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INVARIANT SUBSPACES FOR POLYNOMIALLY COMPACT ALMOST SUPERDIAGONAL OPERATORS ON $l(p_i)$

ARTHUR D. GRAINGER

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It is shown that almost superdiagonal, polynomially compact operators on the sequence space $l(p_i)$ have nontrivial, closed invariant subspaces if the *nonlocally convex* linear topology $\tau(p_i)$ is locally bounded.

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1. Introduction. The purpose of this paper is to show that almost superdiagonal, polynomially compact operators on the sequence space $l(p_i)$ have nontrivial, closed invariant subspaces if the *nonlocally convex* linear topology $\tau(p_i)$ is locally bounded. The proofs and arguments of this paper are stated within the framework of nonstandard analysis (see [4, Theorem 6.3 and Proposition 5.5]).

1.1. Preliminaries. Let $\{p_i\}_{i=1}^{\infty}$ be a sequence of real numbers such that $0 < p_i \le 1$ for each $i \in \mathbb{N}_+$, the set of positive integers. Let

$$l(p_i) = \left\{ (\xi_i) \mid \sum_{i=1}^{\infty} |\xi_i|^{p_i} < \infty \right\}, \tag{1.1}$$

where $\xi_i \in \mathbb{C}$, the complex numbers. Since $|\lambda+\beta|^p \le |\lambda|^p + |\beta|^p$ and $|\lambda\beta|^p \le \max(1,|\lambda|)|\beta|^p$ are valid for all $\lambda,\beta \in \mathbb{C}$ and $0 < p \le 1$, it follows that $l(p_i)$ is a vector space over $\mathbb{C}$. Also,

$$\rho(p_i)(x,y) = \sum_{i=1}^{\infty} |\xi_i - \zeta_i|^{p_i}, \tag{1.2}$$

where $x = (\xi_i)$ and $y = (\zeta_i)$, defines a translation invariant metric on $l(p_i)$. Let $\tau(p_i)$ denote the topology generated on $l(p_i)$ by $\rho(p_i)$. If $p_i = p \in (0,1]$ for all $i \in \mathbb{N}_+$, then we denote $l(p_i)$ by l^p and $\tau(p_i)$ by τ_p.

For $(l(p_i),\tau(p_i))$, the following facts are known:

(1) $(l(p_i),\tau(p_i))$ *is a complete topological vector space*;

(2) $(l(p_i),\tau(p_i))$ *is a locally convex space if and only if* $l(p_i) = l^1$;

(3) *the following three conditions on* $\{p_i\}_{i=1}^{\infty} \subset (0,1]$ *are equivalent*:

(a) $\liminf p_i > 0$,

(b) *a subset* B of $l(p_i)$ *is bounded in* $\tau(p_i)$ *if and only if it is bounded in the metric* $\rho(p_i)$,

(c) $(l(p_i), \tau(p_i))$ *is locally bounded, that is, there exists a* $\tau(p_i)$*-bounded neighborhood of* 0.

(See [5, Lemmas 1 and 2, Theorems 5 and 6].)

Unless stated otherwise, it will be assumed that $0 < p_i \le 1$, for $i \in \mathbb{N}_+$, and $0 < p \le \liminf p_i$.

The sequence $\{e_i\}$ (where $e_i = (\varepsilon_{ij})$, $\varepsilon_{ii} = 1$, and $\varepsilon_{ij} = 0$ for $i \ne j$) will denote the natural Schauder basis for $(l(p_i), \tau(p_i))$ and $\{\pi_i \mid i \in \mathbb{N}_+\}$, $\{P_j \mid j \in \mathbb{N}_+\}$, and $\{E_j \mid j \in \mathbb{N}_+\}$ will denote the sequences of coordinate functionals, projections, and coordinate spaces, respectively, generated in $l(p_i)$ by $\{e_i\}$. Also, a τ or ρ, when used, will symbolize $\tau(p_i)$ and $\rho(p_i)$, respectively.

Let $\mathcal{F}[l(p_i)]$ symbolize the collection of all functions mapping $l(p_i)$ into $l(p_i)$ and let $[l(p_i)]$ and $\mathcal{L}(l(p_i))$ designate the vector spaces of $\tau(p_i)$-continuous linear transformation and linear transformations on $l(p_i)$, respectively. If $T, U \in \mathcal{L}(l(p_i))$, then TU denotes the composite map of T and U. For $n \in \mathbb{N}$, the set of natural numbers, and $T \in \mathcal{L}(l(p_i))$, define T^n in the usual manner, that is, $T^0 = I$, the identity map, $T^1 = T$, and $T^n = TT^{n-1}$ for $1 \le n$. If $q(\lambda) = \sum_{k=0}^{n} c_k \lambda^k$ is a polynomial over $\mathbb{C}$, then we define $q(T) = \sum_{k=0}^{n} c_k T^k$ for $T \in \mathcal{L}(l(p_i))$.

Let $x \in l(p_i)$, which implies $x = \sum_{j=1}^{\infty} \pi_j(x) e_j$. If $T \in [l(p_i)]$, then $Tx = \sum_{j=1}^{\infty} \pi_j(x) T e_j$. Note that $\pi_i(Tx) = \pi_i(\sum_{j=1}^{\infty} \pi_j(x) T e_j) = \sum_{j=1}^{\infty} \pi_i(Te_j)\pi_j(x)$, for $i \in \mathbb{N}_+$, by the continuity of π_i. Consequently, if $a_{ij} = \pi_i(Te_j)$, then $Tx = \sum_{i=1}^{\infty} \pi_i(Tx) e_i = \sum_{i=1}^{\infty} [\sum_{j=1}^{\infty} a_{ij} \pi_j(x)] e_i$. Therefore, for $T \in [l(p_i)]$, a double sequence $[a_{ij}] \subset \mathbb{C}$ will be called the *matrix of* T *with respect to* $\{e_i\}$ (or simply the matrix of T) if and only if $a_{ij} = \pi_i(Te_j)$.

