m)\\&\le\lim_{m}\phi(\max\{\rho^r(gz,gz,gz_m),\delta^r(Fz,Fz,gz_m),\delta^r(gz,Fz,gz_m)\})\\&=\phi(\max\{0,0,0\})\\&=0.\end{aligned} \tag{3.28}$$

Similarly, $\lim_n\delta^r(z,Fz_n,z)=0$. Substituting these estimates in (3.27) yields that

$$\lim_{m,n}\delta^r(Fz_m,Fz_n,Fz)=0 \tag{3.29}$$

or

$$\lim_{m,n}\delta(Fz_m,Fz_n,Fz)=0. \tag{3.30}$$

Now from (3.25), it follows that

$$\lim_{m,n}\kappa(Fz_m,Fz_n,Fz)=0, \tag{3.31}$$

and so F is continuous at the common fixed point z of F and g. This completes the proof. □

Letting $g = I$, the identity map on X and $r = 1$, in Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *Let $F : X \to \mathrm{CB}(X)$ be a multivalued mapping satisfying*

$$\begin{aligned}\delta(Fx,Fy,Fz) \le \phi\big(&\rho(x,y,z),\delta(Fx,Fy,z),\delta(x,Fx,z),\\ &\delta(y,Fy,z),\delta(x,Fy,z),\delta(y,Fx,z)\big)\end{aligned} \tag{3.32}$$

for all $x,y,z \in X$, where $\phi \in \Phi$. Further if X is F-orbitally bounded and F-orbitally complete D-metric space, then F has a unique fixed point $u \in X$ such that $Fu = \{u\}$ and F is continuous at u.

Corollary 3.3. *Let $F : X \to \mathrm{CB}(X)$ be a multivalued mapping satisfying*

$$\begin{aligned}\delta(Fx,Fy,Fz) \le \lambda \max\{&\rho(x,y,z),\delta(Fx,Fy,z),\delta(x,Fx,z),\\ &\delta(y,Fy,z),\delta(x,Fy,z),\delta(y,Fx,z)\}\end{aligned} \tag{3.33}$$

for all $x,y,z \in X$, where $0 \le \lambda < 1$. Further if X is F-orbitally bounded and F-orbitally complete D-metric space, then F has a unique fixed point $u \in X$ such that $Fu = \{u\}$ and F is continuous at u.

Corollary 3.3 includes the following fixed point of Dhage [3] as a special case.

Corollary 3.4 (see [3]). *Let X be a bounded and complete D-metric space and let $F : X \to \mathrm{CB}(X)$ be a multivalued mapping satisfying*

$$\delta(Fx,Fy,Fz) \le \lambda\rho(x,y,z) \tag{3.34}$$

for all $x,y,z \in X$, where $0 \le \lambda < 1$. Then F has a unique fixed point $u \in X$ such that $Fu = \{u\}$ and F is continuous at u.

Corollary 3.5. *Let $f,g : X \to X$ be two mappings satisfying*

$$\begin{aligned}&\rho^r(fx,fy,fz)\\ &\quad\le \phi\big(\max\{\rho^r(gx,gy,gz),\rho^r(fx,fy,gz),\rho^r(gx,fx,gz),\\ &\qquad\qquad \rho^r(gy,fy,gz),\rho^r(gx,fy,gz),\rho^r(gy,fx,gz)\}\big)\end{aligned} \tag{3.35}$$

for all $x,y,z \in X$, where $\phi \in \Phi$. Suppose that

(a) *$f(X) \subseteq g(X)$,*

(b) *$\{f,g\}$ is limit coincidentally commuting,*

(c) *f or g is continuous.*

Further if X is (f/g)-orbitally bounded and (f/g)-orbitally complete D-metric space, then f and g have a unique common fixed point $u \in X$. Moreover, if g is continuous at u, then f is also continuous at u.

Remark 3.6. Note that Corollary 3.5 includes the class of pairs of fixed-point mappings of Dhage [7] characterized by the inequality

$$\begin{aligned}\rho^r(fx,fy,fz)&\\ \le \phi(\max\{&\rho^r(gx,gy,gz),\rho^r(gx,fx,gz),\\ &\rho^r(gy,fy,gz),\rho^r(gx,fy,gz),\rho^r(gy,fx,gz)\})\end{aligned} \tag{3.36}$$

for all $x,y,z\in X$ and $\phi\in\Phi$.

