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MIRROR SYMMETRY FOR CONCAVEX VECTOR BUNDLES ON PROJECTIVE SPACES

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Let $X \subset Y$ be smooth, projective manifolds. Assume that $\iota : X \hookrightarrow \mathbb{P}^s$ is the zero locus of a generic section of $V^+ = \oplus_{i\in I}\mathbb{O}(k_i)$, where all the k_i's are positive. Assume furthermore that $\mathcal{N}_{X/Y} = \iota^*(V^-)$, where $V^- = \oplus_{j\in J}\mathbb{O}(-l_j)$ and all the l_j's are negative. We show that under appropriate restrictions, the generalized Gromov-Witten invariants of X inherited from Y can be calculated via a modified Gromov-Witten theory on $\mathbb{P}^s$. This leads to local mirror symmetry on the A-side.

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1. Introduction. Let $V^+ = \oplus_{i\in I}\mathbb{O}(k_i)$ and $V^- = \oplus_{j\in J}\mathbb{O}(-l_j)$ be vector bundles on $\mathbb{P}^s$ with k_i and l_j positive integers. Suppose that $X \overset{\iota}{\hookrightarrow} \mathbb{P}^s$ is the zero locus of a generic section of V^+ and Y is a projective manifold such that $X \overset{j}{\hookrightarrow} Y$ with normal bundle $\mathcal{N}_{X/Y} = \iota^*(V^-)$. The relations between Gromov-Witten theories of X and Y are studied here by means of a suitably defined equivariant Gromov-Witten theory in $\mathbb{P}^s$. We apply mirror symmetry to the latter to evaluate the gravitational descendants of Y supported in X.

Section 2 is a collection of definitions and techniques that will be used throughout this paper. In Section 3, using an idea from Kontsevich, we introduce a modified equivariant Gromov-Witten theory in $\mathbb{P}^s$ corresponding to $V = V^+ \oplus V^-$. The corresponding $\mathfrak{D}$-module structure [4, 11, 22] is computed in Section 4. It is generated by a single function $\tilde{J}_V$. In general, the equivariant quantum product does not have a nonequivariant limit. It is shown in Lemma 4.3 that the generator $\tilde{J}_V$ does have a limit J_V which takes values in $H^*\mathbb{P}^m[[q,t]]$. It is this limit that plays a crucial role in this work.

Let Y be a smooth, projective manifold. The generator J_Y of the pure $\mathfrak{D}$-module structure of Y encodes one-pointed gravitational descendants of Y. It takes values in the completion of H^*Y along the semigroup (Mori cone) of the rational curves of Y. The pullback map $j^* : H^*Y \to H^*X$ extends to a map between the respective completions. In Theorem 4.7, we describe one aspect of the relation between pure Gromov-Witten theory of $X \overset{j}{\hookrightarrow} Y$ and the modified Gromov-Witten theory of $\mathbb{P}^s$. Under natural restrictions, the pullback $j^*(J_Y)$ pushes forward to J_V. It follows that although defined on $\mathbb{P}^s$, J_V encodes the gravitational descendants of Y supported in X, hence the contribution of X to the Gromov-Witten invariants of Y.

The only way that X remembers the ambient variety Y in this context is by the normal bundle, Y can therefore be substituted by a local manifold. This suggests that there should be a local version of mirror symmetry (see the remark at the end of Section 4). This was first realized by Katz et al. [15]. The principle of local mirror symmetry in general has yet to be understood. Some interesting calculations that contribute toward this goal can be found in [6].

In Section 5, we give a proof of the mirror theorem which allows us to compute J_V. A hypergeometric series I_V that corresponds to the total space of V is defined. The mirror Theorem 5.1 states that $I_V = J_V$ up to a change of variables. Hence, the gravitational descendants of Y supported on X can be computed in $\mathbb{P}^s$.

Two examples of local Calabi-Yau threefolds are considered in Section 6. For $X = \mathbb{P}^1$ and $V = \mathbb{O}(-1) \oplus \mathbb{O}(-1)$, we obtain the Aspinwall-Morrison formula for multiple covers. If $X = \mathbb{P}^2$ and $V = \mathbb{O}(-3)$, the quantum product of Y pulls back to the modified quantum product in $\mathbb{P}^2$. The mirror theorem in this case yields the virtual number of plane curves on a Calabi-Yau threefold.

The rich history of mirror symmetry started in 1990 with a surprising conjecture by Candelas et al. [5] that predicts the number n_d of degree d rational curves on a quintic threefold. In [11], Givental presented a clever argument which, as shown later by Bini et al. in [4] and Pandharipande in [22], yields a proof of the mirror conjecture for Fano and Calabi-Yau (convex) complete intersections in projective spaces. Meanwhile, in a very well-written paper [20], Lian et al. used a different approach to obtain a complete proof of mirror theorem for concavex complete intersections on projective spaces. An alternative proof of the convex mirror theorem has been given by Bertram [3]. In this paper, we use Givental's approach to study the local nature of mirror symmetry and to present a proof of the concavex mirror theorem.

2. Stable maps and localization

2.1. Genus zero stable maps. Let $\overline{M}_{0,n}(X,\beta)$ be the Deligne-Mumford moduli stack of pointed stable maps to X. For an excellent reference on the construction and its properties, we refer the reader to [10]. We recall some of the features on $\overline{M}_{0,n}(X,\beta)$ and establish some notation. For each marking point x_i, let $e_i : \overline{M}_{0,n}(X,\beta) \to X$ be the evaluation map at x_i, and $\mathcal{L}_i$ the cotangent line bundle at x_i. The fiber of this line bundle over a moduli point $(C, x_1, \dots, x_n, f)$ is the cotangent space of the curve C at x_i. Let $\pi_k : \overline{M}_{0,n}(X,\beta) \to \overline{M}_{0,n-1}(X,\beta)$ be the morphism that forgets the kth marked point. The obstruction theory of the moduli stack $\overline{M}_{0,n}(X,\beta)$ is described locally by the following exact sequence:

$$\begin{aligned} 0 \longrightarrow \operatorname{Ext}^0\left(\Omega_C\left(\sum_{i=1}^n x_i\right),\mathcal{O}_C\right) &\longrightarrow H^0(C,f^*TX) \longrightarrow \mathcal{T}_M \\ \longrightarrow \operatorname{Ext}^1\left(\Omega_C\left(\sum_{i=1}^n x_i\right),\mathcal{O}_C\right) &\longrightarrow H^1(C,f^*TX) \longrightarrow \Upsilon \longrightarrow 0. \end{aligned} \tag{2.1}$$

(Here and thereafter, we are naming sheaves after their fibres.) To understand the geometry behind this exact sequence, we note that $\mathcal{T}_M = \operatorname{Ext}^1(f^*\Omega_X \to \Omega_C,\mathcal{O}_C)$ and $\Upsilon = \operatorname{Ext}^2(f^*\Omega_X \to \Omega_C,\mathcal{O}_C)$ are, respectively, the tangent space and the obstruction space at the moduli point $(C,x_1,\dots,x_n,f)$. The spaces $\operatorname{Ext}^0(\Omega_C(\sum_{i=1}^n x_i),\mathcal{O}_C)$ and $\operatorname{Ext}^1(\Omega_C(\sum_{i=1}^n x_i),\mathcal{O}_C)$ describe, respectively, the infinitesimal automorphisms and infinitesimal deformations of the marked source curve. It follows that the expected dimension of $\overline{M}_{0,n}(X,\beta)$ is $-K_X\cdot\beta+\dim X+n-3$.

A smooth projective manifold X is called *convex* if $H^1(\mathbb{P}^1,f^*TX)=0$ for any morphism $f:\mathbb{P}^1\to X$. For a convex X, the obstruction bundle Υ vanishes and the moduli stack is unobstructed and of the expected dimension. Examples of convex varieties are homogeneous spaces G/P.

In general, this moduli stack may behave badly and have components of larger dimensions. In this case, a Chow homology class of the expected dimension has been constructed [2, 18]. It is called the virtual fundamental class and denoted by $[\overline{M}_{0,n}(X,\beta)]^{\text{virt}}$. Although its construction is quite involved, we mainly use two relatively easy properties. The virtual fundamental class is preserved when pulled back by the forgetful map π_n. A proof of this fact can be found in [7, Section 7.1.5]. If the obstruction sheaf Υ is free, the virtual fundamental class refines the top Chern class of Υ. This fact is proven in [2, Proposition 5.6].

2.2. Equivariant cohomology and localization theorem. The notion of equivariant cohomology and the localization theorem is valid for any compact connected Lie group. For a detailed exposition on this subject, we suggest [7, Chapter 9]. Below, we state without proof the results that are used in this work.

The complex torus $T=(\mathbb{C}^*)^{s+1}$ is classified by the principal T-bundle

$$ET=\left(\mathbb{C}^{\infty+1}-\{0\}\right)^{s+1}\longrightarrow BT=\left(\mathbb{CP}^\infty\right)^{s+1}. \tag{2.2}$$

Let $\lambda_i=c_1(\pi_i^*(\mathcal{O}(1)))$ and $\lambda:=(\lambda_0,\dots,\lambda_s)$. We use $\mathcal{O}(\lambda_i)$ for the line bundle $\pi_i^*(\mathcal{O}(1))$. Clearly, $H^*(BT)=\mathbb{C}[\lambda]$. If T acts on a variety X, we let $X_T:=X\times_T ET$.

Definition 2.1. The equivariant cohomology of X is

$$H_T^*(X):=H^*(X_T). \tag{2.3}$$

If $X = x$ is a point, then $X_T = BT$ and $H_T^*(x) = \mathbb{C}[\lambda]$. For an arbitrary X, the equivariant cohomology $H_T^*(X)$ is a $\mathbb{C}[\lambda]$-module via the equivariant morphism $X \to x$.

Let $\mathcal{U}$ be a vector bundle over X. If the action of T on X can be lifted to an action on $\mathcal{U}$, which is linear on the fibers, $\mathcal{U}$ is an equivariant vector bundle and $\mathcal{U}_T$ is a vector bundle over X_T. The equivariant Chern classes of E are $c_k^T(\mathcal{U}) := c_k(\mathcal{U}_T)$. We use $E(\mathcal{U})$ ($E_T(\mathcal{U})$) to denote the nonequivariant (equivariant) top Chern class of $\mathcal{U}$.

Let $X^T = \cup_{j\in J} X_j$ be the decomposition of the fixed point locus into its connected components. The components X_j are smooth for all j and X_j is smooth for all j and the normal bundle N_j of X_j in X is equivariant. Let $i_j : X_j \to X$ be the inclusion. The following form of the localization theorem will be used extensively here.

Theorem 2.2. *Let $\alpha \in H_T^*(X) \otimes \mathbb{C}(\lambda)$. Then,*

$$\int_{X_T} \alpha = \sum_{j\in J} \int_{(X_j)_T} \frac{i_j^*(\alpha)}{E_T(N_j)}. \tag{2.4}$$

A basis for the characters of the torus is given by $\varepsilon_i(t_0,\dots,t_s) = t_i$. There is an isomorphism between the character group of the torus and $H^2(BT)$ sending ε_i to λ_i. We say that *the weight* of the character ε_i is λ_i.

For an equivariant vector bundle $\mathcal{U}$ over X, it may happen that the restriction of $\mathcal{U}$ on a fixed-point component X_j is trivial (e.g., if X_j is an isolated point). In that case, $\mathcal{U}$ decomposes as a direct sum $\oplus_{i=1}^m \mu_i$ of characters of the torus. If the weight of μ_i is ρ_i, then the restriction of $c_k^T(\mathcal{U})$ on X_j is the symmetric polynomial $\sigma_k(\rho_1,\dots,\rho_m)$.

Our interest here is for $X = \mathbb{P}^s$. For any action of T on $\mathbb{P}^s$, we denote

$$\mathcal{P} := H_T^*\mathbb{P}^s, \qquad \mathcal{R} =: \mathcal{P} \otimes \mathbb{C}(\lambda). \tag{2.5}$$

Consider the diagonal action of $T = (\mathbb{C}^*)^{s+1}$ on $\mathbb{P}^s$ with weights $(-\lambda_0,\dots,-\lambda_s)$, that is,

$$(t_0,t_1,\dots,t_s)\cdot(z_0,z_1,\dots,z_s) = (t_0^{-1}z_0,\dots,t_s^{-1}z_s). \tag{2.6}$$

Then, $\mathbb{P}_T^s = \mathbb{P}(\oplus_i \mathbb{O}(-\lambda_i))$. There is an obvious lifting of the action of T on the tautological line bundle $\mathbb{O}(-1)$. It follows that $\mathbb{O}(k)$ is equivariant for all k. Let $p = c_1^T(\mathbb{O}_{\mathbb{P}^s}(1))$ be the equivariant hyperplane class. We obtain $\mathcal{P} = \mathbb{C}[\lambda,p]/\prod_i(p-\lambda_i)$ and $\mathcal{R} = \mathbb{C}(\lambda)[p]/\prod_i(p-\lambda_i)$. The locus of the fixed points consists of points p_j for $j = 0,1,\dots,s$, where p_j is the point whose jth coordinate is 1 and all the other ones are 0. On the level of the cohomology, the map i_j^* sends p to λ_j. A basis for $\mathcal{R}$ as a $\mathbb{C}(\lambda)$-vector space is given by

$\phi_j = \prod_{k\neq j}(p-\lambda_k)$ for $j = 0,1,\dots,s$. Also, $i_j^*(\phi_j) = \prod_{k\neq j}(\lambda_j - \lambda_k) = \mathrm{Euler}_T(N_j)$. The localization Theorem 2.2 says that for any polynomial $F(p) \in \mathbb{C}(\lambda)[p]/\prod_{i=1}^s(p-\lambda_i)$

$$\int_{\mathbb{P}^s_T} F(p) = \sum_j \frac{F(\lambda_j)}{\prod_{k\neq j}(\lambda_j - \lambda_k)}. \tag{2.7}$$

