

AN OPTIMAL ORDER YIELDING DISCREPANCY PRINCIPLE FOR SIMPLIFIED REGULARIZATION OF ILL-POSED PROBLEMS IN HILBERT SCALES

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Recently, Tautenhahn and Hämarik (1999) have considered a monotone rule as a parameter choice strategy for choosing the regularization parameter while considering approximate solution of an ill-posed operator equation $Tx = y$, where T is a bounded linear operator between Hilbert spaces. Motivated by this, we propose a new discrepancy principle for the simplified regularization, in the setting of Hilbert scales, when T is a positive and selfadjoint operator. When the data y is known only approximately, our method provides optimal order under certain natural assumptions on the ill-posedness of the equation and smoothness of the solution. The result, in fact, improves an earlier work of the authors (1997).

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1. Introduction. Tikhonov regularization (cf. [2]) is one of the most widely used procedures to obtain stable approximate solution to an ill-posed operator equation

$$Tx = y, \tag{1.1}$$

where $T : X \rightarrow Y$ is a bounded linear operator between Hilbert spaces X and Y . Suppose that the data y is not exactly known, but only an approximation of it, namely \tilde{y} , is available. Then, the regularized solution \tilde{x}_α , by Tikhonov regularization, is obtained by minimizing the map

$$x \mapsto \|Tx - \tilde{y}\|^2 + \alpha\|x\|^2 \tag{1.2}$$

for $\alpha > 0$. For $y \in R(T) + R(T)^\perp$, if \hat{x} is the generalized solution of (1.1), that is, $\hat{x} = T^\dagger y$, where T^\dagger is the *Moore-Penrose generalized inverse* of T , then estimates for the error $\|\hat{x} - \tilde{x}_\alpha\|$ are obtained by choosing the regularization parameter α appropriately. It is known that (see, e.g., [2]) if $\hat{x} \in ((T^*T)^\nu)$ for some $\nu > 0$ and if $\|y - \tilde{y}\| \leq \delta$ for some noise level $\delta > 0$, then the optimal order for the above error is $O(\delta^\mu)$, where $\mu = \min\{2\nu/(2\nu + 1), 2/3\}$.

In order to improve the error estimates available in Tikhonov regularization, Natterer [9] carried out error analysis in the framework of Hilbert scales. Subsequently, many authors extended, modified, and generalized Natterer's work

to obtain error bounds under various contexts (see, e.g., Natterer [9], Hegland [3], Schröter and Tautenhahn [12], Mair [6], Nair et al. [8], and Nair [7]).

If T is a positive and selfadjoint operator on a Hilbert space, then the simplified regularization introduced by Lavrentiev is better suited than Tikhonov regularization in terms of speed of convergence and condition number in the case of finite-dimensional approximations (cf. Schock [11]).

In [1], simplified regularization in the framework of Hilbert scales was studied for the first time and obtained error estimates under a priori and a posteriori parameter choice strategies. The a posteriori choice of the parameter in that paper has a drawback that it can yield the optimal rate only under certain restricted smoothness assumption on the solution.

In this paper, we propose a new discrepancy principle, for choosing the regularization parameter α , for simplified regularization in the setting of Hilbert scales, which eliminates the drawback of the method in [1] yielding the optimal order for a range of values of smoothness. The discrepancy principle of this paper is motivated by a recent procedure adopted by Tautenhahn and Hämarik [13].

2. Preliminaries. Let H be a Hilbert space and let $A : H \rightarrow H$ be a bounded, positive and selfadjoint operator on H . Recall that A is said to be a positive operator if $\langle Ax, x \rangle \geq 0$ for every $x \in H$. For $y \in R(A)$, the range of A , consider the operator equation

$$Ax = y. \tag{2.1}$$

Let \hat{x} be the minimal norm solution of (2.1). It is well known that if $R(A)$ is not closed in H , then the problem of solving (2.1) for \hat{x} is ill-posed in the sense that small perturbation in the data y can cause large deviations in the solution.