Corollary 3.7. *Let $f,g:X\to X$ be two mappings satisfying for some positive real numbers p, q, and r,*

$$\begin{aligned}\rho^r(f^px,f^py,f^pz)&\\ \le \phi(\max\{&\rho^r(g^qx,g^qy,g^qz),\rho^r(f^px,f^py,g^qz),\\ &\rho^r(g^qx,f^px,g^qz),\rho^r(g^qy,f^py,g^qz),\\ &\rho^r(g^qx,f^py,g^qz),\rho^r(g^qy,f^px,g^qz)\})\end{aligned} \tag{3.37}$$

for all $x,y,z\in X$, where $\phi\in\Phi$. Suppose that

(a) $f^p(X)\subseteq g^q(X)$,
(b) $\{f,g\}$ *is commuting,*
(c) f *or* g *is continuous.*

Further if X is an (f^p/g^q)-orbitally bounded and (f^p/g^q)-orbitally complete D-metric space, then f and g have a unique common fixed point $u\in X$. Moreover, if g is continuous at u, then f^p is also continuous at u.

Proof. Let $S=f^p$ and $T=g^q$. Then by Corollary 3.5, S and T have a unique common fixed point $u\in X$, that is, $Su=f^pu=u=g^qu=Tu$. Now by commutativity of f and g, we obtain

$$fu=f(f^pu)=f^p(fu),\qquad fu=f(g^qu)=g^q(fu). \tag{3.38}$$

This shows that fu is again a common fixed point of f^p and g^q. By the uniqueness of u, we have $fu=u$. Similarly it is proved that $gu=u$. Thus f and g have a unique common fixed point $u\in X$. Further if g is continuous on X, g^q is continuous on X and by application of Corollary 3.5 yields that f^p is continuous at u. This completes the proof. □

Corollary 3.7 includes the class of pairs of fixed-point mappings of Dhage [7] characterized by the inequality

$$\begin{aligned}\rho^r(f^px,f^py,f^pz)&\\ \le \phi(\max\{&\rho^r(g^qx,g^qy,g^qy),\rho^r(g^qx,f^px,g^qz),\\ &\rho^r(g^qy,f^py,g^qz),\rho^r(g^qx,f^py,g^qz),\rho^r(g^qy,f^px,g^qz)\})\end{aligned} \tag{3.39}$$

for all $x,y,z\in X$ and $\phi\in\Phi$.

Corollary 3.8. *Let f be a self-map of a D-metric space X satisfying*

$$\begin{aligned}\rho(fx,fy,fz) \leq \lambda \max\{&\rho(x,y,z),\rho(fx,fy,z),\rho(x,fx,z),\\ &\rho(y,fy,z),\rho(x,fy,z),\rho(y,fx,z)\}\end{aligned} \tag{3.40}$$

for all $x,y,z \in X$, where $0 \leq \lambda < 1$. Further if X is f-orbitally bounded and f-orbitally complete, then f has a unique fixed point $u \in X$ and f is continuous at u.

Corollary 3.9. *Let f be a self-map of a D-metric space X satisfying, for some positive real number p,*

$$\begin{aligned}&\rho(f^px,f^py,f^pz)\\ &\quad\leq \lambda \max\{\rho(x,y,z),\rho(f^px,f^py,z),\rho(x,f^px,z),\\ &\qquad\qquad \rho(y,f^py,z),\rho(x,f^py,z),\rho(y,f^px,z)\}\end{aligned} \tag{3.41}$$

for all $x,y,z \in X$, where $0 \leq \lambda < 1$. Further if X is f-orbitally bounded and f-orbitally complete, then f has a unique fixed point $u \in X$, f^p is continuous, and f is f-orbitally continuous at u.

Note that Corollaries 3.8 and 3.9 include the fixed-point theorems of Rhoades [12] and Dhage [9] for the mappings characterized by the inequalities

$$\begin{aligned}\rho(fx,fy,fz) \leq \lambda \max\{&\rho(x,y,z),\rho(x,fx,z),\\ &\rho(y,fy,z),\rho(x,fy,z),\rho(y,fx,z)\},\end{aligned} \tag{3.42}$$

$$\begin{aligned}\rho(f^px,f^py,f^pz) \leq \lambda \max\{&\rho(x,y,z),\rho(x,f^px,z),\\ &\rho(y,f^py,z),\rho(x,f^py,z),\rho(y,f^px,z)\},\end{aligned} \tag{3.43}$$

for all $x,y,z \in X$ and $0 \leq \lambda < 1$.