Translating the target of a stable map, we get an action of T on $\overline{M}_{0,n}(\mathbb{P}^s,d)$. In [17], Kontsevich identified the fixed-point components of this action in terms of decorated graphs. If $f:(C,x_1,\dots,x_n)\to\mathbb{P}^s$ is a fixed stable map, then $f(C)$ is a fixed curve in $\mathbb{P}^s$. The marked points, collapsed components, and nodes are mapped to the fixed-points p_i of the T-action on $\mathbb{P}^s$. A noncontracted component must be mapped to a fixed line $\overline{p_ip_j}$ on $\mathbb{P}^s$. The only branch points are the two fixed points p_i and p_j and the restriction of the map f to this component is determined by its degree. The graph Γ corresponding to the fixed-point component containing such a map is constructed as follows. The vertices correspond to the connected components of $f^{-1}\{p_0,p_1,\dots,p_s\}$. The edges correspond to the noncontracted components of the map. The graph is decorated as follows. Edges are marked by the degree of the map on the corresponding component, and vertices are marked by the fixed point of $\mathbb{P}^s$ where the corresponding component is mapped to. To each vertex, we associate a leg for each marked point that belongs to the corresponding component. For a vertex v, let $n(v)$ be the number of legs or edges incident to that vertex. Also, for an edge e, let d_e be the degree of the stable map on the corresponding component. Let

$$\overline{\mathcal{M}}_\Gamma := \prod_v \overline{M}_{0,n(v)}. \tag{2.8}$$

There is a finite group of automorphisms G_Γ acting on $\overline{M}_\Gamma$ [7, 12]. The order of the automorphism group G_Γ is

$$a_\Gamma = \prod_e d_e \cdot |\mathrm{Aut}(\Gamma)|. \tag{2.9}$$

The fixed-point component corresponding to the decorated graph Γ is

$$\overline{M}_\Gamma = \overline{\mathcal{M}}_\Gamma / G. \tag{2.10}$$

Let $i_\Gamma : \overline{M}_\Gamma \hookrightarrow \overline{M}_{0,n}(\mathbb{P}^s,d)$ be the inclusion of the fixed-point component corresponding to Γ and N_Γ its normal bundle. This bundle is T-equivariant. Let α be an equivariant cohomology class in $H_T^*(\overline{M}_{0,n}(\mathbb{P}^s,d))$ and $\alpha_\Gamma := i_\Gamma^*(\alpha)$. Theorem 2.2 says that

$$\int_{\overline{M}_{0,n}(\mathbb{P}^s,d)_T} \alpha = \sum_\Gamma \int_{(\overline{M}_\Gamma)_T} \frac{\alpha_\Gamma}{a_\Gamma \mathrm{Euler}_T(N_\Gamma)}. \tag{2.11}$$

Explicit formulas for $\mathrm{Euler}_T(N_\Gamma)$ in terms of Chern classes of cotangent line bundles in $H_T^*(\overline{M}_\Gamma)$ have been found by Kontsevich in [17].

2.3. Linear and nonlinear sigma models for a projective space. Two compactifications of the space of degree d maps $\mathbb{P}^1 \to \mathbb{P}^s$ are very important in this paper. $M_d := \overline{M}_{0,0}(\mathbb{P}^s \times \mathbb{P}^1, (d,1))$ is called the degree-d *nonlinear sigma model* of $\mathbb{P}^s$, and $N_d := \mathbb{P}(H^0(\mathbb{P}^1, \mathbb{O}_{\mathbb{P}^1}(d))^{s+1})$ is called the degree-d *linear sigma model* of the projective space $\mathbb{P}^s$. An element in $H^0(\mathbb{P}^1, \mathbb{O}_{\mathbb{P}^1}(d))^{s+1}$ is an $(s+1)$-tuple of degree d homogeneous polynomials in two variables w_0 and w_1. As a vector space, $H^0(\mathbb{P}^1, \mathbb{O}_{\mathbb{P}^1}(d))^{s+1}$ is generated by the vectors $v_{ir} = (0, \dots, 0, w_0^r w_1^{d-r}, 0, \dots, 0)$ for $i = 0, 1, \dots, s$ and $r = 0, 1, \dots, d$. The only nonzero component of v_{ir} is the ith one.

The action of $T' := T \times \mathbb{C}^*$ in $\mathbb{P}^s \times \mathbb{P}^1$ with weights $(-\lambda_0, \dots, -\lambda_s)$ in the $\mathbb{P}^s$ factor and $(-\hbar, 0)$ in the $\mathbb{P}^1$ factor gives rise to an action of T' in M_d by translation of maps. T' also acts in N_d as follows. For $\bar{t} = (t_0, \dots, t_s) \in T$ and $t \in \mathbb{C}^*$,

$$(\bar{t}, t) \cdot [P_0(w_0, w_1), \dots, P_s(w_0, w_1)] = [t_0 P_0(t w_0, w_1), \dots, t_s P_s(t w_0, w_1)]. \tag{2.12}$$

There is a T'-equivariant morphism $\psi : M_d \mapsto N_d$. Here is a set-theoretical description of this map (for a proof that it is a morphism, see [11] or [19]). Let q_i for $i = 1, 2$ be the projection maps on $\mathbb{P}^s \times \mathbb{P}^1$. For a stable map $(C, f) \in M_d$, let C_0 be the unique component of C such that $q_2 \circ f : C_0 \to \mathbb{P}^1$ is an isomorphism. Let $C_1, \dots, C_n$ be the irreducible components of $C - C_0$ and d_i the degree of the restriction of $q_1 \circ f$ on C_i. Choose coordinates on $C_0 \cong \mathbb{P}^1$ such that $q_2 \circ f(y_0, y_1) = (y_1, y_0)$. Let $C_0 \cap C_i = (a_i, b_i)$ and $q_1 \circ f = [f_0 : f_1 : \cdots : f_s] : C_0 \mapsto \mathbb{P}^s$. Then,

$$\psi(C, f) := \prod_{i=1}^{n} (b_i w_0 - a_i w_1)^{d_i} [f_0 : f_1 : \cdots : f_s]. \tag{2.13}$$

Let p_{ir} be the points of N_d corresponding to the vectors v_{ir}. The fixed-point loci of the T'-action on N_d consists of the points p_{ir}. We write κ for the equivariant hyperplane class of N_d. The restriction of κ at the fixed point p_{ir} is $\lambda_i + r\hbar$. The restriction of the equivariant Euler class of the tangent space TN_d at p_{ir} is [19]

$$E_{ir} = \prod_{(j,t) \neq (i,r)} (\lambda_i - \lambda_j + r\hbar - t\hbar). \tag{2.14}$$

Fixed-point components of M_d are obtained as follows. Let $\Gamma_{d_j}^i$ be the graph of a T-fixed point component in $\overline{M}_{0,1}(\mathbb{P}^s, d_j)$, where the marking is mapped to p_i and $d_1 + d_2 = d$. Let (d_1, d_2) be a partition of d. We identify $\overline{M}_{\Gamma_{d_1}^i} \times \overline{M}_{\Gamma_{d_2}^i}$ with a T'-fixed point component $M_{d_1 d_2}^i$ in M_d in the following manner. Let $(C_1, x_1, f_1) \in \overline{M}_{\Gamma_{d_1}^i}$ and $(C_2, x_2, f_2) \in \overline{M}_{\Gamma_{d_2}^i}$. Let C be the nodal curve obtained by gluing C_1 with $\mathbb{P}^1$ at x_1 and $0 \in \mathbb{P}^1$ and C_2 with $\mathbb{P}^1$ at x_2 and $\infty \in \mathbb{P}^1$. Let $f : C \to \mathbb{P}^s \times \mathbb{P}^1$ map C_1 to the slice $\mathbb{P}^s \times \infty$ by means of f_1 and C_2 to $\mathbb{P}^s \times 0$ by means of f_2. Finally, f maps $\mathbb{P}^1$ to $p_i \times \mathbb{P}^1$ by permuting coordinates and ψ maps $M^i_{d_1 d_2}$ to $p_{id_2} \in N_d$, hence the equivariant restriction of $\psi^*(\kappa)$ in $M^i_{d_1 d_2}$ is $\lambda_i + d_2\hbar$. The normal bundle $N_{\Gamma^i_{d_1 d_2}}$ of this component in the above identification can be found by splitting it in five pieces: smoothing the nodes x_1 and x_2 and deforming the restriction of the map to C_1, C_2, $\mathbb{P}^1$. Using Kontsevich's calculations, Givental obtained [11]

$$\frac{1}{E_T\left(N_{\Gamma^i_{d_1 d_2}}\right)} = \frac{1}{\prod_{k \neq i}(\lambda_i - \lambda_k)} \frac{1}{E_T\left(N_{\Gamma^i_{d_1}}\right)} \frac{1}{E_T\left(N_{\Gamma^i_{d_2}}\right)} \frac{e_1^*(\phi_i)}{-\hbar(-\hbar - c_1)} \frac{e_1^*(\phi_i)}{\hbar(\hbar - c_2)}, \tag{2.15}$$

where c_j, $j = 1,2$ is the first Chern class of the cotangent line bundle on $\overline{M}_{\Gamma^i_{d_j}}$.

3. A Gromov-Witten theory induced by a vector bundle

3.1. The obstruction class of a concavex vector bundle. The notion of concavex vector bundle is due to Lian et al. [19] and is central to this work.

Definition 3.1. (1) A line bundle $\mathcal{L}$ on X is called *convex* if $H^1(C, f^*(\mathcal{L})) = 0$ for any genus zero stable map $(C, x_1, \ldots, x_n, f)$.

(2) A line bundle $\mathcal{L}$ on X is called *concave* if $H^0(C, f^*(\mathcal{L})) = 0$ for any non-constant genus zero stable map $(C, x_1, \ldots, x_n, f)$.

(3) A direct sum of convex and concave line bundles on X is called a *concavex* vector bundle.

A concavex vector bundle V in a projective space $\mathbb{P}^s$ has the form

$$V = V^+ \oplus V^- = \left(\oplus_{i \in I} \mathcal{O}(k_i)\right) \oplus \left(\oplus_{j \in J} \mathcal{O}(-l_j)\right), \tag{3.1}$$

where k_i and l_j are positive numbers. Denote $\mathbb{E}^+ := E(V^+)$ and $\mathbb{E}^- := E(V^-)$.

Let $d > 0$. Consider the following diagram:

$$\begin{array}{ccc} \overline{M}_{0,n+1}(\mathbb{P}^s, d) & \xrightarrow{e_{n+1}} & \mathbb{P}^s \\ \Big\downarrow{\scriptstyle \pi_{n+1}} & & \\ \overline{M}_{0,n}(\mathbb{P}^s, d). & & \end{array} \tag{3.2}$$

Since V is concavex, the sheaf

$$V_d := V_d^+ \oplus V_d^- = \pi_{n+1\,*} e_{n+1}^*(V^+) \oplus R^1 \pi_{n+1\,*} e_{n+1}^*(V^-) \tag{3.3}$$

is locally free.

Definition 3.2. The obstruction class corresponding to V is defined to be

$$\mathbb{E}_d := E(V_d) = E(V_d^+)E(V_d^-) := \mathbb{E}_d^+\mathbb{E}_d^-. \tag{3.4}$$

For a T-action on $\mathbb{P}^s$ that lifts to a linear action on the fibers of $V = V^+ \oplus V^-$, let $E^+ := E_T(V^+)$ and $E^- := E_T(V^-)$. Assume that E^- is invertible.

Definition 3.3. The modified equivariant integral $\omega_V : \mathcal{R} \to \mathbb{C}(\lambda)$ corresponding to V is defined as follows:

$$\omega_V(\alpha) := \int_{\mathbb{P}_T^m} \alpha \cup \frac{E^+}{E^-}. \tag{3.5}$$

Consider the trivial action of $T = (\oplus_{i\in I} C^*) \oplus (\oplus_{j\in J} C^*)$ on $\mathbb{P}^s$. In this case, $\mathbb{P}_T^s = \mathbb{P}^s \times (\oplus_{i\in I}\mathbb{P}^\infty) \times (\oplus_{j\in J}\mathbb{P}^\infty)$ and $\overline{M}_{0,n}(\mathbb{P}^s,d)_T = \overline{M}_{0,n}(\mathbb{P}^s,d) \times (\oplus_{i\in I}\mathbb{P}^\infty) \times (\oplus_{j\in J}\mathbb{P}^\infty)$. It follows that $\mathcal{P} = H^*(\mathbb{P}^s,\mathbb{C}[\lambda])$ and $\mathcal{R} = H^*(\mathbb{P}^s,\mathbb{C}(\lambda))$. Let p denote the equivariant hyperplane class. The T-action lifts to a linear action on the fibers of V with weights $((-\lambda_i)_{i\in I},(-\lambda_j)_{j\in J})$. Let q_i and q_j denote the projection maps on $\overline{M}_{0,n}(\mathbb{P}^s,d)_T$. Both V_d^+ and V_d^- are T-equivariant bundles and

$$\begin{aligned}(V_d^+)_T &= V_d^+ \otimes \big(\oplus_{i\in I} q_i^*\mathcal{O}_{\mathbb{P}^\infty}(-\lambda_i)\big),\\ (V_d^-)_T &= V_d^- \otimes \big(\oplus_{j\in J} q_j^*\mathcal{O}_{\mathbb{P}^\infty}(-\lambda_j)\big).\end{aligned} \tag{3.6}$$

The equivariant obstruction class is

$$E_d := E_T(V_d) = E_T(V_d^+)E_T(V_d^-) = E_d^+E_d^-. \tag{3.7}$$

The modified equivariant integral for the trivial action of T on $\mathbb{P}^s$ gives rise to a modified perfect pairing in $\mathcal{R}$

$$\langle a,b\rangle_V := \omega_V(a\cup b). \tag{3.8}$$

Let $T_0 = 1$, $T_1 = p,\dots,T^s = p^s$ be a basis of $\mathcal{R}$ as a $\mathbb{C}(\lambda)$-vector space. The intersection matrix $(g_{rt}) := (\langle T_r,T_t\rangle_V)$ has an inverse (g^{rt}). Let $T^i = \sum_{j=0}^s g^{ij}T_j$ be the dual basis with respect to this pairing. Clearly,

$$T^i = T_{m-i}\cdot\left(\frac{E^-}{E^+}\right). \tag{3.9}$$

This implies that, in $H^*(\mathbb{P}^s\times\mathbb{P}^s)\otimes\mathbb{C}(\lambda)$, we have

$$\sum_{i=1}^s T_i\otimes T^i = \Delta\cdot\left(1\otimes\frac{E^-}{E^+}\right), \tag{3.10}$$

where $\Delta = \sum_{i=0}^s T_i\otimes T_{s-i}$ is the class of the diagonal in $\mathbb{P}^s\times\mathbb{P}^s$.