A prototype of (2.1) is an integral equation of the first kind,

$$\int_0^1 k(s, t)x(t)dt = y(s), \quad 0 \leq s \leq 1, \tag{2.2}$$

where $k(\cdot, \cdot)$ is a nondegenerate kernel which is square integrable, that is,

$$\int_0^1 \int_0^1 |k(s, t)|^2 dt ds < \infty, \tag{2.3}$$

satisfying $k(s, t) = k(t, s)$ for all s, t in $[0, 1]$, and such that the eigenvalues of the corresponding integral operator $A : L^2[0, 1] \rightarrow L^2[0, 1]$,

$$(Ax)(s) = \int_0^1 k(s, t)x(t)dt, \quad 0 \leq s \leq 1, \tag{2.4}$$

are all nonnegative. For example, consider the kernel $k(\cdot, \cdot)$ defined by

$$k(s, t) = \begin{cases} (1-s)t, & \text{if } 0 \leq s \leq t \leq 1, \\ (1-t)s, & \text{if } 0 \leq t \leq s \leq 1. \end{cases} \tag{2.5}$$

Clearly, $k(s, t) = k(t, s)$, so that $A : L^2[0, 1] \rightarrow L^2[0, 1]$, defined as in (2.4), is a selfadjoint operator. Moreover, the eigenvalues of this operator are $1/n^2\pi^2$ for $n = 1, 2, \dots$ (see Limaye [5, page 329]).

For considering the regularization of (2.1) in the setting of Hilbert scales, we consider a Hilbert scale $\{H_t\}_{t \in \mathbb{R}}$ generated by a strictly positive operator $L : D(L) \rightarrow H$ with its domain $D(L)$ dense in H satisfying

$$\|Lx\| \geq \|x\|, \quad x \in D(L). \tag{2.6}$$

Recall (cf. [4]) that the space H_t is the completion of $D := \bigcap_{k=0}^\infty D(L^k)$ with respect to the norm $\|x\|_t$, induced by the inner product

$$\langle u, v \rangle_t = \langle L^t u, L^t v \rangle, \quad u, v \in D. \tag{2.7}$$

Moreover, if $\beta \leq \gamma$, then the embedding $H_\gamma \hookrightarrow H_\beta$ is continuous, and therefore the norm $\|\cdot\|_\beta$ is also defined in H_γ and there is a constant $c_{0,1}$ such that

$$\|x\|_\beta \leq c_{0,1} \|x\|_\gamma, \quad x \in H_\gamma. \tag{2.8}$$

We assume that the ill-posed nature of the operator A is related to the Hilbert scale $\{H_t\}_{t \in \mathbb{R}}$ according to the relation

$$c_1 \|x\|_{-a} \leq \|Ax\| \leq c_2 \|x\|_{-a}, \quad x \in H, \tag{2.9}$$

for some positive reals a, c_1 , and c_2 .

For the example of an integral operator given in the previous paragraph, one may take L to be defined by

$$Lx := \sum_{j=1}^\infty j^2 \langle x, u_j \rangle u_j, \tag{2.10}$$

where $u_j(t) := \sqrt{2} \sin(j\pi t)$, $j \in \mathbb{N}$, and the domain of L is

$$D(L) := \left\{ x \in L^2[0, 1] : \sum_{j=1}^\infty j^4 |\langle x, u_j \rangle|^2 < \infty \right\}. \tag{2.11}$$

In this case, it can be seen that

$$H_t = \left\{ x \in L^2[0, 1] : \sum_{j=1}^\infty j^{4t} |\langle x, u_j \rangle|^2 < \infty \right\} \tag{2.12}$$

and the constants a , c_1 , and c_2 in (2.9) are given by $a = 1$ and $c_1 = c_2 = 1/\pi^2$ (see Schröter and Tautenhahn [12, Section 4]).

As in [1], we consider the regularized solution of (1.1) as the solution of the well-posed equation

$$(A + \alpha L^s)x_\alpha = y, \quad \alpha > 0, \tag{2.13}$$

where s is a fixed nonnegative real number.