Define $\mathcal{CF}[l(p_i)] \subset [l(p_i)]$ as follows: $T \in \mathcal{CF}[l(p_i)]$ if and only if $T \in [l(p_i)]$ and for $j \in \mathbb{N}_+$, there exists $n \in \mathbb{N}_+$, depending on j, such that $Te_j \in E_n$. If $T \in \mathcal{CF}[l(p_i)]$ and $[a_{ij}]$ is the matrix of T, then there exists $n \in \mathbb{N}_+$ such that $a_{ij} = 0$ for $n < i$. Also, $T, U \in \mathcal{CF}[l(p_i)]$ implies $TU \in \mathcal{CF}[l(p_i)]$. For $T \in \mathcal{CF}[l(p_i)]$, T is said to be *almost superdiagonal (a.sd.)* if and only if $Te_j \in E_{j+1}$ for each $j \in \mathbb{N}_+$.

For $T \in [l(p_i)]$, $n \in \mathbb{N}$, and $[a_{ij}]$, the matrix of T, let the matrix of T^n be denoted by $[a_{ij}^{(n)}]$. It can be shown that if T is almost superdiagonal, then

$$a_{j+n,j}^{(m)} = 0 \quad \text{for } m < n, \qquad a_{j+n,j}^{(n)} = \prod_{i=0}^{n-1} a_{j+i+1,j+i} \quad \text{for } j, m, n \in \mathbb{N}_+ \tag{1.3}$$

(see [1, Section 3, Theorem 3.6]).

An asterisk appended to the upper-left corner of a symbol indicates the *nonstandard extension* of the object represented by the symbol. The notation

$\mu(0)$ will denote the set of *infinitesimals* of $^*\mathbb{C}$. An element λ of $^*\mathbb{C}$ is said to be *finite* if and only if $|\lambda| \le {}^*\delta$ for some positive $\delta \in \mathbb{R}$; otherwise, λ is said to be *infinite*. It is customary to consider $\mathbb{C} \subset {}^*\mathbb{C}$, that is, the elements of $\mathbb{C}$ are identified with their nonstandard extensions; therefore, the asterisk notation is mostly not used for these elements. However, to bring clarity to some arguments, the asterisk notation for nonstandard extensions of elements of $\mathbb{C}$ will be used occasionally.

The notation $\mathcal{N}_\tau(0)$ denotes the $\tau(p_i)$-neighborhood filter of zero in $l(p_i)$ and $\mu_\tau(0)$ denotes the *monad* of the $\tau(p_i)$-neighborhoods of zero in $l(p_i)$. An element $z \in {}^*l(p_i)$ is called *near standard* if and only if there exists a (unique) element in $l(p_i)$, denoted by $^\circ z$, such that $z - {}^\circ z \in \mu_\tau(0)$. Also, for $A \subset {}^*l(p_i)$, the set $^\circ A \subset l(p_i)$, called the *standard part* of A, is defined as follows: $x \in {}^\circ A$ if and only if there is a $z \in A$ such that $z - {}^*x \in \mu_\tau(0)$. It can be shown that if F is an *internal* vector subspace of $^*l(p_i)$, then $^\circ F$ is a $\tau(p_i)$-closed linear subspace of $l(p_i)$ (see [3, Proposition 1.7]). If $T \in [l(p_i)]$, then $^*T[\mu_\tau(0)] \subseteq \mu_\tau(0)$, $^*T(z)$ is *near standard* if $z \in {}^*l(p_i)$ is *near standard*, and $^\circ[{}^*T(z)] = T({}^\circ z)$. We will denote the (*external*) set of all *near standard* points of $^*l(p_i)$ by the notation $\mathrm{ns}_\tau({}^*l(p_i))$. The (*external*) set $\{z \in {}^*l(p_i) \mid \lambda z \in \mu_\tau(0)$ for each $\lambda \in \mu(0)\}$ is called the set of *finite points* of $^*l(p_i)$ and is denoted by $\mathrm{fin}_\tau({}^*l(p_i))$. Clearly,

$$\mathrm{ns}_\tau({}^*l(p_i)) \subset \mathrm{fin}_\tau({}^*l(p_i)). \tag{1.4}$$

Finally, if Y is any set belonging to the *superstructure* generated by $\mathbb{C} \cup l(p_i)$, then $\triangle(Y)$ denotes the collection of all finite subsets of the set Y. Also, the elements of $^*\triangle(Y)$ are called $*$-*finite* subsets of $^*Y$.

2. Nonstandard properties of $l(p_i)$.

The purpose of this section is to state some of the (nonstandard) properties of $(l(p_i), \tau(p_i))$ that will be used in later arguments. These facts were developed in [2, 3]. The reader is referred to these references for the proofs.

Recall that for $x \in l(p_i)$,

$$P_i(x) = \sum_{j=1}^{i} \pi_j(x) e_j \in E_i, \tag{2.1}$$

where $\{e_i\}$ is the natural Schauder basis for $l(p_i)$, $E_i = \mathrm{sp}(e_1, \dots, e_i)$, and $\{\pi_i\}$ is the sequence of scalar projections generated by $\{e_i\}$. Let $\mathcal{FS}(l(p_i))$ be the collection of all finite-dimensional linear subspaces of $l(p_i)$. We will let $d : \mathcal{FS}(l(p_i)) \to \mathbb{N}$ denote the dimension function. In other words, $d(F) = n$, for $F \in \mathcal{FS}(l(p_i))$, if and only if $F = \mathrm{sp}(x_1, \dots, x_n)$ for some linearly independent $\{x_i\}_{i=1}^n \subset l(p_i)$. In particular,

$$d(E_i) = i \quad \text{for each } i \in \mathbb{N}_+. \tag{2.2}$$

Proposition 2.1 (see [2, Propositions III.1 and III.3]). *If $\alpha \in {}^*\mathbb{N}_+ - \mathbb{N}_+$, then the (internal) projection $P_\alpha : {}^*l(p_i) \to E_\alpha$ satisfies the following two conditions:*

(1) *for $W \in \mathcal{N}_\tau(0)$, there exists $V \in \mathcal{N}_\tau(0)$ such that $P_\alpha[{}^*V] \subset {}^*W$;*

(2) *if $x \in l(p_i)$, then $P_\alpha({}^*x) - {}^*x \in \mu_\tau(0)$ (i.e., ${}^\circ[P_\alpha({}^*x)] = x$).*