Recall that the morphism $\pi_k : \overline{M}_{0,n}(\mathbb{P}^s, d) \to \overline{M}_{0,n-1}(\mathbb{P}^s, d)$ forgets the kth marked point.

Lemma 3.4. *The forgetful morphisms satisfy* $\pi_k{}^*(E_d) = E_d$ *and* $\pi_k{}^*(\mathbb{E}_d) = \mathbb{E}_d$.

Proof. For simplicity, we consider the case $V = \mathcal{O}(k) \oplus \mathcal{O}(-l)$ and $k = n$. The general case is similar. Let $M_k = \overline{M}_{0,k}(\mathbb{P}^s, d)$ and $M_{n,n} = M_n \times_{M_{n-1}} M_n$. Consider the following equivariant commutative diagram:

$$\begin{gathered} M_{n+1}\, \pi_{n+1} \mu^{e_{n+1} \pi_n}, \\ M^{\beta}_{n,n} \alpha \mathbb{P}^s, \\ M_n \pi_n M_{n_{e_n}}^{\pi_n}, \\ M_{n-1}. \end{gathered} \tag{3.11}$$

We compute

$$\pi_{n+1*} e^*_{n+1} \mathcal{O}(k) = \pi_{n+1*} \pi^*_n e_n{}^* \mathcal{O}(k) = \beta_* \mu_* \mu^* \alpha^* e_n{}^* \mathcal{O}(k). \tag{3.12}$$

By the projection formula,

$$\mu_* \mu^* \alpha^* e_n{}^* \mathcal{O}(k) = \alpha^* e_n{}^* \mathcal{O}(k) \otimes \mu_*(\mathcal{O}_{M_{n+1}}). \tag{3.13}$$

Since the map μ is birational and M_{n+1} is normal $\mu_*(\mathcal{O}_{M_{n+1}}) = \mathcal{O}_{M_{n,n}}$, hence

$$\mu_* \mu^* \alpha^* e_n{}^* \mathcal{O}(k) = \alpha^* e_n{}^* \mathcal{O}(k). \tag{3.14}$$

Substituting into (3.12) and applying base extension properties (π_n is flat) yields

$$\pi_{n+1*} e^*_{n+1} \mathcal{O}(k) = \beta_* \alpha^* e_n{}^* \mathcal{O}(k) = \pi_n{}^* (\pi_{n*} e_n{}^* \mathcal{O}(k)). \tag{3.15}$$

For the case of a negative line bundle, we have

$$R^1 \pi_{n+1*} e^*_{n+1} \mathcal{O}(-l) = R^1 \pi_{n+1*} \pi_n{}^* e_n{}^* \mathcal{O}(-l) = R^1 \pi_{n+1*} \mu^* \alpha^* e_n{}^* \mathcal{O}(-l). \tag{3.16}$$

We now use the spectral sequence

$$R^p \beta_* (R^q \mu_* \mathcal{F}) \Longrightarrow R^{p+q} \pi_{n+1*} \mathcal{F}, \tag{3.17}$$

where $\mathcal{F}$ is a sheaf of $\mathcal{O}_{\mathcal{M}_{n+1}}$-modules. The map μ is birational. If we think of M_n as the universal map of M_{n-1}, then the map μ has nontrivial fibers only over pairs of stable maps in M_n that represent the same special point (i.e., node or marked point) of a stable map in M_{n-1}. These nontrivial fibers are isomorphic to $\mathbb{P}^1$. Since $\mathcal{F} = e^*_{n+1} \mathcal{O}(-l)$, we obtain $R^q \mu_* \mathcal{F} = 0$ for $q > 0$. It follows that this spectral sequence degenerates, giving

$$R^1 \pi_{n+1*} e^*_{n+1} \mathcal{O}(-l) = R^1 \beta_* \mu_* \mu^* \alpha^* e_n{}^* \mathcal{O}(-l). \tag{3.18}$$

Now, we proceed as in (3.14) to conclude

$$R^1\pi_{n+1*}e_{n+1}^*\mathbb{O}(-l) = \pi_n{}^*(R^1\pi_{n*}e_n{}^*\mathbb{O}(-l)). \tag{3.19}$$

The lemma is proven. □

Remark 3.5. The previous lemma justifies the omission of n from the notation of the obstruction class.

3.2. Modified equivariant correlators and quantum cohomology. Let $\gamma_i \in \mathcal{R}$ for $i = 1,\dots,n$ and $d > 0$. Introduce the following modified equivariant Gromov-Witten invariants:

$$\tilde{I}_d(\gamma_1,\dots,\gamma_n) := \int_{\overline{M}_{0,n}(\mathbb{P}^m,d)_T} e_1^*(\gamma_1)\cup\cdots\cup e_n^*(\gamma_n)\cup E_d \in \mathbb{C}(\lambda). \tag{3.20}$$

Now, $\overline{M}_{0,n}(\mathbb{P}^s,0) = \overline{M}_{0,n}\times\mathbb{P}^s$ and all the evaluation maps equal the projection q_2 to the second factor. The integrals in this case are defined as follows:

$$\tilde{I}_0(\gamma_1,\dots,\gamma_n) := \int_{\overline{M}_{0,n}(\mathbb{P}^s,0)} e_1^*(\gamma_1)\cup\cdots\cup e_n^*(\gamma_n)\cup q_2^*(E(V)) \in \mathbb{C}(\lambda). \tag{3.21}$$

The modified equivariant gravitational descendants are defined similarly to Gromov-Witten invariants

$$\begin{aligned}\tilde{I}_d&(\tau_{k_1}\gamma_1,\dots,\tau_{k_n}\gamma_n)\\ &:= \int_{\overline{M}_{0,n}(\mathbb{P}^s,d)_T} c_1^{k_1}(\mathcal{L}_1)\cup e_1^*(\gamma_1)\cup\cdots\cup c_1^{k_n}(\mathcal{L}_n)\cup e_n^*(\gamma_n)\cup E_d.\end{aligned} \tag{3.22}$$

Lemma 3.4 is essential in proving that the modified correlators satisfy the same properties, such as fundamental class property, divisor property, point mapping axiom, and so on, that the usual Gromov-Witten invariants do. The proofs are similar to the ones in pure Gromov-Witten theory. As an illustration, we prove one of these properties.

Fundamental class property. Let $\gamma_n = 1$ and $d \neq 0$. The forgetful morphism $\pi_n : \overline{M}_{0,n}(\mathbb{P}^s,d) \to \overline{M}_{0,n-1}(\mathbb{P}^s,d)$ is equivariant. Using Lemma 3.4, we obtain

$$\begin{aligned}e_1^*(\gamma_1)\cup\cdots\cup e_{n-1}^*(\gamma_{n-1})\cup e_n^*(1)\cup E_d&\\ = \pi^*(e_1^*(\gamma_1)\cup\cdots\cup e_{n-1}^*(\gamma_{n-1})\cup E_d)&.\end{aligned} \tag{3.23}$$

Therefore,

$$\begin{aligned}\tilde{I}_d(\gamma_1,\dots,\gamma_{n-1},1) &= \int_{\overline{M}_{0,n}(\mathbb{P}^s,d)} \pi^*(e_1^*(\gamma_1)\cup\cdots\cup e_{n-1}^*(\gamma_{n-1})\cup E_d)\\ &= \int_{\pi_{n*}(\overline{M}_{0,n}(\mathbb{P}^s,d))} e_1^*(\gamma_1)\cup\cdots\cup e_{n-1}^*(\gamma_{n-1})\cup E_d = 0.\end{aligned} \tag{3.24}$$

The last equality is because the fibers of π_n are positive dimensional. If $d = 0$, by the point mapping property we know that the integral is zero unless $n = 3$. In that case, $\tilde{I}_0(\gamma_1,\gamma_2,1) = \langle\gamma_1,\gamma_2\rangle$.

We will now prove a technical lemma that will be very useful later. Let $A\cup B$ be a partition of the set of markings and $d = d_1 + d_2$. Let $D = D(A,B,d_1,d_2)$ be the closure in $\overline{M}_{0,n}(\mathbb{P}^s,d)$ of stable maps of the following type. The source curve is a union $C = C_1 \cup C_2$ of two lines meeting at a node x. The marked points corresponding to A are on C_1, and those corresponding to B are on C_2. The restriction of the map f on C_i has degree d_i for $i = 1,2$. D is a boundary divisor in $\overline{M}_{0,n}(\mathbb{P}^s,d)$. Let $M_1 := \overline{M}_{0,|A|+1}(\mathbb{P}^s,d_1)$ and $M_2 := \overline{M}_{0,|B|+1}(\mathbb{P}^s,d_2)$. Let e_x and $\tilde{e_x}$ be the evaluation maps at the additional marking in M_1 and M_2 and $\mu := (e_x,\tilde{e_x})$. The boundary divisor D is obtained from the following fibre diagram:

$$\begin{array}{ccc} D & \xrightarrow{\iota} & M_1 \times M_2 \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \mu} \\ \mathbb{P}^s & \xrightarrow{\delta} & \mathbb{P}^s \times \mathbb{P}^s \end{array} \tag{3.25}$$

where ν is the "evaluation map at the node x" and δ is the diagonal map.

Lemma 3.6. *For any classes $\gamma_1,\dots,\gamma_n$ in $\mathcal{R}$,*

$$\int_D \prod_{i=1}^n e_i^*(\gamma_i)E_d = \sum_{a=0}^s \left(\int_{M_1}\prod_{i\in A} e_i^*(\gamma_i)e_x^*(T_a)E_{d_1}\right)\times\left(\int_{M_2}\prod_{j\in B} e_j^*(\gamma_j)\tilde{e}_x^*(T^a)E_{d_2}\right). \tag{3.26}$$

Proof. This lemma is the analogue of [10, Lemma 16]. The proof needs a minor modification. Let $\alpha : D \to \overline{M}_{0,n}(\mathbb{P}^s,d)$. Consider the normalization sequence at x

$$0 \longrightarrow \mathbb{O}_C \longrightarrow \mathbb{O}_{C'} \oplus \mathbb{O}_{C''} \longrightarrow \mathbb{O}_x \longrightarrow 0. \tag{3.27}$$

Twisting it by $f^*(V^+)$ and $f^*(V^-)$ and taking the cohomology sequence yield the following identities on D:

$$\alpha^*(E_d^+)\nu^*(E^+) = \iota^*(E_{d_1}^+ \times E_{d_2}^+), \tag{3.28}$$

$$\alpha^*(E_d^-) = \iota^*(E_{d_1}^- \times E_{d_2}^-)\nu^*(E^-). \tag{3.29}$$

Combining (3.28) and (3.29), we obtain the restriction of E_d in the divisor D

$$\alpha^*(E_d) = \iota^*(E_{d_1} \times E_{d_2})\nu^*\left(\frac{E^-}{E^+}\right). \tag{3.30}$$

Using formula (3.10), we obtain

$$\iota_* \nu^* \left(\frac{E^-}{E^+}\right) = \mu^* \left(1 \otimes \frac{E^-}{E^+}\right) \mu^*(\Delta). \tag{3.31}$$

Therefore,

$$\begin{aligned}
&\int_D \prod_{i=1}^n e_i^*(\gamma_i) \cup E_d \\
&\quad = \int_{M_1 \times M_2} \prod_{i=1}^n e_i^*(\gamma_i) \cup E_{d_1} \cup E_{d_2} \cup \mu^* \left(1 \otimes \frac{E^-}{E^+}\right) \cup \mu^*(\Delta) \\
&\quad = \int_{M_1 \times M_2} \prod_{i=1}^n e_i^*(\gamma_i) \cup E_{d_1} \cup E_{d_2} \cup \mu^* \left(\sum_a T_a \otimes T^a\right) \\
&\quad = \sum_{a=0}^m \left(\int_{M_1} \prod_{i=1}^{n_1} e_i^*(\gamma_i) \cup e_x^*(T_a) \cup E_{d_1}\right) \times \left(\int_{M_2} \prod_{j=1}^{n_2} e_j^*(\gamma_j) \cup \tilde{e}_x^*(T^a) \cup E_{d_2}\right).
\end{aligned} \tag{3.32}$$

The lemma is proven. □

The same proof can be used to show that the previous splitting lemma is true for gravitational descendants as well.

Corollary 3.7. *The following modified topological recursion relations hold:*

$$\begin{aligned}
&\tilde{I}_d \left(\tau_{k_1+1}\gamma_1, \tau_{k_2}\gamma_2, \tau_{k_3}\gamma_3, \prod_{i=4}^n \tau_{s_i}\omega_i \right) \\
&\quad = \sum \tilde{I}_{d_1} \left(\tau_{k_1}\gamma_1, \prod_{i \in I_1} \tau_{s_i}\omega_i, T_a \right) \tilde{I}_{d_2} \left(T^a, \tau_{k_2}\gamma_2, \tau_{k_3}\gamma_3, \prod_{i \in I_2} \tau_{s_i}\omega_i \right),
\end{aligned} \tag{3.33}$$

where the sum is over all splittings $d_1 + d_2 = d$ and partitions $I_1 \cup I_2 = \{4, \dots, n\}$ and over all indices a.