Suppose that the data $y \neq 0$ is known only approximately, say $\tilde{y} \neq 0$ with $\|y - \tilde{y}\| \leq \delta$ for a known error level $\delta > 0$. Then, in place of (2.13), we consider

$$(A + \alpha L^s)\tilde{x}_\alpha = \tilde{y}. \tag{2.14}$$

It can be seen that the solution \tilde{x}_α of the above equation is the unique minimizer of the function

$$x \mapsto \langle Ax, x \rangle - 2\langle \tilde{y}, x \rangle + \alpha \langle L^s x, x \rangle, \quad x \in D(L). \tag{2.15}$$

We also observe that taking

$$A_s := L^{-s/2}AL^{-s/2}, \tag{2.16}$$

(2.13) and (2.14) take the forms

$$L^{s/2}(A_s + \alpha I)L^{s/2}x_\alpha = y, \quad L^{s/2}(A_s + \alpha I)L^{s/2}\tilde{x}_\alpha = \tilde{y}, \tag{2.17}$$

respectively. Note that the operator A_s defined above is positive and selfadjoint bounded operator on H .

One of the crucial results for proving the results in [1] as well as the results in this paper is the following result, where functions f and g are defined by

$$f(t) = \min \{c_1^t, c_2^t\}, \quad g(t) = \max \{c_1^t, c_2^t\}, \quad t \in \mathbb{R}, |t| \leq 1, \tag{2.18}$$

respectively.

PROPOSITION 2.1 (see [1, Proposition 3.1]). *For $s \geq 0$ and $|v| \leq 1$,*

$$f\left(\frac{v}{2}\right)\|x\|_{-v(s+a)/2} \leq \|A_s^{v/2}x\| \leq g\left(\frac{v}{2}\right)\|x\|_{-v(s+a)/2}, \quad x \in H. \tag{2.19}$$

Using the above proposition, the following result has been proved in [1].

THEOREM 2.2 (see [1, Theorem 3.2]). *Suppose that $\hat{x} \in H_t$, $0 < t \leq s + a$, and $\alpha > 0$. Then*

$$\|\hat{x} - \tilde{x}_\alpha\| \leq \phi(s, t)\alpha^{t/(s+a)}\|x\|_t + \psi(s)\alpha^{-a/(s+a)}\delta, \tag{2.20}$$

where

$$\phi(s, t) = \frac{g((s - 2t)/(2s + 2a))}{f(s/(2s + 2a))}, \quad \psi(s) = \frac{g(-s/(2s + 2a))}{f(s/(2s + 2a))}. \tag{2.21}$$

In particular, if $\alpha = c_0 \delta^{(s+a)/(t+a)}$ for some constant $c_0 > 0$, then

$$\|\hat{x} - \tilde{x}_\alpha\| \leq \eta(s, t) \delta^{t/(t+a)}, \tag{2.22}$$

where

$$\eta(s, t) = \max \left\{ \phi(s, t) \|\hat{x}\|_t c_0^{t/(t+a)}, \psi(s) c_0^{-a/(s+a)} \right\}. \tag{2.23}$$

Let $R_\alpha = (A_s + \alpha I)^{-1}$. We will make use of the relation

$$\|R_\alpha A_s^\tau\| \leq \alpha^{\tau-1}, \quad \alpha > 0, \quad 0 < \tau \leq 1, \tag{2.24}$$

which follows from the spectral properties of the selfadjoint operator $A_s, s > 0$.

In [1], the authors considered parameter choice strategies, a priori and a posteriori, which yield the optimal rate $O(\delta^{t/(t+a)})$ if $\hat{x} \in H_t$ for certain specific values of t . The a posteriori parameter choice strategy in [1] is to choose α such that

$$\alpha^{p+1} \|(A_s + \alpha I)^{-p-1} L^{-s/2} x\| = k \delta, \tag{2.25}$$

where $k > 1$ and $\tilde{y} \in X$ satisfy $0 < k \delta \leq \|\tilde{y}\|_{-s/2}$. Under the above procedure, the optimal order $O(\delta^{t/(t+a)})$ is obtained for $t = s + p(s + a)$.