Note that Proposition 2.1 implies that for $x \in l(p_i)$ and $\alpha \in {}^*\mathbb{N}_+ - \mathbb{N}_+$, we have $l(p_i) = {}^\circ E_\alpha$ and $P_\alpha[\mu_\tau(0)] \subseteq \mu_\tau(0)$. It can be shown that

$$z \in \mathrm{ns}_\tau({}^*l(p_i)) \text{ implies } z - P_\alpha(z) \in \mu_\tau(0) \tag{2.3}$$

(see [2, Proposition II.2]). Also, for $T \in [l(p_i)]$ and $\alpha \in {}^*\mathbb{N}_+ - \mathbb{N}_+$, if we define $T_\alpha = P_\alpha({}^*T)P_\alpha$, then $T_\alpha \in {}^*\mathcal{L}(l(p_i))$ (i.e., T_α is an *internal* linear transformation on ${}^*l(p_i)$), $T_\alpha : {}^*l(p_i) \to E_\alpha$, and $T_\alpha[\mu_\tau(0)] \subseteq \mu_\tau(0)$. In addition, ${}^\circ[T_\alpha({}^*x)] = T(x)$ for $x \in l(p_i)$ and ${}^\circ[T_\alpha(z)] = T({}^\circ z)$ for any *near standard* $z \in {}^*l(p_i)$ (see [2, Propositions II.4 and II.5]). Finally, if $F \in {}^*\mathcal{FS}(l(p_i))$ such that $F \subseteq E_\alpha$ and $T_\alpha[F] \subseteq F$, then $T[{}^\circ F] \subseteq {}^\circ F$ (see [2, Proposition II.6]).

Proposition 2.2 (see [2, Theorem II.1]). *There exists a function $\nabla : \mathcal{FS}(l(p_i)) \to \mathcal{F}[l(p_i)]$ that satisfies the following conditions:*

(1) *if $F \in \mathcal{FS}(l(p_i))$, then $\nabla(F) : l(p_i) \to F$;*

(2) *for each $V \in N_\tau(0)$ and any nonzero $x \in l(p_i)$, there exists a positive $\lambda \in \mathbb{R}$ such that $\nabla(F)(\lambda x) \in V$ for all $F \in \mathcal{FS}(l(p_i))$;*

(3) *if $x \in l(p_i)$ such that $x \in {}^\circ F$ for $F \in {}^*\mathcal{FS}(l(p_i))$, then ${}^*\nabla(F)({}^*x) - {}^*x \in \mu_\tau(0)$ (i.e., ${}^\circ[{}^*\nabla(F)({}^*x)] = x$).*

Let $[\mathcal{FS}(l(p_i))]$ be the collection of all linear transformations Q with $\mathcal{D}(Q)$, $\mathcal{R}(Q) \in \mathcal{FS}(l(p_i))$, where $\mathcal{D}(Q)$ and $\mathcal{R}(Q)$ are the domain and range of Q, respectively, (i.e., the domain and range of linear transformation Q are *finite* dimensional for $Q \in [\mathcal{FS}(l(p_i))]$). Since the scalar field of $l(p_i)$ is complex, the following sentence is true.

If $E \in \mathcal{FS}(l(p_i))$ and $Q \in [\mathcal{FS}(l(p_i))]$ such that $Q : E \to E$, then for $n = d(E)$, there exists $\{F_j\}_{j=0}^n \in \triangle(\mathcal{FS}(l(p_i)))$ such that

(a) *$F_0 = \{0\}$ and $F_n = E$,*

(b) *$F_{j-1} \subset F_j$ for $j = 1, \ldots, n$,*

(c) *$d(F_j) = d(F_{j-1}) + 1$ for $j = 1, \ldots, n$,*

(d) *$Q[F_j] \subset F_j$ for $j = 0, \ldots, n$.*

Note that the (logical) constants of the previous statement are $\mathcal{FS}(l(p_i))$, $[\mathcal{FS}(l(p_i))]$, $\triangle(\mathcal{FS}(l(p_i)))$, and d, the dimension function. Therefore, by the *transfer principle*, the following sentence is true.

If $E \in {}^*\mathcal{FS}(l(p_i))$ and $Q \in {}^*[\mathcal{FS}(l(p_i))]$ such that $Q : E \to E$, then for $\alpha = {}^*d(E)$, there exists $\{F_\iota\}_{\iota=0}^\alpha \in {}^*\triangle(\mathcal{FS}(l(p_i)))$ such that

(a) *$F_0 = \{0\}$ and $F_\alpha = E$,*

(b) *$F_{\iota-1} \subset F_\iota$ for $\iota = 1, \ldots, \alpha$,*

(c) *${}^*d(F_\iota) = {}^*d(F_{\iota-1}) + 1$ for $\iota = 1, \ldots, \alpha$,*

(d) *$Q[F_\iota] \subset F_\iota$ for $\iota = 0, \ldots, \alpha$.*

In [2], this fact was used to obtain the following proposition.

Proposition 2.3 (see [2, Definition II.2 and Lemma II.8]). *Let $\nabla : \mathcal{FS}(l(p_i)) \to \mathcal{F}[l(p_i)]$ be the function established by Proposition 2.2 and let $\alpha \in {}^*\mathbb{N}_+ - \mathbb{N}_+$. If $T \in \mathcal{L}(l(p_i))$, then there exists an internal family $\{F_\iota\}_{\iota=0}^{\alpha} \in {}^*\triangle(\mathcal{FS}(l(p_i)))$ such that the following conditions are fulfilled:*

(1) $F_0 = \{0\}$, $F_\alpha = E_\alpha$, *and* $F_{\iota-1} \subset F_\iota$ *for* $\iota = 1,\dots,\alpha$,

(2) ${}^*d(F_\iota) = {}^*d(F_{\iota-1}) + 1$ *for* $\iota = 1,\dots,\alpha$,

(3) $T_\alpha[F_\iota] \subset F_\iota$ *for* $\iota = 0,\dots,\alpha$, *where* $T_\alpha = P_\alpha({}^*T)P_\alpha$,

(4) $\{F_\iota\}_{\iota=0}^{\alpha}$ *and* $\{{}^*\nabla(F_\iota)\}_{\iota=0}^{\alpha}$ *are* $*$*-finite,*

(5) ${}^*\nabla(F_\iota) : {}^*l(p_i) \to F_\iota$ *such that* $x \in {}^\circ F_\iota$ *implies* ${}^*\nabla(F_\iota)({}^*x) - {}^*x \in \mu_\tau(0)$ *(i.e.,* ${}^\circ[{}^*\nabla(F_\iota)({}^*x)] = x$*) for each* $\iota \in \{0,\dots,\alpha\}$.