Proof. Let A and B be two disjoint subsets of $\{1, 2, \dots, n\}$. We denote by $D(A, B)$ the sum of boundary divisors $D(E, F, d_1, d_2)$ such that E, F is a partition of $\{1, 2, \dots, n\}$ and $A \subset E$, $B \subset F$, and $d_1 + d_2 = d$. The notation $D(A, B)$ reflects neither the number n of marked points nor the degree d of the maps, but they will be clear from the context. Consider the morphism $\pi : \overline{M}_{0,n}(\mathbb{P}^s, d) \to \overline{M}_{0,3}$ that forgets the map and all but the first 3 markings. Since $\overline{M}_{0,3}$ is a point, the cotangent line bundle at the first marking is trivial. But $\pi^*(\mathcal{L}_1) = \mathcal{L}_1 - D(\{1\}, \{2, 3\})$; therefore, $\mathcal{L}_1 = D(\{1\}, \{2, 3\})$ in $\overline{M}_{0,n}(\mathbb{P}^s, d)$. Multiply both sides of the previous equation by $\prod_{j=1}^3 c_1(\mathcal{L}_j)^{k_j} \cup e_j^*(\gamma_j) \cup \prod_{i=4}^n c_1(\mathcal{L}_i)^{s_i} \cup e_i^*(\omega_i) \cup E_d$ and integrate. The corollary follows from the splitting lemma for gravitational descendants. □

In the process of finding solutions to the WDVV equations, Kontsevich suggested the following modified equivariant Gromov-Witten potential:

$$\tilde{\Phi}(t_0,t_1,\dots,t_m) := \sum_{n\geq 3}\sum_{d\geq 0}\frac{1}{n!}\tilde{I}_d(\gamma^{\otimes n}), \tag{3.34}$$

where $\gamma = t_0 + t_1 p + \cdots + t_s p^s$ and $t_i \in \mathbb{C}(\lambda)$. Let $\tilde{\Phi}_{ijk} = \partial^3\tilde{\Phi}/\partial t_i \partial t_j \partial t_k$.

Definition 3.8. The modified, equivariant quantum product on $\mathcal{R}$ is defined to be the linear extension of

$$T_i *_V T_j := \sum_{k=0}^{m}\tilde{\Phi}_{ijk}T^k. \tag{3.35}$$

Theorem 3.9. *The algebra $QH_V^*\mathbb{P}_T^s := (\mathcal{R}, *_V)$ is a commutative, associative algebra with unit T_0.*

Proof. A simple calculation shows that

$$\tilde{\Phi}_{ijk} = \sum_{n\geq 0}\sum_{d\geq 0}\frac{1}{n!}\tilde{I}_d(T_i,T_j,T_k,\gamma^{\otimes n}). \tag{3.36}$$

The commutativity of the modified, equivariant quantum product follows from the symmetry of the new integrals. T_0 is the unit due to the fundamental class property for the modified Gromov-Witten invariants. To prove the associativity, we proceed as in [9, Theorem 4]. Let $\tilde{\Phi}_{ijk} = \partial^3\tilde{\Phi}/\partial t_i\partial t_j\partial t_k$. We compute

$$\begin{aligned}(T_i *_V T_j) *_V T_k &= \sum\sum \tilde{\Phi}_{ije}g^{ef}\tilde{\Phi}_{fkl}g^{ld}T_d,\\ T_i *_V (T_j *_V T_k) &= \sum\sum \tilde{\Phi}_{jke}g^{ef}\tilde{\Phi}_{fil}g^{ld}T_d.\end{aligned} \tag{3.37}$$

Since the matrix (g^{ld}) is nonsingular, $(T_i *_V T_j) *_V T_k = T_i *_V (T_j *_V T_k)$ is equivalent to

$$\sum_{e,f}\tilde{\Phi}_{ije}g^{ef}\tilde{\Phi}_{fkl} = \sum_{e,f}\tilde{\Phi}_{jke}g^{ef}\tilde{\Phi}_{fil}. \tag{3.38}$$

Equation (3.38) is the WDVV equation for the modified potential $\tilde{\Phi}$. To prove this equation, let q, r, s, t be four different integers in $\{1,2,\dots,n\}$. There exists an equivariant morphism

$$\pi : \overline{M}_{0,n}(\mathbb{P}^s, d) \longrightarrow \overline{M}_{0,4} = \mathbb{P}^1 \tag{3.39}$$

that forgets the map and all the marked points but q, r, s, t. Obviously, the divisors $D(\{q,r\},\{s,t\})$ and $D(\{q,s\},\{r,t\})$ are linearly equivalent in $\overline{M}_{0,4}$, hence, via the pullback π^*, they are linearly equivalent in $\overline{M}_{0,n}(\mathbb{P}^s,d)$. Now, integrate the class

$$\prod_{i=1}^{n-4}(e_i^*(\gamma))\cup e_{n-3}^*(T_i)\cup e_{n-2}^*(T_j)\cup e_{n-1}^*(T_k)\cup e_n^*(T_l)\cup E_d \tag{3.40}$$

over $D(\{q,r\},\{s,t\})$, and use Lemma 3.6 to obtain WDVV equation; hence, the associativity □

If we restrict $\tilde{\Phi}_{ijk}$ to the divisor classes $\gamma = tp$ and use the divisor property for the modified Gromov-Witten invariants, we obtain the *small product*

$$T_i *_V T_j := T_i \cup T_j + \sum_{d>0} q^d \sum_{k=0}^{m} \tilde{I}_d(T_i,T_j,T_k)T^k. \tag{3.41}$$

Here, $q = e^t$. We extend this product to $\mathcal{R}\otimes_{\mathbb{C}}\mathbb{C}[[q]]$ to obtain the small equivariant quantum cohomology ring $SQH_V^*\mathbb{P}^s{}_T$. We use $*_V$ to denote both the small and the big quantum product. The difference is clear from the context.

Remark 3.10. (i) Equation (3.30) and Lemma 3.4 are the basis for building a modified equivariant Gromow-Witten theory similar to pure Gromov-Witten theory.

(ii) We can see from (3.9) that the only potential problem with the existence of the nonequivariant limit of (3.41) is the presence of E^+ in the denominator of T^k. Hence, if $V = V^-$ is a pure negative line bundle, the nonequivariant limit of this product exists. An example of this situation is treated in the last section.

4. A $\mathcal{D}$-module structure induced by V

4.1. Equivariant quantum differential equations. Recall from Section 2.3 the generator $\hbar$ of $H^2(BC^*)$. Consider the system of first-order differential equations on the modified, big quantum cohomology ring $QH_V^*(\mathbb{P}_T^s)$

$$\hbar\frac{\partial}{\partial t_i} = T_i *_V, \quad i = 1,\dots,m. \tag{4.1}$$

Theorem 4.1. *The space of solutions of these equations has the following basis:*

$$\begin{aligned} s_a &= T_a + \sum_{j=0}^{s}\sum_{n=0}^{\infty}\sum_{d=0}^{\infty}\sum_{k=0}^{\infty}\frac{\hbar^{-(k+1)}}{n!}\tilde{I}_d(\tau_k T_a, T_j, \gamma^{\otimes n})T^j \\ &= T_a + \sum_{j=0}^{m}\sum_{d=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{n!}\tilde{I}_d\left(\frac{T_a}{\hbar - c}, T_j, \gamma^{\otimes n}\right)T^j, \end{aligned} \tag{4.2}$$

where c is a formal symbol that stands for $c_1{}^T(\mathcal{L}_1)$ and $T_a/(\hbar - c)$ should be expanded in powers of $c/\hbar$.

Proof. On the one hand,

$$\hbar \frac{\partial s_a}{\partial t_i} = \sum_{j=0}^{m} \sum_{n=0}^{\infty} \sum_{d=0}^{\infty} \sum_{k=0}^{\infty} \frac{\hbar^{-k}}{n!} \tilde{I}_d(\tau_k T_a, T_j, T_i, \gamma^{\otimes n}) T^j. \tag{4.3}$$

On the other hand,

$$\begin{aligned} T_i * s_a &= T_i * T_a + \sum_{j=0}^{m} \sum_{n=0}^{\infty} \sum_{d=0}^{\infty} \sum_{k=0}^{\infty} \frac{\hbar^{-(k+1)}}{n_1!} \tilde{I}_d(\tau_k T_a, T_j, \gamma^{\otimes n})(T_i * T^j) \\ &= \sum_{n,d,e} \frac{1}{n!} \tilde{I}_d(T_i, T_a, T_e, \gamma^{\otimes n}) T^e + \sum_{j=0}^{m} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{d_1} \frac{\hbar^{-(k+1)}}{n!} \tilde{I}_{d_1}(\tau_k T_a, T_j, \gamma^{\otimes n}) \\ &\qquad \times \sum_{m,d_2,e} \frac{1}{m!} \tilde{I}_{d_2}(T_i, T^j, T_e, \gamma^{\otimes m}) T^e. \end{aligned} \tag{4.4}$$

The theorem follows from the topological recursion relations (3.33). □

Restrictions $\tilde{s}_a$ of the sections s_a to $\gamma \in H^0(\mathbb{P}^m) \oplus H^2(\mathbb{P}^m)$ are solutions of

$$\hbar \frac{\partial}{\partial t_i} = T_i *_V : i = 0, 1. \tag{4.5}$$

Repeated use of the divisor axiom yields

$$\tilde{s}_a = e^{(t_0 + pt_1)/\hbar} \cup T_a + \sum_{d=1}^{\infty} \sum_{j=0}^{m} q^d \tilde{I}_d \left(\frac{e^{(t_0+pt_1)/\hbar} \cup T_a}{\hbar - c}, T_j \right) T^j, \tag{4.6}$$

where $q := e^{t_1}$.

Definition 4.2. The module of differential operators that annihilate $\langle \tilde{s}_a, 1 \rangle_V$ for all a is called the modified equivariant $\mathcal{D}$-module of $\mathbb{P}^s$ induced by V.

This module is generated by the following $\mathcal{R}[[t_0, t_1, q]]$-valued function

$$\tilde{J}_V = \sum_{a=0}^{s} \langle \tilde{s}_a, 1 \rangle_V T^a. \tag{4.7}$$

Recall that $e_1 : \overline{M}_{0,2}(\mathbb{P}^s, d) \to \mathbb{P}^s$ is the evaluation map at the first marked point and c is the Chern class of the cotangent line bundle at the first marked point.

Substituting (4.6) into (4.7) and using the projection formula, we obtain

$$\tilde{J}_V = \exp\left(\frac{t_0 + pt_1}{\hbar}\right) \cdot \left(1 + \sum_{d>0} q^d PD^{-1}\left(e_{1*}\left(\left(\frac{E_d}{\hbar - c} \cap [\overline{M}_{0,2}(\mathbb{P}^s, d)]\right)\right) \cup \left(\frac{E^-}{E^+}\right)\right)\right). \tag{4.8}$$

In the above expression, $PD : H^*(\overline{M}_{0,2}(\mathbb{P}^s, d)) \to H_{s+d+sd-1-*}(\overline{M}_{0,2}(\mathbb{P}^s, d))$ is the Poincaré duality isomorphism.

It is convenient for us to work with the moduli space of one pointed stable maps. To that end we note that

$$e_{1*}\left(\frac{E_d}{\hbar - c} \cap [\overline{M}_{0,2}(\mathbb{P}^s, d)]\right) = e_{1*}\left(\frac{E_d}{\hbar(\hbar - c)} \cap [\overline{M}_{0,1}(\mathbb{P}^s, d)]\right). \tag{4.9}$$

This identity follows easily from the fact that if $\pi_2 : \overline{M}_{0,2}(\mathbb{P}^s, d) \to \overline{M}_{0,1}(\mathbb{P}^s, d)$ forgets the second marked point and D is the image of the universal section of π induced by the marked point, then $c = \pi_2{}^*(c) + D$ and $\mathbb{E}_d = \pi_2{}^*(\mathbb{E}_d)$.

The final expression for $\tilde{J}_V$ is

$$\tilde{J}_V = \exp\left(\frac{t_0 + pt_1}{\hbar}\right) \cdot \left(1 + \sum_{d>0} q^d PD^{-1}\left(e_{1*}\left(\frac{E_d}{\hbar(\hbar - c)} \cap [\overline{M}_{0,1}(\mathbb{P}^s, d)]\right)\right) \cup \left(\frac{E^-}{E^+}\right)\right). \tag{4.10}$$

From this presentation, we see that the presence of the equivariant class E^+ in the denominator of $\tilde{J}_V$ is a potential problem for the existence of the nonequivariant limit.