In the present paper, we propose a new discrepancy for choosing the regularization parameter α which yields the optimal rate

$$\|\hat{x} - \tilde{x}_\alpha\| = O(\delta^{t/(t+a)}). \tag{2.26}$$

3. The discrepancy principle. Let s and a be fixed positive real numbers. For $\alpha > 0$ and nonzero $x \in H$, let

$$\Phi(\alpha, x) := \frac{\alpha \|R_\alpha^{3/2} A_s^{-s/(2s+2a)} L^{-s/2} x\|^2}{\|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} x\|}. \tag{3.1}$$

Note that, by assumption (2.9), $\|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} x\|$ is nonzero for every nonzero $x \in H$ so that the function $\Phi(\alpha, x)$ is well defined for every $\alpha > 0$ and for every nonzero $x \in H$.

We assume that the available data \tilde{y} is nonzero and

$$\|y - \tilde{y}\| \leq \delta \tag{3.2}$$

for some known error level $\delta > 0$. Our idea is to prove the existence of a unique α such that

$$\Phi(\alpha, \tilde{y}) = c\delta \quad (3.3)$$

for some known $c > 0$.

In due course we will make use of the relation

$$f\left(\frac{-s}{2s+2a}\right)\|x\| \leq \|A_s^{-s/(2s+2a)}L^{-s/2}x\| \leq g\left(\frac{-s}{2s+2a}\right)\|x\| \quad (3.4)$$

which can be easily derived from [Proposition 2.1](#).

First we prove the monotonicity of the function $\Phi(\alpha, x)$ defined in (3.1).

THEOREM 3.1. *For each nonzero $x \in H$, the function $\alpha \mapsto \Phi(\alpha, x)$ for $\alpha > 0$, defined in (3.1), is increasing and it is continuously differentiable with $\Phi'(\alpha, x) \geq 0$. In addition*

$$\lim_{\alpha \rightarrow 0} \Phi(\alpha, x) = 0, \quad \lim_{\alpha \rightarrow \infty} \Phi(\alpha, x) = \|A_s^{-s/(2s+2a)}L^{-s/2}x\|. \quad (3.5)$$

PROOF. Using (3.1), one can write

$$\begin{aligned} & \frac{d}{d\alpha} \Phi(\alpha, x) \\ &= \frac{(d/d\alpha)(\Phi^2(\alpha, x))}{2\Phi(\alpha, x)} \\ &= \frac{2\alpha \|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\|^2 \|R_\alpha^{3/2} A_s^{-s/(2s+2a)}L^{-s/2}x\|^2}{2\alpha \|R_\alpha^{3/2} A_s^{-s/(2s+2a)}L^{-s/2}x\|^2} \\ & \quad \times \frac{(d/d\alpha) \left[\alpha \|R_\alpha^{3/2} A_s^{-s/(2s+2a)}L^{-s/2}x\|^2 \right]}{\|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\|^3} \\ & \quad - \frac{\alpha^2 \|R_\alpha^{3/2} A_s^{-s/(2s+2a)}L^{-s/2}x\|^4 (d/d\alpha) \left[\|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\|^2 \right]}{2\alpha \|R_\alpha^{3/2} A_s^{-s/(2s+2a)}L^{-s/2}x\|^2 \|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\|^3}. \end{aligned} \quad (3.6)$$

Thus,

$$\begin{aligned} & \frac{d}{d\alpha} \Phi(\alpha, x) \\ &= \frac{\|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\|^2 (d/d\alpha) \left[\alpha \|R_\alpha^{3/2} A_s^{-s/(2s+2a)}L^{-s/2}x\|^2 \right]}{\|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\|^3} \\ & \quad - \frac{\alpha \|R_\alpha^{3/2} A_s^{-s/(2s+2a)}L^{-s/2}x\|^2 (d/d\alpha) \left[\|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\|^2 \right]}{2\|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\|^3}. \end{aligned} \quad (3.7)$$

Let $\{E_\lambda : 0 \leq \lambda \leq a\}$ be the spectral family of A_s , where $a = \|A_s\|$. Then

$$\begin{aligned} & \frac{d}{d\alpha} \left(\alpha \|R_\alpha^{3/2} A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2 \right) \\ &= \frac{d}{d\alpha} \int_0^a \frac{\alpha}{\lambda^{s/(s+a)} (\lambda + \alpha)^3} d \langle E_\lambda L^{-s/2} \mathbf{x}, L^{-s/2} \mathbf{x} \rangle \\ &= \int_0^a \left[\frac{1}{\lambda^{s/(s+a)} (\lambda + \alpha)^3} - \frac{3\alpha}{\lambda^{s/(s+a)} (\lambda + \alpha)^4} \right] d \langle E_\lambda L^{-s/2} \mathbf{x}, L^{-s/2} \mathbf{x} \rangle \\ &= \|R_\alpha^{3/2} A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2 - 3\alpha \|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2. \end{aligned} \quad (3.8)$$