Note that ${}^\circ F_\alpha = {}^\circ E_\alpha = l(p_i)$ and from continuity and Proposition 2.3(3), we infer that $T[{}^\circ F_\iota] \subset {}^\circ F_\iota$ for $T \in [l(p_i)]$ and $\iota \in \{0,\dots,\alpha\}$. Also, given $F_{\iota-1}$ and F_ι, $\iota = 1,\dots,\alpha$, it can be shown that Proposition 2.3(2) implies that for $x_1, x_2 \in {}^\circ F_\iota$, either $x_1 = \zeta_1 x_2 + y_1$ or $x_2 = \zeta_2 x_1 + y_2$ for some $\zeta_1, \zeta_2 \in \mathbb{C}$ and $y_1, y_2 \in {}^\circ F_{\iota-1}$. In other words, any two points of ${}^\circ F_\iota$ are linearly dependent modulo ${}^\circ F_{\iota-1}$ (see [2, Proposition I.21]).

Observe that for $T \in [l(p_i)]$, Proposition 2.3 produces a chain of closed invariant linear subspaces for T, namely $\{{}^\circ F_\iota\}_{\iota=0}^{\alpha}$. The problem is that we could have ${}^\circ F_\iota = \{0\}$ for $\iota \in \{0,\dots,\alpha\} \cap \mathbb{N}$ (i.e., the *finite* elements of $\{0,\dots,\alpha\}$) and ${}^\circ F_\iota = l(p_i)$ for $\iota \in \{0,\dots,\alpha\} \cap ({}^*\mathbb{N} - \mathbb{N})$ (i.e., the *infinite* elements of $\{0,\dots,\alpha\}$). However, if we could find $\nu \in \{1,\dots,\alpha\}$ such that ${}^\circ F_{\nu-1} \neq l(p_i)$ and ${}^\circ F_\nu \neq \{0\}$, then either ${}^\circ F_{\nu-1}$ or ${}^\circ F_\nu$ is a closed nontrivial linear subspace of T since $l(p_i)$ is infinite dimensional and any two points of ${}^\circ F_\nu$ are linearly dependent modulo ${}^\circ F_{\nu-1}$. The next proposition gives sufficient conditions for the existence of such a ν.

Proposition 2.4 (see [2, Definition II.2 and Lemma II.9]). *Let $T \in \mathcal{L}(l(p_i))$, $\nabla : \mathcal{FS}(l(p_i)) \to \mathcal{F}[l(p_i)]$ be the function established by Proposition 2.2, and let $\alpha \in {}^*\mathbb{N}_+ - \mathbb{N}_+$. Let the collection $[\{F_\iota\}_{\iota=0}^{\alpha} : \{{}^*\nabla(F_\iota)\}_{\iota=0}^{\alpha}]$ satisfy the conditions of Proposition 2.3 with respect to T, ∇, and α. Let $U \in {}^*\mathcal{L}(l(p_i))$ such that $U[\mu_\tau(0)] \subset \mu_\tau(0)$. If there exists $x \in l(p_i)$ such that $U({}^*x) \notin \mu_\tau(0)$ and $U({}^*\nabla(F_\iota)({}^*x)) \in F_\iota \cap \mathrm{ns}_\tau({}^*l(p_i))$ for each $\iota \in \{0,\dots,\alpha\}$, then there exists $\nu \in \{1,\dots,\alpha\}$ such that ${}^\circ F_{\nu-1} \neq l(p_i)$ and ${}^\circ F_\nu \neq \{0\}$.*

We close this section with a useful characterization of $\mathrm{fin}_\tau({}^*l(p_i))$, the *finite points* of ${}^*l(p_i)$.

Proposition 2.5. *If $z \in \mathrm{fin}_\tau({}^*l(p_i))$, then $\pi_\iota(z)$ is finite for $\pi_\iota \in {}^*\{\pi_i \mid i \in \mathbb{N}_+\}$.*

Proof. Since $0 < p \le \liminf p_i$ implies that $\tau(p_i)$ is locally bounded (see [5, Theorem 6]), there exists a positive $\delta_0 \in \mathbb{R}$ such that

$$V_0 = S(\rho(p_i);\delta_0) = \{x \in l(p_i) \mid \rho(p_i)(x,0) \le \delta_0\} \tag{2.4}$$

is $\tau(p_i)$-bounded. Let $z \in \text{fin}_\tau({}^*l(p_i))$. Hence, ${}^*g_{V_0}(z) \le {}^*\delta$ for some positive $\delta \in \mathbb{R}$, where g_{V_0} is the gauge of V_0 (see [2, Proposition I.14]). So, $z \in {}^*(\delta V_0) \subset {}^*S(\rho(p_i);\lambda)$ for $\lambda = \max(\delta\delta_0, \delta_0, 1)$ since V_0 is closed, balanced and $\delta S(\rho(p_i);\delta_0) \subset S(\rho(p_i);\lambda)$. Therefore, ${}^*\rho(p_i)(z,0) \le {}^*\lambda$, which implies $|\pi_\iota(z)|^{p_\iota} \le {}^*\lambda$ for $\pi_\iota \in {}^*\{\pi_i \mid i \in \mathbb{N}_+\}$. It suffices to consider the case when $1 \le |\pi_\iota(z)|$ for $\iota \in {}^*\mathbb{N}_+$. Since $p \in \mathbb{R}$ and $0 < p_\iota^{-1} \le {}^*(p^{-1})$ for $p_\iota \in {}^*\{p_i\}$, we have $|\pi_\iota(z)| \le ({}^*\lambda)^{p_\iota^{-1}} \le {}^*(\lambda^{p^{-1}})$ for $\pi_\iota \in {}^*\{\pi_i \mid i \in \mathbb{N}_+\}$. We infer that $\pi_\iota(z)$ is finite for $\pi_\iota \in {}^*\{\pi_i \mid i \in \mathbb{N}_+\}$. □