Lemma 4.3. *The generator $\tilde{J}_V \in \mathcal{P}[[q]]$; therefore, it has a nonequivariant limit.*

Proof. Let V'_d be the subbundle of V^+_d whose fiber consists of those sections of $H^0(C, f^*(V^+))$ that vanish at the marked point. Let $E'_d := c^T_{\text{top}}(V'_d)$. There is an exact sequence of equivariant bundles on $\overline{M}_{0,1}(\mathbb{P}^s, d)$

$$0 \longrightarrow V'_d \longrightarrow V^+_d \longrightarrow e_1{}^*(V^+) \longrightarrow 0. \tag{4.11}$$

Taking the top Chern classes, we obtain

$$E^+_d = E'_d \cdot e_1^*(E^+). \tag{4.12}$$

We compute

$$\begin{aligned} PD^{-1}e_{1*}&\left(\frac{E_d}{\hbar(\hbar-c)}\cap[\overline{M}_{0,1}(\mathbb{P}^s,d)]\right)\\ &=PD^{-1}\left(e_{1*}\left(\frac{E'_de_1^*(E^+)E^-_d}{\hbar(\hbar-c)}\cap[\overline{M}_{0,1}(\mathbb{P}^s,d)]\right)\right)\\ &=PD^{-1}\left(E^+\cap e_{1*}\left(\frac{E'_dE^-_d}{\hbar(\hbar-c)}\cap[\overline{M}_{0,1}(\mathbb{P}^s,d)]\right)\right)\\ &=E^+\cup PD^{-1}\left(e_{1*}\left(\frac{E'_dE^-_d}{\hbar(\hbar-c)}\cap[\overline{M}_{0,1}(\mathbb{P}^s,d)]\right)\right). \end{aligned} \tag{4.13}$$

Therefore,

$$\tilde{J}_V=\exp\left(\frac{t_0+pt_1}{\hbar}\right)\cdot\left(1+\sum_{d>0}q^dPD^{-1}e_{1*}\left(\frac{E'_dE^-_d}{\hbar(\hbar-c)}\cap[\overline{M}_{0,1}(\mathbb{P}^s,d)]\right)\cup E^-\right). \tag{4.14}$$

It is now visible from this presentation that $\tilde{J}_V\in\mathcal{P}[[q]]$ and

$$\begin{aligned} J_V&:=\lim_{\lambda\to 0}\tilde{J}_V\\ &=\exp\left(\frac{t_0+Ht_1}{\hbar}\right)\left(1+\sum_{d>0}q^dPD^{-1}e_{1*}\left(\frac{\mathbb{E}'_d\mathbb{E}^-_d}{\hbar(\hbar-c)}\cap[\overline{M}_{0,1}(\mathbb{P}^s,d)]\right)\cup\mathbb{E}^-\right). \end{aligned} \tag{4.15}$$

The lemma is proven. □

4.2. A local property of the J-function. Let Y be a smooth projective variety and $j:\mathbb{P}^s\hookrightarrow Y$ an embedding. Suppose that $\mathcal{N}_{\mathbb{P}^s/Y}=V^-=\mathbb{O}(-l)$ for some $l>0$. Let C be a curve in $\mathbb{P}^s$. The map j gives rise to an embedding

$$\overline{M}_{0,n}(\mathbb{P}^s,[C])\hookrightarrow\overline{M}_{0,n}(Y,j_*([C])). \tag{4.16}$$

Lemma 4.4. *Let C be a degree-d rational curve in $\mathbb{P}^s$. Then, $\overline{M}_{0,n}(\mathbb{P}^s,d)=\overline{M}_{0,n}(Y,j_*([C]))$.*

Proof. Let $(C',x_1,\dots,x_n,f)\in\overline{M}_{0,n}(Y,j_*([C]))$ and $f(C')=C_1\cup C_2\cup\cdots\cup C_p$ be the irreducible decomposition. Then, $d[\text{line}]=[C_1]+\cdots[C_p]$. Let $I_1=\{i:C_i\subset\mathbb{P}^s\}$ and $I_2=\{1,2,\dots,n\}-I_1$. Assume that I_2 is nonempty. If

$$d[\text{line}]-\sum_{i\in I_1}[C_i] \tag{4.17}$$

has nonpositive degree in $\mathbb{P}^s$, we intersect with an ample divisor in Y to see that

$$d[\text{line}]-\sum_{i\in I_1}[C_i]=\sum_{i\in I_2}[C_i] \tag{4.18}$$

is impossible. Otherwise, we intersect with $[\mathbb{P}^s]$ to get the same contradiction. Hence, I_2 is empty and all the curves C_i lie in $\mathbb{P}^s$. It follows that f factors through $\mathbb{P}^s$, and, therefore, $(C',x_1,\dots,x_n,f)\in\overline{M}_{0,n}(\mathbb{P}^s,d)$. On the other hand, $\overline{M}_{0,n}(\mathbb{P}^s,d)$ is a component of $\overline{M}_{0,n}(Y,j_*([C]))$ (cf. [7, Section 7.4.4]). These two arguments imply the lemma. □

Denote $\overline{M}_{0,n}(Y,d):=\overline{M}_{0,n}(Y,j_*([C]))$, where C is any rational curve of degree d in $\mathbb{P}^s$. The following lemma is a special case of a conjecture by Cox et al. in [8] which was proved in [16].

Lemma 4.5. *The following identity holds* $[\overline{M}_{0,n}(Y,d)]^{\text{virt}}=\mathbb{E}_d\cap[\overline{M}_{0,n}(\mathbb{P}^s,d)]$.

At this point we introduce a new object. For any smooth projective variety Y and any ring $\mathcal{A}$, we define the formal completion of $\mathcal{A}$ along the semigroup of the Mori cone of Y to be

$$\mathcal{A}[[q^\beta]]:=\Big\{\sum_\beta a_\beta q^\beta,\ a_\beta\in\mathcal{A},\ \beta\text{—effective}\Big\},\tag{4.19}$$

where $\beta\in H_2(Y,\mathbb{Z})$ is effective if it is a positive linear combination of algebraic curves. This new ring behaves like a power series since, for each β, the set of α such that α and $\beta-\alpha$ are both effective is finite. For example, in the case of $\mathbb{P}^s$, we obtain the power series $\mathcal{A}[[q]]$.

Choose generators $D_1,\dots,D_r$ of $H^2(Y,\mathbb{Q})$ such that $j^*(D_1)=H$ and $j^*(D_i)=0$ for $i\ge 2$. Elements of $H^0(Y,\mathbb{Q})\oplus H^2(Y,\mathbb{Q})$ are of the form $t_0+tD:=t_0+t_1D_1+\cdots+t_rD_r$. It is shown in [11] that the generator of the quantum $\mathcal{D}$-module for the pure Gromov-Witten theory of Y is

$$J_Y=\exp\Big(\frac{t_0+tD}{\hbar}\Big)\sum_{\beta\in H_2(Y,\mathbb{Q})}q^\beta PD^{-1}\bigg(e_{1*}\bigg(\frac{[\overline{M}_{0,1}(Y,\beta)]^{\text{virt}}}{\hbar(\hbar-c)}\bigg)\bigg).\tag{4.20}$$

The moduli spaces $\overline{M}_{0,1}(Y,\beta)$ are empty unless β is effective. Hence, we consider J_Y as an element of the ring $H^*Y[[t_0,t_1,\dots,t_r]][[q^\beta]]$.

We extend the map $j^*:H^*Y\to H^*\mathbb{P}^s$ to a homomorphism

$$j^*:H^*Y[[t_0,t_1,\dots,t_r]][[q^\beta]]\longrightarrow H^*\mathbb{P}^s[[t_0,t_1]][[q]]\tag{4.21}$$

by defining $j^*(t_i)=0$ for $i>1$ and $j^*(q^\beta)=q^\beta$ for $\beta\in j_*(H_2(\mathbb{P}^s,\mathbb{Z}))$ and $j^*(q^\beta)=0$ for $\beta\in H_2(Y,\mathbb{Z})-j_*(H_2(\mathbb{P}^s,\mathbb{Z}))$. The following results show that J-function is local.

Theorem 4.6. *The generator j is local in the sense that $j^*(J_Y)=J_V$.*

Proof. Notice that

$$j^* J_Y = \exp\left(\frac{t_0 + t_1 H}{\hbar}\right) \sum_{d_1=0}^{\infty} q_1^{d_1} j^* PD^{-1}\left(e_{1*}\left(\frac{[\overline{M}_{0,1}(Y,d_1)]^{\mathrm{virt}}}{\hbar(\hbar - c)}\right)\right). \tag{4.22}$$

Consider the following fiber diagram:

$$\begin{array}{ccc} \overline{M}_{0,1}(\mathbb{P}^s, d_1) & \xrightarrow{=} & \overline{M}_{0,1}(Y,d_1) \\ \downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e_1} \\ \mathbb{P}^s & \xrightarrow{j} & Y. \end{array} \tag{4.23}$$

By excess intersection theory [9] and the previous lemma,

$$j^*\left(e_{1*}\left(\frac{[\overline{M}_{0,1}(Y,d_1)]^{\mathrm{virt}}}{\hbar(\hbar - c)}\right)\right) = \mathbb{E}^- \cap e_{1*}\left(\frac{\mathbb{E}_{d_1}}{\hbar(\hbar - c)} \cap [\overline{M}_{0,1}(\mathbb{P}^s, d_1)]\right). \tag{4.24}$$

The theorem follows easily. □

Theorem 4.7. *Let $V = V^+ \oplus V^- = \mathbb{O}(k) \oplus \mathbb{O}(-l)$ on $\mathbb{P}^s$. Let $\iota : X \hookrightarrow \mathbb{P}^s$ be the zero locus of a generic section of V^+. Assume that X is smooth and* $\dim X > 2$. *Let Y be a smooth projective variety such that $j : X \hookrightarrow Y$ with $\mathcal{N} = \mathcal{N}_{X/Y} = \iota^*(V^-)$. Assume that if $C \subset Y$ is a curve with $[C] \in MX$, then all the irreducible components C_i of C satisfy $C_i \subset X$. Let j^* be the map constructed as in (4.21). Let J_Y be the generator of the pure $\mathfrak{D}$-module of Y* [13]. *Then,*

$$\iota_!(j^*(J_Y)) = E(V^+) J_V, \tag{4.25}$$

where $\iota_!$ is the Gysin map on cohomology.

Proof. Since $\dim X > 2$, it follows that $H^2 X$ is generated by $\iota^*(H)$. Let β_1 be the Poincaré dual to $\iota^*(H)$, and let $D_1, D_2, \dots, D_r$ be a set of generators of $H^2(Y, \mathbb{Q})$. We may assume that $j^*(D_1) = \iota^*(H)$ and $j^*(D_i) = 0$ for $i > 1$. Let $tD := t_1 D_1 + \cdots + t_r D_r$. Now,

$$J_Y = \exp\left(\frac{t_0 + tD}{\hbar}\right) \sum_{\beta \in H^2(Y,\mathbb{Q})} q^\beta PD^{-1} e_{1*}\left(\frac{[\overline{M}_{0,1}(Y,\beta)]^{\mathrm{virt}}}{\hbar(\hbar - c)}\right). \tag{4.26}$$

Consider the following diagram:

$$
\begin{array}{ccccc}
\overline{M}_{0,1}(Y,d_1\beta_1) & \xleftarrow{\ j_1\ } & \overline{M}_{0,1}(X,d_1\beta_1) & \xrightarrow{\ \iota_1\ } & \overline{M}_{0,1}(\mathbb{P}^s,d_1) \\
\downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e_1} \\
Y & \xleftarrow{\ \ j\ \ } & X & \xrightarrow{\ \ \iota\ \ } & \mathbb{P}^s.
\end{array}
\tag{4.27}
$$

The square on the left is a fibre diagram. We repeatedly use the projection formula

$$
\begin{aligned}
&\iota_*(j^*(J_Y)) \\
&= \iota_!\left(\exp\left(\frac{t_0+t_1\iota^*(H)}{\hbar}\right)\sum_{d_1=0}^{\infty} q_1^{d_1} PD^{-1} j^* e_{1*}\left(\frac{[\overline{M}_{0,1}(Y,d_1 j_*(\beta_1))]^{\text{virt}}}{\hbar(\hbar-c)}\right)\right) \\
&= \iota_!\left(\exp\left(\frac{t_0+t_1\iota^*(H)}{\hbar}\right)\sum_{d_1=0}^{\infty} q_1^{d_1} \iota^*(\mathbb{E}^-)\cup PD^{-1} e_{1*} {j_1}^*\left(\frac{[\overline{M}_{0,1}(Y,d_1 j_*(\beta_1))]^{\text{virt}}}{\hbar(\hbar-c)}\right)\right) \\
&= \exp\left(\frac{t_0+t_1H}{\hbar}\right)\left(\mathbb{E}^+ + \sum_{d_1=1}^{\infty} q_1^{d_1}(\mathbb{E}^-)\cup PD^{-1}\iota_* e_{1*} {j_1}^*\left(\frac{[\overline{M}_{0,1}(Y,d_1 j_*(\beta_1))]^{\text{virt}}}{\hbar(\hbar-c)}\right)\right) \\
&= \exp\left(\frac{t_0+t_1H}{\hbar}\right)\left(\mathbb{E}^+ + \sum_{d_1=1}^{\infty} q_1^{d_1}(\mathbb{E}^-)\cup PD^{-1} e_{1*}\iota_{1*} {j_1}^*\left(\frac{[\overline{M}_{0,1}(Y,d_1 j_*(\beta_1))]^{\text{virt}}}{\hbar(\hbar-c)}\right)\right).
\end{aligned}
\tag{4.28}
$$

The equality in the second row follows from excess intersection theory in the left square. An argument similar to Lemma 4.4 implies that

$$\overline{M}_{0,1}(X,d_1\beta_1) = \overline{M}_{0,1}(Y,d_1 j_*(\beta_1)). \tag{4.29}$$

There are two obstruction theories in this moduli stack corresponding to the moduli problems of maps to X and Y, respectively. They differ exactly by the bundle $R^1\pi_{2*}e_2^*(\mathcal{N})$, where

$$\pi_2 : \overline{M}_{0,2}(X,d_1\beta_1) \longrightarrow \overline{M}_{0,1}(X,d_1\beta_1) \tag{4.30}$$

is the map that forgets the second marked point and $\mathcal{N} = \mathcal{N}_{X/Y}$. It follows that

$${j_1}^*\left([\overline{M}_{0,1}(Y,d_1 j_*(\beta_1))]^{\text{virt}}\right) = E(R^1\pi_{2*}e_2^*(\mathcal{N})) \cap [\overline{M}_{0,1}(X,d_1\beta_1)]^{\text{virt}}. \tag{4.31}$$

Consider the following commutative diagram:

$$\begin{array}{ccc} \overline{M}_{0,2}(X,d_1\beta_1) & \xrightarrow{e_2} & X \\ \Big\downarrow{\scriptstyle \iota_2} & & \Big\downarrow{\scriptstyle \iota} \\ \overline{M}_{0,2}(\mathbb{P}^s,d_1) & \xrightarrow{e_2} & \mathbb{P}^s. \end{array} \tag{4.32}$$

We compute

$$e_2^*(\mathcal{N}) = e_2^*(\iota^*(\mathbb{O}(-l))) = \iota_2{}^* e_2{}^*(\mathbb{O}(-l)). \tag{4.33}$$

There is the following fibre square:

$$\begin{array}{ccc} \overline{M}_{0,2}(X,d_1\beta_1) & \xrightarrow{\iota_2} & \overline{M}_{0,2}(\mathbb{P}^s,d_1) \\ \Big\downarrow{\scriptstyle \pi_2} & & \Big\downarrow{\scriptstyle \pi_2} \\ \overline{M}_{0,1}(X,d_1\beta_1) & \xrightarrow{\iota_1} & \overline{M}_{0,1}(\mathbb{P}^s,d_1). \end{array} \tag{4.34}$$