Similarly

$$\frac{d}{d\alpha} (\|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|) = -4 \|R_\alpha^{5/2} A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2. \quad (3.9)$$

Therefore, from (3.7), using (3.8) and (3.9), we get

$$\begin{aligned} & \frac{d}{d\alpha} \Phi(\alpha, \mathbf{x}) \\ &= \|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2 \\ & \quad \times \frac{\left[\|R_\alpha^{3/2} A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2 - 3\alpha \|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2 \right]}{\|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^3} \\ & \quad + \frac{2\alpha \|R_\alpha^{3/2} A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2 \|R_\alpha^{5/2} A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2}{\|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^3}. \end{aligned} \quad (3.10)$$

The above equation can be rewritten as

$$\frac{d}{d\alpha} \Phi(\alpha, \mathbf{x}) = \Psi_1(\alpha, \mathbf{x}) + \Psi_2(\alpha, \mathbf{x}), \quad (3.11)$$

where

$$\begin{aligned} & \Psi_1(\alpha, \mathbf{x}) \\ &= \|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2 \\ & \quad \times \frac{\left[\|R_\alpha^{3/2} A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2 - \alpha \|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2 \right]}{\|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^3}, \\ & \Psi_2(\alpha, \mathbf{x}) \\ &= \left(2\alpha \left[\|R_\alpha^{3/2} A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2 \right. \right. \\ & \quad \left. \left. \times \|R_\alpha^{5/2} A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2 - \|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^4 \right] \right) \\ & \quad \times \frac{1}{\|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^3}. \end{aligned} \quad (3.12)$$

Since

$$\begin{aligned} & \|R_\alpha^{3/2} A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2 \\ &= \langle (A_s + \alpha I)^{-3} A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}, A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x} \rangle, \\ & \|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2 \\ &= \langle (A_s + \alpha I)^{-3} A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}, (A_s + \alpha I)^{-1} A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x} \rangle, \end{aligned} \tag{3.13}$$

we have

$$\begin{aligned} & \|R_\alpha^{3/2} A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2 - \alpha \|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2 \\ &= \|A_s^{a/(2s+2a)} R_\alpha^2 L^{-s/2} \mathbf{x}\|^2. \end{aligned} \tag{3.14}$$

Also,

$$\begin{aligned} & \|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^4 \\ &= [\langle R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}, R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x} \rangle]^2 \\ &= [\langle R_\alpha^{3/2} A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}, R_\alpha^{5/2} A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x} \rangle]^2 \\ &\leq \|R_\alpha^{3/2} A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2 \|R_\alpha^{5/2} A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2. \end{aligned} \tag{3.15}$$

Hence

$$\Psi_1(\alpha, \mathbf{x}) \geq 0, \quad \Psi_2(\alpha, \mathbf{x}) \geq 0, \tag{3.16}$$

so that

$$\frac{d}{d\alpha}(\Phi(\alpha, \mathbf{x})) = \Psi_1(\alpha, \mathbf{x}) + \Psi_2(\alpha, \mathbf{x}) \geq 0. \tag{3.17}$$

To prove the last part of the theorem we observe that

$$\begin{aligned} & \alpha^2 \|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\| - \Phi(\alpha, \mathbf{x}) \\ &= \frac{\alpha^2 \|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2 - \alpha \|R_\alpha^{3/2} A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2}{\|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|}. \end{aligned} \tag{3.18}$$