3.2. Coincidentally commuting mappings. The coincidentally commuting mappings require some stronger condition than limit coincidentally commuting mappings and a good number of mathematicians have studied them on metric and D-metric spaces for the existence of their common fixed point. See [5, 11] and the references therein. The novelty of the fixed-point theorems for these coincidentally commuting mappings lies in the fact that here we do not require any of the maps under consideration to be continuous. Below, we prove a result in this direction and derive some interesting corollaries.

Theorem 3.10. *Let X be a D-metric space and let $F : X \to \mathrm{CB}(X)$ and $g : X \to X$ be two mappings satisfying (3.2). Further suppose that*

(a) *$F(X) \subseteq g(X)$,*
(b) *$g(X)$ is bounded and complete,*
(c) *$\{F,g\}$ is coincidentally commuting.*

Then F and g have a unique common fixed point $u \in X$ such that $Fu = \{u\} = gu$. Moreover, if g is continuous at u, then F is also continuous at u in the Kasubai D-metric on X.

Proof. Let $x \in X$ be arbitrary and define a sequence $\{y_n\} \subset X$ by (3.3). Clearly the sequence $\{y_n\}$ is well defined since $F(X) \subseteq g(X)$. Further we note that $\{y_n\} \subseteq g(X)$. We prove the conclusion of the theorem in two cases.

Case 1. Suppose that $y_r = y_{r+1}$ for some $r \in \mathbb{N}$. Then proceeding with the arguments similar to Case 1 of the proof of Theorem 3.1, it is proved that $y_r = u$ is a common fixed point of F and g such that $Fu = \{u\} = gu$.

Case 2. Assume that $y_n \neq y_{n+1}$ for each $n \in \mathbb{N}$. Then following Case 2 of the proof of Theorem 3.1, it is shown that $\{y_n\}$ is a D-Cauchy sequence. Since $g(X)$ is complete, there is a point $z \in g(X)$ such that $\lim_n y_n = z = \lim_n gx_n$. We will show that $\lim_n Fx_n = \{z\}$.

Now,

$$\begin{aligned}
&\lim_n \delta^r(Fx_n, Fx_n, z)\\
&\quad = \lim_n \delta^r(Fx_n, Fx_n, y_{n+1})\\
&\quad \le \lim_n \delta^r(Fx_n, Fx_n, Fx_n)\\
&\quad \le \lim_n \phi(\max\{\rho^r(gx_n, gx_n, gx_n), \delta^r(Fx_n, Fx_n, gx_n), \delta^r(gx_n, Fx_n, gx_n)\})\\
&\quad = \phi\Big(\max\Big\{0, \lim_n \delta^r(Fx_n, Fx_n, z)\Big\}\Big)\\
&\quad = \phi\Big(\lim_n \delta^r(Fx_n, Fx_n, z)\Big),
\end{aligned} \tag{3.44}$$

which gives that $\lim_n Fx_n = \{z\}$.

Since $z \in g(X)$, there is a point $u \in X$ such that $gu = u$. We will show that $Fu = \{z\} = gu$. Now,

$$\begin{aligned}
&\delta^r(Fu, z, z)\\
&\quad = \lim_n \delta^r(Fu, Fx_n, Fx_n)\\
&\quad = \lim_n \delta^r(Fx_n, Fx_n, Fu)\\
&\quad \le \lim_n \phi(\max\{\rho^r(gu, gx_n, gx_n), \delta^r(Fx_n, Fx_n, gu), \delta^r(gx_n, Fx_n, gu)\})\\
&\quad = \phi(\max\{0,0,0\})\\
&\quad = \phi(0)\\
&\quad = 0
\end{aligned} \tag{3.45}$$

and so $Fu = gu = \{z\}$. Thus u is a coincidence point of F and g. The rest of the proof is similar to Case 2 of the proof of Theorem 3.1. We omitted the details. □

As a consequence of Theorem 3.10, we obtain the following corollaries.