We apply [14, Proposition 9.3] to obtain

$$R^1\pi_{2*}e_2^*(\mathcal{N}) = R^1\pi_{2*}\iota_2{}^*\tilde{e_2}{}^*(\mathbb{O}(-l)) = \iota_1{}^*(R^1\pi_{2*}\tilde{e_2}{}^*(\mathbb{O}(-l))) = \iota_1{}^*\left(V_{d_1}^-\right). \tag{4.35}$$

Therefore,

$$\begin{aligned} j_1{}^*\left([\overline{M}_{0,1}(Y,d_1j_*(\beta_1))]^{\text{virt}}\right) &= E(R^1\pi_{2*}e_2^*(\mathcal{N})) \cap [\overline{M}_{0,1}(X,d_1\beta_1)]^{\text{virt}} \\ &= \iota_1^*(E_d^-) \cap [\overline{M}_{0,1}(X,d_1\beta_1)]^{\text{virt}}. \end{aligned} \tag{4.36}$$

On the other hand, [7, Proposition 11.2.3] says that

$$\iota_{1*}[\overline{M}_{0,1}(X,d_1\beta_1)]^{\text{virt}} = \mathbb{E}_d^+ \cap [\overline{M}_{0,1}(\mathbb{P}^s,d_1)]. \tag{4.37}$$

Substituting (4.36) and (4.37) into (4.28), we obtain

$$\begin{aligned} \iota_*(j^*(J_Y)) = \exp\left(\frac{t_0+t_1H}{\hbar}\right) \\ \cdot \left(\mathbb{E}^+ + \sum_{d=1}^{\infty} q_1^d PD^{-1} e_{1*}\left(\frac{\mathbb{E}_d}{\hbar(\hbar-c)} \cap [\overline{M}_{0,1}(\mathbb{P}^s,d)]\right) \cup (\mathbb{E}^-)\right). \end{aligned} \tag{4.38}$$

Recall that, on $H^*(\overline{M}_{0,1}(\mathbb{P}^s,d))$, we have $\mathbb{E}_d = \mathbb{E}'_d\mathbb{E}^-_d e_1^*(\mathbb{E}^+)$. Substituting this into (4.38) and using the projection formula, we obtain

$$\iota_!(j^*(J_Y)) = \exp\left(\frac{t_0+t_1H}{\hbar}\right) \cup \mathbb{E}^+ \cup \left(1+\sum_{d=1}^{\infty} q_1^d PD^{-1}e_{1*}\left(\frac{\mathbb{E}'_d\mathbb{E}^-_d}{\hbar(\hbar-c)}\cap[\overline{M}_{0,1}(\mathbb{P}^s,d)]\right)\cup(\mathbb{E}^-)\right). \tag{4.39}$$

The theorem is proven. □

Remark 4.8. This naturally leads to local mirror symmetry. For example, let Y be a Calabi-Yau threefold that contains $X = \mathbb{P}^2$. By adjunction formula, the normal bundle of $\mathbb{P}^2$ in X is $K_{\mathbb{P}^2} = \mathbb{O}_{\mathbb{P}^2}(-3)$. The last theorem asserts that the restriction of J_Y in X depends only on $V = \mathbb{O}_{\mathbb{P}^2}(-3)$, that is, in a neighborhood of X in Y. Hence, J_V encodes Gromov-Witten correlators of the total space of $\mathbb{O}_{\mathbb{P}^2}(-3)$ which is a local Calabi-Yau. In the next section, we see that mirror symmetry can be applied to J_V, establishing that mirror symmetry is local at least on the A-side. Interesting calculations in this direction can be found in [6].

5. Mirror theorem. In this section, we formulate and prove the mirror theorem which computes the generator J_V. Recall that $V = (\oplus_{i\in I}\mathbb{O}(k_i))\oplus(\oplus_{j\in J}\mathbb{O}(-l_j)) = V^+\oplus V^-$ with $k_i, l_j > 0$ for all $i \in I$ and $j \in J$. Consider the $H^*\mathbb{P}^s$-valued hypergeometric series

$$I_V(t_0,t_1) := \exp\left(\frac{t_0+t_1H}{\hbar}\right) \times \sum_{d=0}^{\infty} q^d \frac{\prod_{i\in I}\prod_{m=1}^{k_id}(kH+m\hbar)\prod_{j\in J}\prod_{m=0}^{l_jd-1}(-l_jH-m\hbar)}{\prod_{m=1}^{d}(H+m\hbar)^{s+1}}. \tag{5.1}$$

Theorem 5.1 (mirror theorem)**.** *Assume that $\sum_{i\in I}k_i + \sum_{j\in J}l_j \le s+1$ and that J is nonempty. If $|J| > 1$ or $\sum_{i\in I}k_i+\sum_{j\in J}l_j < s+1$, then $J_V = I_V$. Otherwise, there exists a power series I_1 of q such that $J_V(t_0,t_1+I_1) = I_V(t_0,t_1)$ as power series of q.*

Remark 5.2. The case in which J is empty has been treated in [3, 4, 11, 22]. It was suggested by Givental that his techniques should apply in the case in which J is nonempty.

5.1. The equivariant mirror theorem. We use Givental's approach for complete intersections in projective spaces [11] to prove an equivariant version of the theorem. For the remainder of this paper, we use the standard diagonal action of $T = (\mathbb{C}^*)^{s+1}$ on $\mathbb{P}^s$ with weights $(-\lambda_0,\dots,-\lambda_s)$. Recall from Section 2.2 that $\mathcal{P} = H_T^*(\mathbb{P}^s) = \mathbb{C}[\lambda][p]/\prod_{i=0}^{s}(p-\lambda_i)$ and $\mathcal{R} = \mathbb{C}(\lambda)[p]/\prod_{i=0}^{s}(p-\lambda_i)$.

Denote

$$\begin{aligned} J_V^{\text{eq}} &:= \exp\left(\frac{t_0+pt_1}{\hbar}\right)\sum_{d=0}^{\infty} q^d e_{1*}\left(\frac{E'_d E^-_d}{\hbar(\hbar-c)}\right)\cup\left(\prod_{j\in J}-l_j p\right) = \exp\left(\frac{t_0+t_1p}{\hbar}\right)S(q,\hbar),\\ I_V^{\text{eq}} &:= \exp\left(\frac{t_0+t_1p}{\hbar}\right)\sum_{d=0}^{\infty} q^d \frac{\prod_{i\in I}\prod_{m=1}^{k_id}(k_ip+m\hbar)\prod_{j\in J}\prod_{m=0}^{l_jd-1}(-l_jp-m\hbar)}{\prod_{m=1}^{d}\prod_{i=0}^{s}(p-\lambda_i+m\hbar)}\\ &= \exp\left(\frac{t_0+t_1p}{\hbar}\right)S'(q,\hbar). \end{aligned} \tag{5.2}$$

Obviously, the nonequivariant limits of J_V^{eq} and I_V^{eq} are, respectively, J_V and I_V. The mirror theorem follows as a nonequivariant limit of the following theorem.

Theorem 5.3 (the equivariant mirror theorem). *The same change of variables from Theorem 5.1 transforms I_V^{eq} into J_V^{eq}.*

Remark 5.4. As the reader will see, the central part of the proof of the mirror theorem (up to Section 5.5) involves lengthy formulas and algebraic manipulations. To simplify the presentation, we assume during this part that $V = \mathbb{O}(k)\oplus\mathbb{O}(-l)$. The general case is similar. We return to the general case $V = V^+\oplus V^-$ in Section 5.5.

Recall that the equivariant Thom classes ϕ_i of the fixed points p_i form a basis of $\mathcal{R}$ as a $\mathbb{C}(\lambda)$-vector space. Let S_i and S'_i be the restrictions of S and S' at the fixed point p_i. By the localization theorem, in $\mathbb{P}^s$, they determine S and S'. By the projection formula,

$$S_i = \int_{\mathbb{P}^s_T} S\cup\phi_i = 1+\sum_{d=1}^{\infty} q^d \int_{\overline{M}_{0,1}(\mathbb{P}^s,d)_T} \frac{e_1^*(-lp\phi_i)}{\hbar(\hbar-c)}E'_dE^-_d. \tag{5.3}$$

The proof of the equivariant mirror theorem is based on exhibiting similar properties of the correlators S_i and S'_i. The extra property $S_i = 1+o(\hbar^{-2})$ determines S_i uniquely. After the change of variables, that property is satisfied by S'_i as well, which implies $S_i = S'_i$.

We now proceed with displaying properties of the correlators S_i and S'_i.

5.2. Linear recursion relations.

The first property is given by the following lemma.

Lemma 5.5. *The correlators S_i satisfy the following linear recursion relations:*

$$S_i = 1+\sum_{d=1}^{\infty} q^d R_{id} + \sum_{d=1}^{\infty}\sum_{j\neq i} q^d C_{ijd} S_j\left(q,\frac{\lambda_j-\lambda_i}{d}\right), \tag{5.4}$$

where $R_{id} \in \mathbb{C}(\lambda)[\hbar^{-1}]$ *are polynomials in* $\hbar^{-1}$ *and*

$$C_{ijd} = \frac{(\lambda_j - \lambda_i)\prod_{m=1}^{kd}(k\lambda_i + m((\lambda_i - \lambda_j)/d))\prod_{m=0}^{ld-1}(-l\lambda_i + m((\lambda_i - \lambda_j)/d))}{d\hbar(d\hbar + \lambda_i - \lambda_j)\prod_{m=1}^{d}\prod_{k=0,(k,m)\neq(j,d)}^{s}(\lambda_i - \lambda_k + m((\lambda_j - \lambda_i)/d))}. \tag{5.5}$$

Proof. We see during the proof that S_j is regular at $\hbar = (\lambda_j - \lambda_i)/d$. The integrals that appear in the formula for S_i can be evaluated using localization theorem

$$\int_{\overline{M}_{0,1}(\mathbb{P}^s,d)_T} \frac{e_1^*(-lp\phi_i)}{\hbar(\hbar - c)} E'_d E^-_d = \sum_\Gamma \int_{(\overline{M}_\Gamma)_T} \frac{1}{a_\Gamma E_T(N_\Gamma)} \left(\frac{e_1^*(-lp\phi_i)}{\hbar(\hbar - c)} E'_d E^-_d\right)_\Gamma. \tag{5.6}$$

There are three types of fixed-point components M_Γ of $\overline{M}_{0,1}^T(\mathbb{P}^s, d)$. The first one consists of those M_Γ where the component of the curve that contains the marked point is collapsed to p_i. We denote the set of these components by $\mathcal{F}_{1,d}^i$. Let $\mathcal{F}_{2,d}^i$ be the set of those M_Γ in which the marked point is mapped at p_i and its incident component is a multiple cover of the line $\overline{p_i, p_j}$ for some $j \neq i$. Finally, let $\mathcal{F}_{0,d}^i$ be the rest of the fixed-point components. Notice first that

$$\sum_{\Gamma \in \mathcal{F}_{0,d}^i} \int_{(\overline{M}_\Gamma)_T} \frac{1}{a_\Gamma E_T(N_\Gamma)} \left(\frac{e_1^*(-lp\phi_i)}{\hbar(\hbar - c)} E'_d E^-_d\right)_\Gamma = 0. \tag{5.7}$$

Indeed, let $\Gamma_j \in \mathcal{F}_{0,d}^i$ represent a fixed-point component with the marked point mapped to the fixed point p_j for some $j \neq i$. Since $(e_1^*(\phi_i))_{\Gamma_j} = 0$, we are done. Next, in each fixed-point component that belongs to $\mathcal{F}_{1,d}^i$, the class c is nilpotent. Indeed, if Γ is the decorated graph that represents such a fixed-point component, let $\overline{M}_{0,k}$ correspond to the vertex of Γ that contains the marked point. Then, $k \le d+1$. There is a morphism

$$\varphi : M_\Gamma \longmapsto \overline{M}_{0,k} \tag{5.8}$$

such that $\varphi^*(c) = c_\Gamma$. For dimension reasons, $c^{d-1} = 0$ on $\overline{M}_{0,k}$, therefore, $1/\hbar(\hbar - c)$ is a polynomial of c in $\overline{M}_\Gamma$. Hence,

$$\sum_{\Gamma \in \mathcal{F}_{1,d}^i} \int_{(\overline{M}_\Gamma)_T} \frac{1}{a_\Gamma E_T(N_\Gamma)} \left(\frac{e_1^*(-lp\phi_i)}{\hbar(\hbar - c)} E'_d E^-_d\right)_\Gamma = R_{id} \tag{5.9}$$

is a polynomial in $\hbar^{-1}$.