Since

$$\begin{aligned} & \alpha^2 \|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2 \\ &= \alpha \langle R_\alpha^3 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}, \alpha R_\alpha A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x} \rangle, \\ & \alpha \|R_\alpha^{3/2} A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|^2 \\ &= \alpha \langle R_\alpha^3 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}, A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x} \rangle, \end{aligned} \tag{3.19}$$

and since $\alpha R_\alpha - I = A_s R_\alpha = R_\alpha A_s$, we have

$$\begin{aligned} & \alpha^2 \|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\| - \Phi(\alpha, \mathbf{x}) \\ &= \frac{-\alpha \langle R_\alpha^3 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}, A_s R_\alpha A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x} \rangle}{\|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|} \\ &= \frac{-\alpha \|A_s^{a/(2s+2a)} R_\alpha^2 L^{-s/2} \mathbf{x}\|^2}{\|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|} \leq 0. \end{aligned} \tag{3.20}$$

Hence

$$\Phi(\alpha, \mathbf{x}) \geq \alpha^2 \|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\| \geq \alpha^2 \frac{\|A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|}{(\|A_s\| + \alpha)^2}. \tag{3.21}$$

Also, we have

$$\begin{aligned} \Phi(\alpha, \mathbf{x}) &= \frac{\alpha \langle R_\alpha A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}, R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x} \rangle}{\|R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|} \\ &\leq \alpha \|R_\alpha A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|. \end{aligned} \tag{3.22}$$

Hence

$$\begin{aligned} & \left(\frac{\alpha}{\|A_s\| + \alpha} \right)^2 \|A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\| \\ &\leq \Phi(\alpha, \mathbf{x}) \leq \alpha \|R_\alpha A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|. \end{aligned} \tag{3.23}$$

From this, it follows that

$$\lim_{\alpha \rightarrow 0} \Phi(\alpha, \mathbf{x}) = 0, \quad \lim_{\alpha \rightarrow \infty} \Phi(\alpha, \mathbf{x}) = \|A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{x}\|. \tag{3.24}$$

This completes the proof. □

For the next theorem, in addition to (3.2), we assume that

$$\|A_s^{-s/(2s+2a)} L^{-s/2} \tilde{\mathbf{y}}\| \geq c\delta \tag{3.25}$$

for some $c > 0$. This assumption will be satisfied if, for example,

$$\delta \leq \frac{\tilde{f}(s)}{c + \tilde{f}(s)} \|\mathbf{y}\|, \quad \tilde{f}(s) := f\left(\frac{-s}{2s+2a}\right) \tag{3.26}$$

since, by (3.2), we have $\|\tilde{\mathbf{y}}\| \geq \|\mathbf{y}\| - \delta$, and by (3.4),

$$\|A_s^{-s/(2s+2a)} L^{-s/2} \tilde{\mathbf{y}}\| \geq f\left(\frac{-s}{2s+2a}\right) \|\tilde{\mathbf{y}}\|, \tag{3.27}$$

where f is as in (2.18).

Now, the following theorem is a consequence of Theorem 3.1.

THEOREM 3.2. *Assume that (3.2) and (3.25) are satisfied. Then there exists a unique $\alpha := \alpha(\delta)$ satisfying*

$$\Phi(\alpha, \tilde{y}) = c\delta. \tag{3.28}$$

4. Error estimates. In order to obtain Hölder-type error bounds, that is, error bounds of the form

$$\|\tilde{x}_\alpha - \hat{x}\| = O(\delta^\tau) \tag{4.1}$$

for some τ , we assume that the solution \hat{x} of (2.1) satisfies the source condition (as in [1, 10]):

$$\hat{x} \in M_{\rho,t} := \{x \in H_t : \|x\|_t \leq \rho\} \tag{4.2}$$

for some $t > 0$.

LEMMA 4.1. *Suppose that \hat{x} belongs to $M_{\rho,t}$ for some $t \leq s$, and $\alpha := \alpha(\delta) > 0$ is the unique solution of (3.28), where $c > g(-s/(2s + 2a))$. Then*

$$\alpha \geq c_0 \delta^{(s+a)/(t+a)}, \quad c_0 = \frac{c - g(-s/(2s + 2a))}{g((s - 2t)/(2s + 2a))\rho}. \tag{4.3}$$