Corollary 3.11. *Let $f,g : X \to X$ be two mappings satisfying (3.35). Suppose that*
(a) $f(X) \subseteq g(X)$,
(b) $g(X)$ *is bounded and complete,*
(c) $\{f,g\}$ *is coincidentally commuting.*
Then f and g have a unique common fixed point u and if g is continuous at u, then f is also continuous at u.

Corollary 3.12. *Let X be a D-metric space and let $f,g : X \to X$ be two mappings satisfying*

$$\begin{aligned}\rho(fx,fy,fz) &\\ \le \lambda \max\{&\rho(gx,gy,gz),\rho(fx,fy,gz),\rho(gx,fx,gz),\\ &\rho(gy,fy,gz),\rho(gx,fy,gz),\rho(gy,fx,gz)\}\end{aligned} \tag{3.46}$$

for all $x,y,z \in X$, where $0 \le \lambda < 1$. Further suppose that hypotheses (a), (b), and (c) of Corollary 3.11 hold. Then f and g have a unique common fixed point $u \in X$ and if g is continuous at u, then f is also continuous at u.

Corollary 3.12 includes a common fixed-point theorem of Dhage [5] for the mappings f and g on a D-metric space characterized by the inequality

$$\begin{aligned}\rho(fx,fy,gz) &\\ \le \lambda \max\{&\rho(gx,gy,gz),\rho(gx,fx,gz),\\ &\rho(gy,fy,gz),\rho(gx,fy,gz),\rho(gy,fx,gz)\}\end{aligned} \tag{3.47}$$

for all $x,y,z \in X$ and $0 \le \lambda < 1$.

Corollary 3.13. *Let X be a D-metric space and let $f,g : X \to X$ be two mappings satisfying (3.37). Further suppose that*
(a) $f^p(X) \subseteq g^q(X)$,
(b) $g^p(X)$ *is bounded and complete,*
(c) $\{f,g\}$ *is commuting.*
Then f and g have a unique common fixed point u and if g^q is continuous at u, then f^p is also continuous at u.

Notice that Corollary 3.13 includes a class of common fixed-point mappings f and g on a D-metric space X characterized by the inequality

$$\begin{aligned}\rho(f^px,f^py,f^pz) &\\ \le \lambda \max\{&\rho(g^qx,g^qy,g^qz),\rho(g^qx,f^px,g^qz),\\ &\rho(g^qy,f^py,g^qz),\rho(g^qx,f^py,g^qz),\rho(g^qy,f^px,g^qz)\}\end{aligned} \tag{3.48}$$

for all $x,y,z \in X$ and $0 \le \lambda < 1$. See [5].

4. Weak commuting mappings in compact D-metric spaces. In this section, we prove some common fixed-point theorems for the pairs of singlevalued and multivalued coincidentally commuting mappings on a D-metric space satisfying a contraction condition more general than (4.3). But in this case the D-metric space under consideration is required to satisfy a stronger condition of compactness and the mappings under consideration are required to satisfy the continuity condition on the D-metric spaces. Our results of this section generalize some earlier known fixed-point theorems such as those of Dhage [9] and Rhoades [12] for single maps as well as for a pair of maps on D-metric spaces.

Theorem 4.1. *Let X be a compact D-metric space and let $F : X \to \mathrm{CB}(X)$ and $g : X \to X$ be two continuous mappings satisfying, for some positive real number r,*

$$\begin{aligned} &\delta^r(Fx,Fy,Fz) \\ &\quad < \max\{\rho^r(gx,gy,gz),\delta^r(Fx,Fy,gz),\delta^r(gx,Fx,gz), \\ &\qquad \delta^r(gy,Fy,gz),\delta^r(gx,Fy,gz),\delta^r(gy,Fx,gz)\} \end{aligned} \tag{4.1}$$

for all $x,y,z \in X$ for which the right-hand side is not zero. Further suppose that
(a) *$F(X) \subseteq g(X)$,*
(b) *$\{F,g\}$ is limit coincidentally commuting.*
Then F and g have a unique common fixed point $u \in X$ such that $Fu = \{u\} = gu$.

Proof. From inequality (4.3), it follows that if F and g have a common fixed point $u \in X$, then it is unique and $Fu = \{u\} = gu$. Since X is compact and δ is continuous, both sides of inequality (4.1) are bounded on X. Now, there are two cases.