We now consider the fixed-point components in $\mathcal{F}^i_{2,d}$. Again, let Γ represent such a component. For a stable map (C,x_1,f) in Γ, let C' be the component of C containing x_1, C'' the rest of the curve, $x = C'\cap C''$, and $f(x) = p_j$ for some $j \neq i$. Let d' be the degree of the map f on the component C' and $d'' = d-d'$. Then, $(C'',x,f|_{C''})$ is a fixed point in $\overline{M}_{0,1}(\mathbb{P}^s,d'')$. Denote its decorated graph by Γ''. Choose the coordinates on C' such that the restriction of f on C' is given by $f(y_0,y_1) = (0,\dots,z_i = y_0^{d'},\dots,z_j = y_1^{d'},\dots,0)$. As Γ moves in $\mathcal{F}^i_{2,d}$, the set of all such Γ'' exhausts all the fixed points in $\overline{M}_{0,1}(\mathbb{P}^s,d'')$, where the first marked point is not mapped to p_i. Since $\operatorname{Aut}(\Gamma) = \operatorname{Aut}(\Gamma'')$, it follows from (2.9) that

$$a_\Gamma = d' a_{\Gamma''}. \tag{5.10}$$

The local coordinate at p_i on the component C' is $z = y_1/y_0$. The weight of the T-action on y_l is λ_l/d' for $l = 0,1$. It follows that the weight of the action on the coordinate z and hence on $T^*_{p_i}C'$ is $(\lambda_j-\lambda_i)/d'$; therefore, $c_\Gamma = (\lambda_j-\lambda_i)/d'$. Now, $E_T(N_\Gamma)$ can be split in three pieces: smoothing the node x, deforming the maps $f|_{C''}$, and $f|_{C'}$. It follows [11] that

$$E_T(N_\Gamma) = \left(\frac{\lambda_j-\lambda_i}{d'} - c''_\Gamma\right) E_T(N_{\Gamma''}) \cdot \prod_{m=0}^{d'-1} \prod_{k=0,(m,k)\neq(0,i)}^{s} \left(\lambda_i - \lambda_k + m\frac{\lambda_j-\lambda_i}{d'}\right). \tag{5.11}$$

Next, we find the localization of E'_d and E^-_d on the fixed-point component $M^i_{d_1d_2}$. Consider the normalization sequence at the node x

$$0 \longrightarrow \mathbb{O}_C \longrightarrow \mathbb{O}_{C'} \oplus \mathbb{O}_{C''} \longrightarrow \mathbb{O}_x \longrightarrow 0. \tag{5.12}$$

Twisting it by $f^*(V^+)$ and $f^*(V^-)$, respectively, and taking the cohomology sequence yield

$$\begin{aligned} (E^-_d)_\Gamma &= (-l\lambda_j)(E^-_{d''})_{\Gamma''}(E^-_{d'})_{\Gamma'}, \\ (E^+_d)_\Gamma &= \frac{(E^+_{d'})_{\Gamma'}(E^+_{d''})_{\Gamma''}}{k\lambda_j}. \end{aligned} \tag{5.13}$$

An explicit basis for $H^1(C',f^*(V^-)) = H^1(\mathbb{O}_{\mathbb{P}^1}(-ld'))$ consists of

$$\frac{y_0^s y_1^{ld'-2-s}}{(y_0y_1)^{ld'-1}} = \frac{1}{y_0^{ld'-s-1}y_1^{1+s}}, \quad s = 0,1,\dots,ld'-2. \tag{5.14}$$

It allows us to compute

$$(E^-_{d'})_{\Gamma'} = \prod_{s=0}^{ld'-2} \left(\frac{1+s-ld'}{d'}\lambda_i - \frac{1+s}{d'}\lambda_j\right) = \prod_{s=1}^{ld'-1} \left(-l\lambda_i + s\frac{\lambda_i-\lambda_j}{d'}\right). \tag{5.15}$$

Therefore, we have

$$(E_d^-)_\Gamma = (-l\lambda_j)\prod_{s=1}^{ld'-1}\left(-l\lambda_i + s\frac{\lambda_i-\lambda_j}{d'}\right)(E_{d''}^-)_{\Gamma''}. \tag{5.16}$$

A basis for $H^0(C', f^*(V^+)) = H^0(\mathbb{O}_{\mathbb{P}^1}(kd'))$ consists of monomials $y_0^s y_1^{kd'-s}$ for $s = 0,\ldots,kd'$. It can be used to calculate $(E_{d'}^+)_{\Gamma'}$ similarly to $(E_{d'}^-)_{\Gamma'}$ above. Recall from (4.12) that $E_d^+ = e_1^*(E(V^+))E_d'$. The line bundle $e_1^*(V^+)$ is trivial on M_Γ, but the torus acts on it with weight $k\lambda_i$. Hence, $(E_d^+)_\Gamma = k\lambda_i(E_d')_\Gamma$. Substituting into (5.13) yields

$$(E_d')_\Gamma = \prod_{r=1}^{kd'}\left(k\lambda_i + r\frac{\lambda_j-\lambda_i}{d'}\right)(E_{d''}')_{\Gamma''}. \tag{5.17}$$

We pause here to show that S_i is regular at $\hbar = (\lambda_j-\lambda_i)/d$ for any $j \neq i$ and any $d > 0$. It follows from (5.7) and (5.9) that

$$S_i = 1 + \sum_{d=1}^{\infty} q^d R_{id} + \sum_{\Gamma\in\mathcal{F}_{2,d}^i} q^d \int_{(\overline{M}_\Gamma)_T} \frac{(-l\lambda_i)\prod_{k\neq i}(\lambda_i-\lambda_k)(E_d'\cdot E_d^-)_\Gamma}{\hbar(\hbar-c_\Gamma)a_\Gamma E_T(N_\Gamma)}. \tag{5.18}$$

From this representation of S_i, it is clear that the coefficients of the power series $S_i = \sum_{d=0}^{\infty} S_{id}q^d$ belong to $\mathbb{Q}(\lambda,\hbar)$. But $c_\Gamma = (\lambda_j-\lambda_i)/d'$ for some $d' \le d$ and R_{id} has poles only at $\hbar = 0$; therefore, S_i is regular at $\hbar = (\lambda_i-\lambda_j)/d$. We use (5.11), (5.17), and (5.16) to compute

$$\begin{aligned}
&\sum_{\Gamma\in\mathcal{F}_{2,d}^i} q^d \int_{(\overline{M}_\Gamma)_T} \frac{(-l\lambda_j)\prod_{k\neq i}(\lambda_i-\lambda_k)(E_d'\cdot E_d^-)_\Gamma}{\hbar(\hbar-c_\Gamma)a_\Gamma E_T(N_\Gamma)}\\
&= \sum_{d'=1}^{\infty}\sum_{j\neq i} q^{d'}C_{ijd'}\sum_{\Gamma''} q^{d''}\int_{(\overline{M}_{\Gamma''})_T} \frac{-l\lambda_j\prod_{k\neq j}(\lambda_j-\lambda_k)(E_{d''}'\cdot E_{d''}^-)_{\Gamma''}}{((\lambda_j-\lambda_i)/d')((\lambda_j-\lambda_i)/d'-c_{\Gamma}'')a_{\Gamma''}E_T(N_{\Gamma''})}\\
&= \sum_{d'=1}^{\infty}\sum_{j\neq i} q^{d'}C_{ijd'}S_j\left(q,\frac{\lambda_j-\lambda_i}{d'}\right).
\end{aligned} \tag{5.19}$$

The lemma follows by substituting the above identity into (5.18). □

Lemma 5.6. *The correlators S_i' satisfy the same linear recursion relations as S_i.*

Proof. We know that $S'_i = \sum_{d=0}^{\infty} q^d S'_{id}$ with

$$S'_{id} = \frac{\prod_{m=1}^{kd}(k\lambda_i + m\hbar)\prod_{m=0}^{ld-1}(-l\lambda_i - m\hbar)}{d\hbar\prod_{m=1}^{d}\prod_{j=0,(j,m)\neq(i,d)}^{s}(\lambda_i - \lambda_j + m\hbar)}. \tag{5.20}$$

Note that $S'_{id} \in \mathbb{C}(\lambda,\hbar)$ is a proper rational expression of $\hbar$. It has multiple poles at $\hbar = 0$ and simple poles at $\hbar = (\lambda_r - \lambda_i)/m$ for any $r \neq i$ and any $1 \leq m \leq d$. Applying calculus of residues in the $\hbar$-variable yields

$$\begin{aligned} S'_{id} = R_{id} + \sum_{m=1}^{d}\sum_{r\neq i}\frac{1}{d\hbar(\lambda_i - \lambda_r + m\hbar)} \\ \times \frac{\prod_{n=1}^{kd}(k\lambda_i + n((\lambda_r - \lambda_i)/m))\prod_{n=0}^{ld-1}(-l\lambda_i - n((\lambda_r - \lambda_i)/m))}{\prod_{n=1,(j,n)\neq(r,m)}^{d}\prod_{j=0,(j,n)\neq(i,d)}^{s}(\lambda_i - \lambda_j + n((\lambda_r - \lambda_i)/m))} \end{aligned} \tag{5.21}$$

for some polynomials $R_{id} \in \mathbb{C}(\lambda)[\hbar^{-1}]$ such that $R_{id}(0) = 0$. Substitute (5.21) into (5.20) to obtain

$$\begin{aligned} S'_i = 1 + \sum_{d=1}^{\infty} q^d R_{id} \\ + \sum_{d=1}^{\infty} q^d \sum_{r\neq i}\sum_{m=1}^{d}\frac{1}{d\hbar(\lambda_i - \lambda_r + m\hbar)} \\ \times \frac{\prod_{n=1}^{kd}(k\lambda_i + n((\lambda_r - \lambda_i)/m))\prod_{n=0}^{ld-1}(-l\lambda_i - n((\lambda_r - \lambda_i)/m))}{\prod_{n=1,(j,n)\neq(r,m)}^{d}\prod_{j=0,(j,n)\neq(i,d)}^{s}(\lambda_i - \lambda_j + n((\lambda_r - \lambda_i)/m))}. \end{aligned} \tag{5.22}$$

Changing the order of summation in the last equation yields

$$\begin{aligned} S'_i - 1 - \sum_{d=1}^{\infty} q^d R_{id} \\ = \sum_{r\neq i}\sum_{m=1}^{\infty} q^m \frac{1}{\hbar(\lambda_i - \lambda_r + m\hbar)} \\ \times \sum_{d=m}^{\infty} q^{d-m}\frac{\prod_{n=1}^{kd}(k\lambda_i + n((\lambda_r - \lambda_i)/m))\prod_{n=0}^{ld-1}(-l\lambda_i - n((\lambda_r - \lambda_i)/m))}{d\prod_{n=1,(j,n)\neq(r,m)}^{d}\prod_{j=0,(j,n)\neq(i,d)}^{s}(\lambda_i - \lambda_j + n((\lambda_r - \lambda_i)/m))}. \end{aligned} \tag{5.23}$$

The lemma follows from the identity

$$
\begin{aligned}
&\sum_{d=m}^{\infty} q^{d-m} \frac{\prod_{n=1}^{kd}\left(k\lambda_i + n((\lambda_r-\lambda_i)/m)\right)\prod_{n=0}^{ld-1}\left(-l\lambda_i - n((\lambda_r-\lambda_i)/m)\right)}{d\prod_{n=1,(j,n)\neq(r,m)}^{d}\prod_{j=0,(j,n)\neq(i,d)}^{s}\left(\lambda_i-\lambda_j+n((\lambda_r-\lambda_i)/m)\right)}\\
&= \frac{((\lambda_r-\lambda_i)/m)\prod_{n=1}^{km}\left(k\lambda_i+n((\lambda_r-\lambda_i)/m)\right)\prod_{n=0}^{lm-1}\left(-l\lambda_i-n((\lambda_r-\lambda_i)/m)\right)}{\prod_{n=1,(j,n)\neq(r,m)}^{m}\prod_{j=0}^{s}\left(\lambda_i-\lambda_j+n((\lambda_r-\lambda_i)/m)\right)}\\
&\quad\times\sum_{u=0}^{\infty} q^{u}\frac{\prod_{n=1}^{ku}\left(k\lambda_r+n((\lambda_r-\lambda_i)/m)\right)\prod_{n=0}^{lu-1}\left(-l\lambda_r-n((\lambda_r-\lambda_i)/m)\right)}{u((\lambda_r-\lambda_i)/m)\prod_{n=1,(j,n)\neq(r,u)}^{u}\prod_{j=0}^{s}\left(\lambda_r-\lambda_j+n((\lambda_r-\lambda_i)/m)\right)}.
\end{aligned}
\tag{5.24}
$$

□

5.3. Double polynomiality. Recall from Section 3.1 that V induces a modified equivariant integral $\omega_V : \mathcal{R} \to \mathbb{C}(\lambda)$ defined as follows:

$$\omega_V(a) := \int_{\mathbb{P}^s_T} a \cup \frac{E^+}{E^-}. \tag{5.25}$$

As we can see, in the case $V = \mathcal{O}(k) \oplus \mathcal{O}(-l)$, this modified equivariant integral simplifies via $E^+/E^- = kp/-lp = k/l$. We have chosen not to simplify this integral in the proof of the following lemma so that it is easier to see how to proceed in the general case.

Lemma 5.7. *If z is a variable, the expression*

$$P(z,\hbar) = \omega_V(e^{pz}S(qe^{z\hbar},\hbar)S(q,-\hbar)) \tag{5.26}$$

belongs to $\mathbb{Q}(\lambda)[\hbar][[q,z]]$.

Proof. In Section 2.3, we introduced the action of $T' = T\times\mathbb{C}^*$ on $\mathbb{P}^s\times\mathbb{P}^1$ with weights $(-\lambda_0,\dots,-\lambda_s)$ on the first factor and $(-\hbar,0)$ in the second factor. Consider the following T'-equivariant diagram:

$$
\begin{array}{ccc}
\overline{M}_{0,1}(\mathbb{P}^s\times\mathbb{P}^1,(d,1)) & \xrightarrow{\;e_1\;} & \mathbb{P}^s\times\mathbb{P}^1\\
\Big\downarrow{\scriptstyle\pi} & & \\
M_d = \overline{M}_{0,0}(\mathbb{P}^s\times\mathbb{P}^1,(d,1)). & &
\end{array}
\tag{5.27}
$$

Define

$$W_d = W_d^+ \oplus W_d^- := \pi_*((e_1)^*(\mathcal{O}_{\mathbb{P}^s}(k)\otimes\mathcal{O}_{\mathbb{P}^1})) \oplus R^1\pi_*((e_1)^*(\mathcal{O}_{\mathbb{P}^s}(-l)\otimes\mathcal{O}_{\mathbb{P}^1})). \tag{5.28}$$

The lemma follows from the identity

$$P(z,\hbar) = \sum_{d=0}^{\infty} q^d \int_{(M_d)_{T'}} e^{z\psi^*\kappa} E_{T'}(W_d), \tag{5.29}$$

where ψ and κ were defined in Section 2.3. The localization formula for the diagonal action of T on $\mathbb{P}^s$ applied to the left side gives

$$P(z,\hbar) = \sum_{i=0}^{s} \frac{S_i(qe^{z\hbar},\hbar)e^{z\lambda_i}S_i(q,-\hbar)}{\prod_{k\neq i}(\lambda_i-\lambda_k)}\left(\frac{k\lambda_i}{-l\lambda_i}\right). \tag{5.30}$$

We recall from identity (5.3)

$$S_i = 1+\sum_{d=1}^{\infty} q^d \int_{\overline{M}_{0,1}(\mathbb{P}^s,d)_T} \frac{e_1^*(-lp\phi_i)}{\hbar(\hbar-c)} E'_d E^-_d. \tag{5.31}$$