3. Polynomially compact almost superdiagonal operators. We want to show that if $T \in \mathscr{CF}[l(p_i)]$ is almost superdiagonal and $q(T)$ is compact for some polynomial $q(\lambda)$ over $\mathbb{C}$, then T has a nontrivial closed invariant linear subspace. Note that for $\alpha \in {}^*\mathbb{N}_+ - \mathbb{N}_+$ and $\nabla : \mathscr{FS}(l(p_i)) \to \mathscr{F}[l(p_i)]$, the function defined by Proposition 2.2, we can use Proposition 2.3 to produce a collection $[\{F_\iota\}_{\iota=0}^{\alpha} : \{{}^*\nabla(F_\iota)\}_{\iota=0}^{\alpha}]$, for some $\alpha \in {}^*\mathbb{N}$, such that $\{{}^\circ F_\iota\}_{\iota=0}^{\alpha}$ is a collection of closed invariant linear subspaces of T. The strategy is to find some $\alpha \in {}^*\mathbb{N}_+ - \mathbb{N}_+$ such that ${}^*q(T_\alpha) \in {}^*\mathscr{L}(l(p_i))$ satisfies the hypotheses of Proposition 2.4, where $T_\alpha = P_\alpha({}^*T)P_\alpha$.

First, however, consider a compact operator U on $l(p_i)$.

As stated in the proof of Proposition 2.5, there exists a positive $\delta_0 \in \mathbb{R}$ such that

$$V_0 = S(\rho(p_i);\delta_0) = \{x \in l(p_i) \mid \rho(p_i)(x,0) \le \delta_0\} \tag{3.1}$$

is $\tau(p_i)$-bounded since $0 < p \le \liminf p_i$ implies that $\tau(p_i)$ is locally bounded (see the known facts about $(l(p_i), \tau(p_i))$ in the first section). Therefore,

$${}^*V_0 \subset \text{fin}_\tau({}^*l(p_i)) \tag{3.2}$$

(see [2, Corollary I.18]). If $U \in [l(p_i)]$ such that $\overline{U[W]}$ is $\tau(p_i)$-compact for some $W \in \mathcal{N}_\tau(0)$, then $\overline{U[V_0]}$ is $\tau(p_i)$-compact since $\lambda V_0 \subset W$ for some positive scalar λ. Unless stated otherwise, $V_0 = S(\rho(p_i);\delta_0)$ will be a fixed, $\tau(p_i)$-bounded neighborhood of 0, with $0 < \delta_0 \le 1$. Thus, we have that $U \in [l(p_i)]$ is compact if and only if $\overline{U[V_0]}$ is $\tau(p_i)$-compact. Also, if $\overline{U[V_0]}$ is $\tau(p_i)$-compact, then ${}^*U[{}^*V_0] \subset \text{ns}_\tau({}^*l(p_i))$ (see [2, Proposition I.1]).

Proposition 3.1. *If $U \in [l(p_i)]$ is compact and $[b_{ij}]$ is the matrix of U, then $b_{\iota\kappa} \in \mu(0)$ for $b_{\iota\kappa} \in {}^*[b_{ij}]$ such that $\iota,\kappa \in {}^*\mathbb{N}_+ - \mathbb{N}_+$.*

Proof. Let $\lambda_\kappa = ({}^*\delta_0)^{p_\kappa^{-1}}$ for $\kappa \in {}^*\mathbb{N}_+$ and $p_\kappa \in {}^*\{p_i\}$ and define $z_\kappa = \lambda_\kappa e_\kappa$ for $\kappa \in {}^*\mathbb{N}_+$ and $e_\kappa \in {}^*\{e_i\}$. We infer that ${}^*\rho(p_i)(z_\kappa,0) = |\lambda_\kappa|^{p_\kappa} = {}^*\delta_0$ for $\kappa \in {}^*\mathbb{N}_+$, which implies $z_\kappa \in {}^*V_0$ for $\kappa \in {}^*\mathbb{N}_+$. Therefore, ${}^*Uz_\kappa \in \text{ns}_\tau({}^*l(p_i))$ for $\kappa \in {}^*\mathbb{N}_+$ since ${}^*U[{}^*V_0] \subset \text{ns}_\tau({}^*l(p_i))$. So, if $\iota \in {}^*\mathbb{N}_+ - \mathbb{N}_+$, then ${}^*Uz_\kappa - P_{\iota-1}({}^*Uz_\kappa) \in \mu_\tau(0)$ for $\kappa \in {}^*\mathbb{N}_+$ by expression (2.3) since $\iota \in {}^*\mathbb{N}_+ - \mathbb{N}_+$ implies $\iota - 1 \in {}^*\mathbb{N}_+ - \mathbb{N}_+$.

Let $\iota,\kappa \in {}^*\mathbb{N}_+ - \mathbb{N}_+$ and let $[b_{ij}]$ be the matrix of U with respect to $\{e_i\}$. Note that $\pi_\iota({}^*Uz_\kappa) = \lambda_\kappa \pi_\iota({}^*Ue_\kappa) = \lambda_\kappa b_{\iota\kappa}$ for $\pi_\iota \in {}^*\{\pi_i \mid i \in \mathbb{N}_+\}$, $e_\iota \in {}^*\{e_i\}$, and $b_{\iota\kappa} \in {}^*[b_{ij}]$. Since

$$|\lambda_\kappa b_{\iota\kappa}|^{p_\iota} = {}^*\rho(p_i)(\lambda_\kappa b_{\iota\kappa} e_\iota, 0) \le {}^*\rho(p_i)({}^*Uz_\kappa - P_{\iota-1}({}^*Uz_\kappa), 0), \tag{3.3}$$

we have $|\lambda_\kappa b_{\iota\kappa}|^{p_\iota} \in \mu(0)$. Consequently, $1 \le p_\iota^{-1} \le {}^*(p^{-1})$ implies $\lambda_\kappa |b_{\iota\kappa}| \in \mu(0)$. Also, $1 \le p_\kappa^{-1} \le {}^*(p^{-1})$ implies ${}^*(\delta_0^{p^{-1}}) \le \lambda_\kappa \le {}^*\delta_0$, which implies $|b_{\iota\kappa}| \in \mu(0)$. Therefore, $b_{\iota\kappa} \in \mu(0)$. □