PROOF. Note that by (3.22), Proposition 2.1, and (2.24), we have

$$\begin{aligned} \Phi(\alpha, \tilde{y}) &\leq \alpha \|R_\alpha A_s^{-s/(2s+2a)} L^{-s/2} \tilde{y}\| \\ &\leq \alpha \|R_\alpha A_s^{-s/(2s+2a)} L^{-s/2} (\tilde{y} - y)\| + \alpha \|R_\alpha A_s^{-s/(2s+2a)} A_s L^{s/2} \hat{x}\| \\ &\leq \alpha \|R_\alpha A_s^{-s/(2s+2a)} L^{-s/2} (\tilde{y} - y)\| + \alpha \|R_\alpha A_s^{(s+2a)/(2s+2a)} L^{s/2} \hat{x}\| \\ &\leq \alpha \|R_\alpha A_s^{-s/(2s+2a)} L^{-s/2} (\tilde{y} - y)\| \\ &\quad + \alpha \|R_\alpha A_s^{(t+a)/(s+a)} A_s^{(s-2t)/(2s+2a)} L^{s/2} \hat{x}\| \\ &\leq \|\alpha R_\alpha\| \|A_s^{-s/(2s+2a)} L^{-s/2} (\tilde{y} - y)\| \\ &\quad + \|\alpha R_\alpha A_s^{(t+a)/(s+a)}\| \|A_s^{(s-2t)/(2s+2a)} L^{s/2} \hat{x}\| \\ &\leq g\left(\frac{-s}{2s+2a}\right) \delta + g\left(\frac{s-2t}{2s+2a}\right) \rho \alpha^{(t+a)/(s+a)}. \end{aligned} \tag{4.4}$$

Thus

$$\left[c - g\left(\frac{-s}{2s+2a}\right) \right] \delta \leq g\left(\frac{s-2t}{2s+2a}\right) \rho \alpha^{(t+a)/(s+a)}, \tag{4.5}$$

which implies

$$\alpha \geq c_0 \delta^{(s+a)/(t+a)}, \quad c_0 = \frac{c - g(-s/(2s + 2a))}{g((s - 2t)/(2s + 2a))\rho}. \tag{4.6}$$

This completes the proof. □

THEOREM 4.2. *Under the assumptions in Lemma 4.1,*

$$\|\hat{x} - x_\alpha\| = O(\delta^\kappa), \quad \kappa := \frac{t}{t+a}. \tag{4.7}$$

PROOF. Since x_α is the solution of (2.13), we have

$$\begin{aligned} \hat{x} - x_\alpha &= \hat{x} - (A + \alpha L^s)^{-1} \mathcal{Y} \\ &= \alpha L^{-s/2} (A_s + \alpha I)^{-1} L^{s/2} \hat{x} = \alpha L^{-s/2} R_\alpha L^{s/2} \hat{x}. \end{aligned} \tag{4.8}$$

Therefore, by (3.4), we have

$$f\left(\frac{s}{2s+2a}\right) \|\hat{x} - x_\alpha\| \leq \|\alpha A_s^{s/(2s+2a)} R_\alpha L^{s/2} \hat{x}\|. \tag{4.9}$$

To obtain an estimate for $\|\alpha A_s^{s/(2s+2a)} R_\alpha L^{s/2} \hat{x}\|$, first we will make use of the following moment inequality

$$\|B^u x\| \leq \|B^v x\|^{u/v} \|x\|^{1-u/v}, \quad 0 \leq u \leq v, \tag{4.10}$$

where B is a positive selfadjoint operator. Precisely, we use (4.10) with

$$\begin{aligned} u &= \frac{t}{a}, \quad v = 1 + \frac{t}{a}, \quad B = \alpha R_\alpha A_s^{a/(s+a)}, \\ x &= \alpha^{1-t/a} R_\alpha^{1-t/a} A_s^{(s-2t)/(2s+2a)} L^{s/2} \hat{x}. \end{aligned} \tag{4.11}$$