Case 1. Suppose that the right-hand side of (4.1) is zero for some $x,y,z \in X$. Then, we have

$$Fx = gx = gz, \qquad Fy = gy = gz. \tag{4.2}$$

Now, proceeding with the arguments similar to Case 1 of the proof of Theorem 3.1, it is proved that $u = Fx = gx$ is a common fixed point of F and g and so it is unique.

Case 2. Suppose that the right-hand side of inequality (4.1) is not zero for all $x,y,z \in X$. Define a mapping $T : X \times X \times X \to (0,\infty)$ by

$$T(x,y,z) = \frac{\delta^r(Fx,Fy,Fz)}{M(x,y,z)}, \tag{4.3}$$

where

$$\begin{aligned} M(x,y,z) = \max\{&\rho^r(gx,gy,gz),\delta^r(Fx,Fy,gz),\delta^r(gx,Fx,gz), \\ &\delta^r(gy,Fy,gz),\delta^r(gx,Fy,gz),\delta^r(gy,Fx,gz)\}. \end{aligned} \tag{4.4}$$

Clearly, the function T is well defined since $M(x,y,z) \neq 0$ for all $x,y,z \in X$. Since F and g are continuous, from the compactness of X it follows that the function T attains its maximum on X^3 at some point $u,v,w \in X$. Call the value c. It is clear from (4.1) that $0 < c < 1$. By the definition of c, we have $T(x,y,z) \le c$ for all $x,y,z \in X$. This further, in view of (4.3), implies that

$$\begin{aligned}
&\delta^r(Fx,Fy,Fz)\\
&\quad \le cM(x,y,z)\\
&\quad = c\max\{\rho^r(gx,gy,gz),\delta^r(Fx,Fy,Fz),\delta^r(gx,Fx,gx),\\
&\qquad\qquad \delta^r(gy,Fy,gz),\delta^r(gx,Fy,gz),\delta^r(gy,Fx,gz)\}
\end{aligned} \tag{4.5}$$

for all $x,y,z \in X$.

As X is compact, it is complete and $g(X)$ is bounded in view of the continuity of g on X. Now, the desired conclusion follows by an application of Theorem 3.1. This completes the proof. □

Now we derive some interesting corollaries.

Corollary 4.2. *Let X be a compact D-metric space and let $F : X \to \mathrm{CB}(X)$ be a continuous mapping satisfying*

$$\begin{aligned}
\delta(Fx,Fy,Fz) < \max\{&\rho(x,y,z),\delta(Fx,Fy,z),\delta(x,Fx,z),\\
&\delta(y,Fy,z),\delta(x,Fy,z),\delta(y,Fx,z)\}
\end{aligned} \tag{4.6}$$

for all $x,y,z \in X$ for which the right-hand side is not zero. Then F has a unique fixed point $u \in X$ such that $Fu = \{u\}$.

Proof. The proof follows by letting $g = I$ in Theorem 4.1, where I is the identity map on X. □

Corollary 4.3 (see [3]). *Let X be a compact D-metric space and let $F : X \to \mathrm{CB}(X)$ be a continuous mapping satisfying*

$$\delta(Fx,Fy,Fz) < \rho(x,y,z) \tag{4.7}$$

for all $x,y,z \in X$ for which $\rho(x,y,z) \neq 0$. Then F has a unique fixed point $u \in X$ such that $Fu = \{u\}$.

Corollary 4.4. *Let X be a compact D-metric space and let $f,g : X \to X$ be two continuous mappings satisfying*

$$\begin{aligned}
\rho(fx,fy,fz) < \max\{&\rho(gx,gy,gz),\rho(fx,fy,gz),\rho(gx,fx,gz),\\
&\rho(gy,fy,gz),\rho(gx,fy,gz),\rho(gy,fx,gz)\}
\end{aligned} \tag{4.8}$$

for all $x,y,z \in X$ *for which the right-hand side is not zero. Further suppose that*

(a) $f(X) \subseteq g(X)$,

(b) $\{f,g\}$ *is limit coincidentally commuting.*

Then f *and* g *have a unique common fixed point.*

Proof. The proof follows by letting $F = \{f\}$, a single-valued mapping in Theorem 4.1. □