Proposition 3.2. *If $U \in [l(p_i)]$ is almost superdiagonal and $q(U)$ is compact for some complex polynomial $q(\lambda)$, then there exists $\alpha \in {}^*\mathbb{N}_+ - \mathbb{N}_+$ such that $a_{\alpha+1,\alpha} \in \mu(0)$ for $a_{\alpha+1,\alpha} \in {}^*[a_{ij}]$, where $[a_{ij}]$ is the matrix of U.*

Proof. Let n be the degree of $q(\lambda) = \sum_{k=0}^n c_k \lambda^k$, which implies $c_n \ne 0$. If $[b_{ij}]$ is the matrix of $q(U)$, with respect to $\{e_i\}$, then $q(U)$ being compact implies $b_{\iota\kappa} \in \mu(0)$ for $b_{\iota\kappa} \in {}^*[b_{ij}]$ such that $\iota,\kappa \in {}^*\mathbb{N}_+ - \mathbb{N}_+$ by Proposition 3.1. Let $\kappa \in {}^*\mathbb{N}_+ - \mathbb{N}_+$. Since U being almost superdiagonal implies $a_{\kappa+n,\kappa}^{(m)} = 0$ for $m < n$ and $a_{\kappa+n,\kappa}^{(n)} = \prod_{i=0}^{n-1} a_{\kappa+i+1,\kappa+i}$ (by expression (1.3) and the *transfer principle*), we have $b_{\kappa+n,\kappa} = {}^*c_n \prod_{i=0}^{n-1} a_{\kappa+i+1,\kappa+i}$. Therefore, if $\kappa \in {}^*\mathbb{N}_+ - \mathbb{N}_+$, then ${}^*c_n \notin \mu(0)$ and $b_{\kappa+n,\kappa} \in \mu(0)$ imply $a_{\kappa+i_0+1,\kappa+i_0} \in \mu(0)$ for some $i_0 \in \{0,\dots,n-1\}$. Let $\alpha = \kappa + i_0$. □

Proposition 3.3. *If $U \in {}^*\mathcal{L}(l(p_i))$ such that $U[\mu_\tau(0)] \subset \mu_\tau(0)$, then*

$$U[\operatorname{fin}_\tau({}^*l(p_i))] \subset \operatorname{fin}_\tau({}^*l(p_i)). \tag{3.4}$$

Proof. Let $z \in \operatorname{fin}_\tau({}^*l(p_i))$. If $\lambda \in \mu(0)$, then $\lambda z \in \mu_\tau(0)$, which implies $\lambda Uz = U(\lambda z) \in \mu_\tau(0)$. Therefore, $Uz \in \operatorname{fin}_\tau({}^*l(p_i))$. □

Let $T \in \mathscr{CF}[l(p_i)]$ be an almost superdiagonal operator such that $q(T)$ is compact for $q(\lambda) = \sum_{k=0}^n c_k\lambda^k$, a polynomial over $\mathbb{C}$ with $c_n \ne 0$. Let $\nabla : \mathscr{FS}(l(p_i)) \to \mathscr{F}[l(p_i)]$ satisfy the conditions of Proposition 2.2 and let $\alpha \in {}^*\mathbb{N}_+ - \mathbb{N}_+$ satisfy the conclusion of Proposition 3.2. Note that the (internal) projection $P_\alpha : {}^*l(p_i) \to E_\alpha$ satisfies the conditions of Proposition 2.1. Define $T_\alpha = P_\alpha({}^*T)P_\alpha$. Observe that

$$\begin{gathered} {}^*q(T_\alpha) \in {}^*\mathcal{L}(l(p_i)), \qquad T_\alpha[\mu_\tau(0)] \subset \mu_\tau(0), \\ {}^*(q(T))[\mu_\tau(0)] \subset \mu_\tau(0), \end{gathered} \tag{3.5}$$

since T and $q(T)$ are continuous.

Proposition 3.4. *If $z \in \mu_\tau(0)$, then ${}^*q(T_\alpha)z \in \mu_\tau(0)$.*

Proof. It suffices to show that $(T_\alpha)^m[\mu_\tau(0)] \subset \mu_\tau(0)$ for $m \in \mathbb{N}_+$ (see [2, Proposition I.6]). Note that from (3.5), $T_\alpha[\mu_\tau(0)] \subset \mu_\tau(0)$. Assume that

$(T_\alpha)^m[\mu_\tau(0)] \subset \mu_\tau(0)$ for $m \in \mathbb{N}_+$. Consequently,

$$(T_\alpha)^{m+1}[\mu_\tau(0)] = T_\alpha[(T_\alpha)^m[\mu_\tau(0)]] \subset T_\alpha[\mu_\tau(0)] \subset \mu_\tau(0). \tag{3.6}$$

Therefore, $(T_\alpha)^m[\mu_\tau(0)] \subset \mu_\tau(0)$ for each $m \in \mathbb{N}_+$ by induction. □

So, one of the conditions of Proposition 2.4 for ${}^*q(T_\alpha)$ has been satisfied, that is, ${}^*q(T_\alpha)[\mu_\tau(0)] \subset \mu_\tau(0)$.