Then since

$$\begin{aligned} \|x\| &\leq \|A_s^{(s-2t)/(2s+2a)} L^{s/2} \hat{x}\| \\ &\leq g\left(\frac{s-2t}{2s+2a}\right) \|L^{s/2} \hat{x}\|_{t-s/2} \leq g\left(\frac{s-2t}{2s+2a}\right) \rho, \end{aligned} \tag{4.12}$$

we have

$$\begin{aligned} &\|\alpha A_s^{s/(2s+2a)} R_\alpha L^{s/2} \hat{x}\| \\ &= \|B^{t/a} x\| \leq \|B^{1+t/a} x\|^{t/(t+a)} \|x\|^{a/(t+a)} \\ &\leq \|\alpha^2 R_\alpha^2 A_s^{(2a+s)/(2s+2a)} L^{s/2} \hat{x}\|^{t/(t+a)} \|x\|^{a/(t+a)} \\ &\leq \|\alpha^2 R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathcal{Y}\|^{t/(t+a)} \|x\|^{a/(t+a)} \\ &\leq g\left(\frac{s-2t}{2s+2a}\right)^{a/(t+a)} \rho^{a/(t+a)} \|\alpha^2 R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathcal{Y}\|^{t/(t+a)}. \end{aligned} \tag{4.13}$$

Further, by (2.24) and (3.20),

$$\begin{aligned} \|\alpha^2 R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathcal{Y}\| &\leq \|\alpha^2 R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} (\mathcal{Y} - \tilde{\mathcal{Y}})\| \\ &\quad + \|\alpha^2 R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \tilde{\mathcal{Y}}\| \\ &\leq \delta + \Phi(\alpha, \tilde{\mathcal{Y}}). \end{aligned} \tag{4.14}$$

Therefore, if $\alpha := \alpha(\delta)$ is the unique solution of (3.28), then we have

$$\|\alpha^2 R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} \mathbf{y}\| \leq (1+c)\delta. \tag{4.15}$$

Now the result follows from (4.9), (4.13), (4.14), and (4.15). □

THEOREM 4.3. *Under the assumptions in Lemma 4.1,*

$$\|\hat{\mathbf{x}} - \tilde{\mathbf{x}}_\alpha\| = O(\delta^\kappa), \quad \kappa := \frac{t}{t+a}. \tag{4.16}$$

PROOF. Let \mathbf{x}_α and $\tilde{\mathbf{x}}_\alpha$ be the solutions of (2.13) and (2.14), respectively. Then by triangle inequality, (2.24), and Proposition 2.1,

$$\begin{aligned} \|\hat{\mathbf{x}} - \tilde{\mathbf{x}}_\alpha\| &\leq \|\hat{\mathbf{x}} - \mathbf{x}_\alpha\| + \|\mathbf{x}_\alpha - \tilde{\mathbf{x}}_\alpha\| \\ &= \|\hat{\mathbf{x}} - \mathbf{x}_\alpha\| + \|L^{-s/2} R_\alpha L^{-s/2} (\mathbf{y} - \tilde{\mathbf{y}})\| \\ &\leq \|\hat{\mathbf{x}} - \mathbf{x}_\alpha\| + \frac{1}{f(s/(2s+2a))} \|A_s^{s/(2s+2a)} R_\alpha L^{-s/2} (\mathbf{y} - \tilde{\mathbf{y}})\| \\ &\leq \|\hat{\mathbf{x}} - \mathbf{x}_\alpha\| + \frac{1}{f(s/(2s+2a))} \|A_s^{s/(s+a)} R_\alpha A_s^{-s/(2s+2a)} L^{-s/2} (\mathbf{y} - \tilde{\mathbf{y}})\| \\ &\leq \|\hat{\mathbf{x}} - \mathbf{x}_\alpha\| + \frac{1}{f(s/(2s+2a))} \|A_s^{s/(s+a)} R_\alpha\| \|A_s^{-s/(2s+2a)} L^{-s/2} (\mathbf{y} - \tilde{\mathbf{y}})\| \\ &\leq \|\hat{\mathbf{x}} - \mathbf{x}_\alpha\| + \frac{g(-s/(2s+2a))}{f(s/(2s+2a))} \delta \alpha^{-a/(s+a)}. \end{aligned} \tag{4.17}$$

The proof now follows from Lemma 4.1 and Theorem 4.2. □

REMARK 4.4. We observe that unlike the discrepancy principle in [1], the discrepancy principle (3.3) gives the optimal order $O(\delta^{t/(t+a)})$ for all $0 < t \leq s$.

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