Corollary 4.5. *Let* X *be a compact* D*-metric space and let* $f : X \to X$ *be a continuous mapping satisfying*

$$\begin{aligned} \rho(fx,fy,fz) < \max\{&\rho(x,y,z),\rho(fx,fy,z),\rho(x,fx,z),\\ &\rho(y,fy,z),\rho(x,fy,z),\rho(y,fx,z)\} \end{aligned} \tag{4.9}$$

for all $x,y,z \in X$ *for which the right-hand side is not zero. Then* f *has a unique fixed point.*

Proof. The conclusion follows by letting $g = I$ in Corollary 4.4, where I is the identity map on X. □

Note that Corollaries 4.4 and 4.5 include the fixed-point theorems of Dhage [5] and Rhoades [12] for the mappings f and g on a D-metric space X characterized by the inequalities

$$\begin{aligned} \rho(fx,fy,fz) < \max\{&\rho(gx,gy,gz),\rho(gx,fx,gz),\\ &\rho(gy,fy,gz),\rho(gx,fy,gz),\rho(gy,fx,gz)\}, \end{aligned} \tag{4.10}$$

$$\begin{aligned} \rho(fx,fy,fz) < \max\{&\rho(x,y,z),\rho(x,fx,z),\\ &\rho(y,fy,z),\rho(x,fy,z),\rho(y,fx,z)\}, \end{aligned} \tag{4.11}$$

respectively.

Theorem 4.6. *Let* X *be a* D*-metric space and let* $F : X \to \mathrm{CB}(X)$, $g : X \to X$ *be two continuous mappings satisfying (4.1). Suppose further that*

(a) $F(X) \subseteq g(X)$,

(b) $g(X)$ *is compact,*

(c) $\{f,g\}$ *is coincidentally commuting.*

Then F *and* g *have a unique common fixed point* $u \in X$ *such that* $Fu = \{u\} = gu$.

Proof. Let $A = g(X)$. Then A is a compact D-metric space and F and g define the maps $F : A \to \mathrm{CB}(A)$ and $g : A \to A$. Now, the desired conclusion follows by an application of Theorem 4.1. □

Corollary 4.7. *Let* X *be a* D*-metric space and let* $f,g : X \to X$ *be two continuous mappings satisfying (4.8). Further suppose that*

(a) $f(X) \subseteq g(X)$,
(b) $g(X)$ *is compact,*
(c) $\{f,g\}$ *is coincidentally commuting.*

Then f *and* g *have a unique common fixed point.*

5. Remarks and conclusion. It has been noted in [6, 10] that the fixed-point theorems for the limit coincidentally commuting mappings have some nice applications to approximation theory, and therefore it is of interest to discuss the fixed-point theorems for a wide class of coincidentally commuting mappings in a D-metric space. The terms "compatible" and "δ-compatible" have been used by Jungck and Rhoades [11] for limit coincidentally commuting and coincidentally commuting mappings, respectively, but our terminologies are natural and more informative than the previous one patterned after [4]. Further we note that a similar study can be made for coincidentally pseudocommuting mappings on a D-metric space and analogously for limit coincidentally pseudocommuting mappings. But in order to prove fixed-point theorems for these classes of weakly pseudocommuting mappings, we require a stronger contraction condition for the mappings F and g under consideration:

$$\begin{aligned}\delta^r(Fx,Fy,Fz) &\\ \leq \phi\big(\max\{&\rho^r(gx,gy,gz),D^r(Fx,Fy,gz),D^r(gx,Fx,gz),\\ &D^r(gy,Fy,gz),D^r(gx,Fy,gz),D^r(gy,Fx,gz)\}\big).\end{aligned} \tag{5.1}$$

Obviously, condition (5.1) implies condition (3.2) on a D-metric space X and hence the fixed-point theorems for weakly pseudocommuting mappings can be obtained very easily with appropriate modifications. Finally, we close this discussion with the following open question.

Open question. Can we prove fixed-point theorems for a class of multivalued mapping F on a D-metric space X satisfying the generalized contraction condition

$$\begin{aligned}\kappa(Fx,Fy,Fz) \leq \lambda\max\{&\rho(x,y,z),D(Fx,Fy,z),D(x,Fx,z),\\ &D(y,Fy,z),D(x,Fy,z),D(y,Fx,z)\}\end{aligned} \tag{5.2}$$

for all $x,y,z \in X$ and $0 \leq \lambda < 1$?

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