Proposition 3.5. *Let $F \in {}^*\mathcal{FS}(l(p_i))$ such that $F \subset E_\alpha$. If $T_\alpha[F] \subset F$, then $(T_\alpha)^m[F] \subset F$ for $m \in \mathbb{N}$.*

Proof. If $(T_\alpha)^m[F] \subset F$ for $m \in \mathbb{N}$, then $(T_\alpha)^{m+1}[F] = T_\alpha[(T_\alpha)^m[F]] \subset T_\alpha[F] \subset F$. Therefore, $(T_\alpha)^m[F] \subset F$ for any $m \in \mathbb{N}$ by induction. □

Consequently, if $F \in {}^*\mathcal{FS}(l(p_i))$ such that $F \subset E_\alpha$ and $T_\alpha[F] \subset F$, then

$${}^*q(T_\alpha)[F] \subset F. \tag{3.7}$$

Proposition 3.6. *If $z \in E_\alpha \cap \operatorname{fin}_\tau({}^*l(p_i))$, then $[{}^*q(T_\alpha)z - {}^*(q(T))z] \in \mu_\tau(0)$.*

Proof. It is sufficient to show that ${}^*(T^m)z - (T_\alpha)^m z \in \mu_\tau(0)$ for $z \in E_\alpha \cap \operatorname{fin}_\tau({}^*l(p_i))$ and $m \in \mathbb{N}$ (see [2, Proposition I.6]). Let $z \in E_\alpha \cap \operatorname{fin}_\tau({}^*l(p_i))$, which implies $z = \sum_{\kappa=1}^{\alpha} \pi_\kappa(z)e_\kappa$ for $\pi_\kappa \in {}^*\{\pi_i \mid i \in \mathbb{N}_+\}$ and $e_\kappa \in {}^*\{e_i\}$. Consequently,

$${}^*Tz = \sum_{\kappa=1}^{\alpha} \pi_\kappa(z){}^*Te_\kappa = \sum_{\kappa=1}^{\alpha} \pi_\kappa(z)\left[\sum_{\iota=1}^{\alpha+1} a_{\iota\kappa}e_\iota\right] = \sum_{\iota=1}^{\alpha+1}\left[\sum_{\kappa=1}^{\alpha} a_{\iota\kappa}\pi_\kappa(z)\right]e_\iota \tag{3.8}$$

since T is almost superdiagonal. Also, $a_{\alpha+1,\kappa} = 0$ for $\kappa < \alpha$, which implies $\sum_{\kappa=1}^{\alpha} a_{\alpha+1,\kappa}\pi_\kappa(z) = a_{\alpha+1,\alpha}\pi_\alpha(z)$. Therefore, ${}^*Tz = \sum_{\iota=1}^{\alpha}[\sum_{\kappa=1}^{\alpha} a_{\iota\kappa}\pi_\kappa(z)]e_\iota + a_{\alpha+1,\alpha}\pi_\alpha(z)e_{\alpha+1}$, which implies ${}^*Tz - P_\alpha({}^*Tz) = a_{\alpha+1,\alpha}\pi_\alpha(z)e_{\alpha+1}$. So,

$${}^*\rho(p_i)({}^*Tz - P_\alpha({}^*Tz), 0) = |a_{\alpha+1,\alpha}|^{p_{\alpha+1}}|\pi_\alpha(z)|^{p_{\alpha+1}}. \tag{3.9}$$

Since $\pi_\alpha(z)$ is finite (by Proposition 2.5), $a_{\alpha+1,\alpha} \in \mu(0)$ (Proposition 3.2), and $0 < {}^*p \le p_{\alpha+1} \le 1$, we infer that $|a_{\alpha+1,\alpha}|^{p_{\alpha+1}}|\pi_\alpha(z)|^{p_{\alpha+1}} \in \mu(0)$, which implies ${}^*Tz - P_\alpha({}^*Tz) \in \mu_\tau(0)$. Therefore, ${}^*Tz - T_\alpha z \in \mu_\tau(0)$ since $z \in E_\alpha$ implies $z = P_\alpha(z)$.

Now, let $m \in \mathbb{N}$ such that $2 \le m$ and assume that ${}^*(T^{m-1})z - (T_\alpha)^{m-1}z \in \mu_\tau(0)$ for $z \in E_\alpha \cap \operatorname{fin}_\tau({}^*l(p_i))$. If $z \in E_\alpha \cap \operatorname{fin}_\tau({}^*l(p_i))$, then

$${}^*(T^m)z - {}^*T\left((T_\alpha)^{m-1}z\right) = {}^*T\left({}^*(T^{m-1})z - (T_\alpha)^{m-1}z\right) \in \mu_\tau(0) \tag{3.10}$$

since $T \in [l(p_i)]$ implies that T is linear and ${}^*T[\mu_\tau(0)] \subseteq \mu_\tau(0)$. If we set $y = (T_\alpha)^{m-1}z$, then $y \in E_\alpha \cap \mathrm{fin}_\tau({}^*l(p_i))$ by Propositions 3.3 and 3.5 since $T_\alpha[E_\alpha] \subset E_\alpha$ and $(T_\alpha)^{m-1}[\mu_\tau(0)] \subseteq \mu_\tau(0)$ (see the proof of Proposition 3.4). Thus,

$$ {}^*T\Big((T_\alpha)^{m-1}z\Big) - (T_\alpha)^m z = {}^*T(y) - T_\alpha y \in \mu_\tau(0) \tag{3.11} $$

(see the first part of the present proof), which implies

$$ \begin{aligned} {}^*(T^m)z - (T_\alpha)^m z = {} & \Big[{}^*(T^m)z - {}^*T\Big((T_\alpha)^{m-1}z\Big)\Big] \\ & + \Big[{}^*T\Big((T_\alpha)^{m-1}z\Big) - (T_\alpha)^m z\Big] \in \mu_\tau(0). \end{aligned} \tag{3.12} $$

Therefore, by induction, it follows that ${}^*(T^m)z - (T_\alpha)^m z \in \mu_\tau(0)$ for $z \in E_\alpha \cap \mathrm{fin}_\tau({}^*l(p_i))$ and $m \in \mathbb{N}$. □

Proposition 3.7. *If $z \in E_\alpha \cap \mathrm{fin}_\tau({}^*l(p_i))$, then ${}^*q(T_\alpha)z \in \mathrm{ns}_\tau({}^*l(p_i))$ (i.e., ${}^*q(T_\alpha)z$ is $\tau(p_i)$-near standard).*

Proof. Let $z \in E_\alpha \cap \mathrm{fin}_\tau({}^*l(p_i))$. There exists $n \in \mathbb{N}$ such that $z \in {}^*(nV_0)$ (see [2, Corollary I.15]). Since $q(T)$ is compact, it follows that

$$ {}^*(q(T))[{}^*(nV_0)] = n{}^*(q(T))[{}^*V_0] \subset n[\mathrm{ns}_\tau({}^*l(p_i))] \subset \mathrm{ns}_\tau({}^*l(p_i)) \tag{3.13} $$

(see [2, Proposition I.1 and Corollary I.10]). Therefore,

$$ \begin{aligned} {}^*q(T_\alpha)z - {}^\circ[{}^*(q(T))z] = {} & [{}^*q(T_\alpha)z - {}^*(q(T))z] \\ & + [{}^*(q(T))z - {}^\circ[{}^*(q(T))z]] \in \mu_\tau(0) \end{aligned} \tag{3.14} $$

since $[{}^*q(T_\alpha)z - {}^*(q(T))z] \in \mu_\tau(0)$ by Proposition 3.6. Therefore, ${}^*q(T_\alpha)z \in \mathrm{ns}_\tau({}^*l(p_i))$. □

We now state and prove the main result.

Theorem 3.8. *Let $0 < p \le p_i \le 1$ and let $T \in [l(p_i)]$ be almost superdiagonal. If $q(\lambda)$ is a polynomial over $\mathbb{C}$ such that $q(T)$ is compact, then T has at least one nontrivial $\tau(p_i)$-closed invariant linear subspace of $l(p_i)$.*

Proof. Let $[a_{ij}]$ be the matrix of T with respect to $\{e_i\}$. Therefore, there exists $\alpha \in {}^*\mathbb{N}_+ - \mathbb{N}_+$ such that $a_{\alpha+1,\alpha} \in \mu(0)$ for $a_{\alpha+1,\alpha} \in {}^*[a_{ij}]$ by Proposition 3.2. Let $\nabla : \mathcal{FS}(l(p_i)) \to \mathcal{F}[l(p_i)]$ satisfy the conditions of Proposition 2.2 and let the collection $[\{F_\iota\}_{\iota=0}^{\alpha} : \{{}^*\nabla(F_\iota)\}_{\iota=0}^{\alpha}]$ satisfy the conclusion of Proposition 2.3 with respect to T, ∇, and α. From Proposition 2.2(2) (and the *transfer principle*), we infer the existence of a nonzero $x_0 \in l(p_i)$ such that ${}^*\nabla(F)({}^*x_0) \in {}^*V_0$ for each $F \in {}^*\mathcal{FS}(l(p_i))$, which implies ${}^*\nabla(F)({}^*x_0) \in F \cap \mathrm{fin}_\tau({}^*l(p_i))$ for each

$F \in {}^*\mathscr{F}\mathscr{S}(l(p_i))$ (see expression (3.2)). Consequently,

$$ {}^*q(T_\alpha)({}^*\nabla(F_\iota)({}^*x_0)) \in F_\iota \cap \mathrm{ns}_\tau({}^*l(p_i)) \quad \text{for } \iota \in \{0,\dots,\alpha\} \tag{3.15} $$

by Proposition 3.7 since, for each $\iota \in \{0,\dots,\alpha\}$, ${}^*\nabla(F_\iota)({}^*x_0) \in F_\iota \subset E_\alpha$, by definitions of ∇, $\{F_\iota\}_{\iota=0}^{\alpha}$ (and the *transfer principle*), and ${}^*q(T_\alpha)[F_\iota] \subset F_\iota$ (see expression (3.7)).

If $\{x_0, Tx_0, \dots, T^m x_0\}$ is linearly *dependent* for some $m \in \mathbb{N}_+$, then the linear space generated by $\{x_0, Tx_0, \dots, T^{m-1}x_0\}$ is nontrivial, closed, and invariant under T.

For the remainder of the proof, we will assume that $\{x_0, Tx_0, \dots, T^m x_0\}$ is linearly *independent* for each $m \in \mathbb{N}_+$. Consequently,

$$ q(T)(x_0) \neq 0. \tag{3.16} $$

Since the (internal) projection P_α satisfies Proposition 2.1, we have ${}^*x_0 - P_\alpha({}^*x_0) \in \mu_\tau(0)$, which implies $[{}^*q(T_\alpha)({}^*x_0) - {}^*q(T_\alpha)(P_\alpha({}^*x_0))] \in \mu_\tau(0)$ by Proposition 3.4 and $[{}^*(q(T))(P_\alpha({}^*x_0)) - {}^*(q(T)(x_0))] \in \mu_\tau(0)$ because $q(T) \in [l(p_i)]$. Also, $P_\alpha({}^*x_0) \in E_\alpha \cap \mathrm{fin}_\tau({}^*l(p_i))$ (see expression (1.4)) implies

$$ [{}^*q(T_\alpha)(P_\alpha({}^*x_0)) - {}^*(q(T))(P_\alpha({}^*x_0))] \in \mu_\tau(0) \tag{3.17} $$

by Proposition 3.6. Therefore,

$$ \begin{aligned} &{}^*q(T_\alpha)({}^*x_0) - {}^*(q(T)(x_0)) \\ &\quad = [{}^*q(T_\alpha)({}^*x_0) - {}^*q(T_\alpha)(P_\alpha({}^*x_0))] \\ &\qquad + [{}^*q(T_\alpha)(P_\alpha({}^*x_0)) - {}^*(q(T))(P_\alpha({}^*x_0))] \\ &\qquad + [{}^*(q(T))(P_\alpha({}^*x_0)) - {}^*(q(T)(x_0))], \end{aligned} \tag{3.18} $$

which implies $[{}^*q(T_\alpha)({}^*x_0) - {}^*(q(T)(x_0))] \in \mu_\tau(0)$. So, $q(T)(x_0) \neq 0$ implies

$$ {}^*q(T_\alpha)({}^*x_0) \notin \mu_\tau(0) \tag{3.19} $$

since $\tau = \tau(p_i)$ is Hausdorff. Therefore, by Propositions 2.4 and 3.4, expressions (3.15) and (3.19), there exists $\nu \in \{1,\dots,\alpha\}$ such that ${}^\circ F_{\nu-1} \neq l(p_i)$ and ${}^\circ F_\nu \neq \{0\}$. Since any two points of ${}^\circ F_\nu$ are linearly dependent modulo ${}^\circ F_{\nu-1}$, we have that either ${}^\circ F_{\nu-1}$ or ${}^\circ F_\nu$ is a closed nontrivial linear subspace of T. □

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Arthur D. Grainger: Department of Mathematics, Morgan State University, 1700 East Cold Spring Lane, Baltimore, MD 21239, USA
E-mail address: agrainge@morgan.edu