To compute the integrals on the right side of (5.29), we use localization for the action of T' on M_d. In Section 2.3, we found that the components of the fixed-point loci have the form $M^i_{d_1d_2} = \overline{M}_{\Gamma^i_{d_1}} \times \overline{M}_{\Gamma^i_{d_2}}$ for some $i = 0,1,\dots,s$ and a splitting $d = d_1+d_2$. We first compute the restriction of $E_{T'}(W_d)$ in such a component. Consider the following normalization sequence:

$$0 \longrightarrow \mathbb{O}_C \longrightarrow \mathbb{O}_{C_0}\oplus\mathbb{O}_{C_1}\oplus\mathbb{O}_{C_2} \longrightarrow \mathbb{O}_{x_1}\oplus\mathbb{O}_{x_2} \longrightarrow 0. \tag{5.32}$$

Twist (5.32) by $f^*(\mathbb{O}(-l)\otimes\mathbb{O}_{\mathbb{P}^1})$ and take the corresponding long exact cohomology sequence. We obtain

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathbb{O}_{x_1}(-l)\oplus\mathbb{O}_{x_2}(-l) \longrightarrow W^-_d \longrightarrow W^-_{d_1}\oplus W^-_{d_2} \longrightarrow 0. \tag{5.33}$$

The first piece is trivial since it comes from the isomorphism

$$(\mathbb{O}_{\mathbb{P}^s}(-l)\times\mathbb{O}_{\mathbb{P}^1})\big|_{C_0} \cong \mathbb{O}_{C_0} \cong \mathbb{C}. \tag{5.34}$$

The left-hand side is generated by $1/{z_i}^l$; therefore, the weight of that piece is $-l\lambda_i$. It follows that

$$E_{T'}(W^-_d) = (-l\lambda_i)E^-_{d_1}E^-_{d_2}. \tag{5.35}$$

Similarly, twisting the normalization sequence (5.32) by $f^*(\mathcal{O}(k)\otimes\mathcal{O}_{\mathbb{P}^1})$ and taking the corresponding cohomology sequence, we obtain

$$E_{T'}(W_d^+) = (k\lambda_i)E'_{d_1}E'_{d_2}. \tag{5.36}$$

We now use the localization theorem to calculate the integrals on the right side of (5.29). The equivariant Euler class of the normal bundle of the fixed-point component $M^i_{d_1 d_2}$ has been calculated at the end of Section 2.3. We have

$$\begin{aligned}
&\int_{(M_d)_{T'}} e^{z\psi^*\kappa}E_{T'}(W_d)\\
&\quad=\sum_{\Gamma^i_{d_1},\Gamma^i_{d_2}}(k\lambda_i)(-l\lambda_i)\frac{e^{z(\lambda_i+d_2\hbar)}}{\prod_{k\neq i}(\lambda_i-\lambda_k)}\int_{\left(\overline{M}_{\Gamma^i_{d_1}}\right)_T}\frac{1}{E_T\left(N_{\Gamma^i_{d_1}}\right)}\left(\frac{e_1^*(\phi_i)E'_{d_1}E^-_{d_1}}{-\hbar(-\hbar-c_1)}\right)_{\Gamma^i_{d_1}}\\
&\qquad\times\int_{\left(\overline{M}_{\Gamma^i_{d_2}}\right)_T}\frac{1}{E_T\left(N_{\Gamma^i_{d_2}}\right)}\left(\frac{e_1^*(\phi_i)E'_{d_2}E^-_{d_2}}{\hbar(\hbar-c_2)}\right)_{\Gamma^i_{d_2}}\\
&\quad=\sum_{\Gamma^i_{d_1},\Gamma^i_{d_2}}\frac{k\lambda_i}{-l\lambda_i}\frac{e^{z(\lambda_i+d_2\hbar)}}{\prod_{k\neq i}(\lambda_i-\lambda_k)}\int_{\left(\overline{M}_{\Gamma^i_{d_1}}\right)_T}\frac{1}{E_T\left(N_{\Gamma^i_{d_1}}\right)}\left(\frac{e_1^*(-l\lambda_i\phi_i)E_{d_1}}{-\hbar(-\hbar-c_1)}\right)_{\Gamma^i_{d_1}}\\
&\qquad\times\int_{\left(\overline{M}_{\Gamma^i_{d_2}}\right)_T}\frac{1}{E_T\left(N_{\Gamma^i_{d_2}}\right)}\left(\frac{e_1^*(-l\lambda_i\phi_i)E_{d_2}}{\hbar(\hbar-c_2)}\right)_{\Gamma^i_{d_2}}.
\end{aligned} \tag{5.37}$$

If we use localization to compute S_i in (5.31) and then substitute in (5.30), we obtain the right side of the last equation. □

Lemma 5.8. *If z is a variable, the expression*

$$P'(z,h) = \omega_V\left(S'(qe^{z\hbar},\hbar)e^{pz}S'(q,-\hbar)\right) \tag{5.38}$$

belongs to $\mathbb{Q}(\lambda)[\hbar][[q,z]]$.

Proof. The lemma will follow from the identity

$$P'(z,h) = \sum_{d=0}^{\infty} q^d\int_{(N_d)_{T'}} e^{z\kappa}\prod_{m=0}^{kd}(k\kappa-m\hbar)\prod_{m=1}^{ld-1}(-l\kappa+m\hbar). \tag{5.39}$$

For $d=0$, the convention

$$\int_{(N_d)_{T'}} e^{z\kappa}\prod_{m=0}^{kd}(k\kappa+m\hbar)\prod_{m=1}^{ld-1}(-l\kappa+m\hbar) = \int_{\mathbb{P}^s_{\mathbf{T}}} e^{pz}\left(\frac{\prod_{i\in I}k_i p}{\prod_{j\in J}-l_j p}\right) \tag{5.40}$$

is taken. Apply the localization formula to the integrals (5.38). We obtain

$$
\begin{aligned}
&P'(z,\hbar)\\
&=\sum_{i=0}^{s}\frac{k\lambda_i e^{\lambda_i z}}{(-l\lambda_i)\prod_{j\neq i}(\lambda_i-\lambda_j)}\sum_{d_1=0}^{\infty}(qe^{z\hbar})^{d_1}\frac{\prod_{m=1}^{kd_1}(k\lambda_i+m\hbar)\prod_{m=0}^{ld_1-1}(-l\lambda_i-m\hbar)}{\prod_{m=1}^{d_1}\prod_{j=0}^{s}(\lambda_i-\lambda_j+m\hbar)}\\
&\quad\times\sum_{d_2=0}^{\infty}q^{d_2}\frac{\prod_{m=1}^{kd_2}(k\lambda_i-m\hbar)\prod_{m=0}^{ld_2-1}(-l\lambda_i+m\hbar)}{\prod_{m=1}^{d_2}\prod_{j=0}^{s}(\lambda_i-\lambda_j-m\hbar)}\\
&=\sum_{i=0}^{s}\frac{1}{\prod_{j\neq i}(\lambda_i-\lambda_j)}\sum_{d_1=0}^{\infty}q^{d_1 z}e^{(\lambda_i+d_1\hbar)z}\frac{\prod_{m=0}^{kd_1}(k\lambda_i+m\hbar)\prod_{m=1}^{ld_1-1}(-l\lambda_i-m\hbar)}{\prod_{m=1}^{d_1}\prod_{j=0}^{s}(\lambda_i-\lambda_j+m\hbar)}\\
&\quad\times\sum_{d_2=0}^{\infty}q^{d_2}\frac{\prod_{m=1}^{kd_2}(k\lambda_i-m\hbar)\prod_{m=0}^{ld_2-1}(-l\lambda_i+m\hbar)}{\prod_{m=1}^{d_2}\prod_{j=0}^{s}(\lambda_i-\lambda_j-m\hbar)}.
\end{aligned}
\tag{5.41}
$$

But for $d_1,d_2>0$

$$
\begin{aligned}
&\frac{\prod_{m=0}^{kd_1}(k\lambda_i+m\hbar)\prod_{m=1}^{ld_1-1}(-l\lambda_i-m\hbar)\prod_{m=1}^{kd_2}(k\lambda_i-m\hbar)\prod_{m=0}^{ld_2-1}(-l\lambda_i+m\hbar)}{\prod_{j\neq i}(\lambda_i-\lambda_j)\prod_{m=1}^{d_1}\prod_{j=0}^{s}(\lambda_i-\lambda_j+m\hbar)\prod_{m=1}^{d_2}\prod_{j=0}^{s}(\lambda_i-\lambda_j-m\hbar)}\\
&\qquad=\frac{\prod_{m=0}^{k(d_1+d_2)}(k(\lambda_i+d_1\hbar)-m\hbar)\prod_{m=1}^{l(d_1+d_2)-1}(-l(\lambda_i+d_1\hbar)+m\hbar)}{\prod_{j=0}^{s}\prod_{m=0,(j,m)\neq(i,d_1)}^{d_1+d_2}(\lambda_i+d_1\hbar-\lambda_j-m\hbar)}.
\end{aligned}
\tag{5.42}
$$

Therefore,

$$
\begin{aligned}
&P'(z,\hbar)\\
&=\sum_{d=0}^{\infty}q^d\sum_{d_1=0}^{d}\sum_{i=0}^{s}e^{(\lambda_i+d_1\hbar)z}\frac{\prod_{m=0}^{kd}(k(\lambda_i+d_1\hbar)-m\hbar)\prod_{m=1}^{ld-1}(-l(\lambda_i+d_1\hbar)+m\hbar)}{\prod_{j=0}^{s}\prod_{m=0,(j,m)\neq(i,d_1)}^{d}(\lambda_i+d_1\hbar-\lambda_j-m\hbar)}.
\end{aligned}
\tag{5.43}
$$

By the localization formula in N_d, we can see that

$$
P'(z,\hbar)=\sum_{d=0}^{\infty}q^d\int_{(N_d)_{T'}}e^{z\kappa}\prod_{m=0}^{kd}(k\kappa-m\hbar)\prod_{m=1}^{ld-1}(-l\kappa+m\hbar). \tag{5.44}
$$

The lemma is proven. □

5.4. Mirror transformation and uniqueness. The following two lemmas carry over from [11]. The first lemma deals with uniqueness.

Lemma 5.9. *Let $S=\sum_{d=0}^{\infty}S_d q^d$ and $S'=\sum_{d=0}^{\infty}S'_d q^d$ be two power series with coefficients in $\mathcal{R}[[\hbar^{-1}]]$ that satisfy the following conditions:*

(1) $S_0 = S_0' = 1$;
(2) *they both satisfy the recursion relations of Lemma 5.5;*
(3) *they both have the double polynomiality property of Lemma 5.8;*
(4) *for any* d, $S_d \equiv S_d' \bmod(\hbar^{-2})$.

Then, $S = S'$.

The second lemma describes a transformation which preserves the properties of Lemma 5.9.

Lemma 5.10. *Let* I_1 *be a power series in* q *whose first term is zero. Then,* $\exp(I_1 p/\hbar) S(qe^{I_1}, \hbar)$ *satisfies conditions (1), (2), and (3) of Lemma 5.9.*

5.5. The conclusion of the proof of mirror theorem. Recall that

$$I_V^{\mathrm{eq}} = \exp\left(\frac{t_0 + t_1 p}{\hbar}\right) \times \left(1 + \sum_{d=1}^{\infty} q^d \frac{\prod_{i\in I}\prod_{m=1}^{k_i d}(k_i p + m\hbar)\prod_{j\in J}\prod_{m=0}^{l_j d-1}(-l_j p - m\hbar)}{\prod_{m=1}^{d}\prod_{i=0}^{s}(p - \lambda_i + m\hbar)}\right). \tag{5.45}$$

We are assuming that there is at least one negative line bundle. We expand the second factor of I_V^{eq} as a polynomial of $\hbar^{-1}$. Each negative line bundle produces a factor of $p/\hbar$. For example, in the case $V = \mathbb{O}(k) \oplus \mathbb{O}(-l)$, the expansion yields

$$I_V^{\mathrm{eq}} = \exp\left(\frac{t_0 + pt}{\hbar}\right)\left(1 + \frac{p}{\hbar}\sum_{d=1}^{\infty} q^d \frac{(-1)^{ld}(ld-1)!(kd)!}{(d!)^{s+1}} \frac{1}{\hbar^{d(s+1-k-l)}} + o\left(\frac{1}{\hbar^2}\right)\right). \tag{5.46}$$

If V contains two or more negative line bundles, it follows that

$$I_V^{\mathrm{eq}} = \exp\left(\frac{t_0 + pt}{\hbar}\right)\left(1 + o\left(\frac{1}{\hbar^2}\right)\right). \tag{5.47}$$

Lemmas 5.9 and 5.10 imply that $J_V^{\mathrm{eq}} = I_V^{\mathrm{eq}}$. If $\sum_{i\in I} k_i + \sum_{j\in J} l_j < s+1$, the presence of $1/\hbar^{d(s+1-k-l)}$ in the above expansion shows, again that,

$$I_V^{\mathrm{eq}} = \exp\left(\frac{t_0 + pt}{\hbar}\right)\left(1 + o\left(\frac{1}{\hbar^2}\right)\right); \tag{5.48}$$

hence, $J^{\mathrm{eq}} = I^{\mathrm{eq}}$. We may assume that $\sum_{i\in I} k_i + \sum_{j\in J} l_j = s+1$ and $|J| = 1$. In this case,

$$I_V^{\mathrm{eq}} = \exp\left(\frac{t_0 + pt}{\hbar}\right)\left(1 + I_1\frac{p}{\hbar} + o\left(\frac{1}{\hbar^2}\right)\right), \tag{5.49}$$

where I_1 is a power series of q whose first term is zero. For example, if $V = \mathbb{O}(k)\oplus\mathbb{O}(-l)$, the power series I_1 is

$$I_1 = \sum_{d=1}^{\infty} q^d \frac{(-1)^{ld}(ld-1)!(kd)!}{(d!)^{s+1}}. \tag{5.50}$$

Recall that $S = 1+o(\hbar^{-2})$. Therefore,

$$\exp\left(\frac{I_1 p}{\hbar}\right)S(qe^{I_1},\hbar) = 1+I_1\frac{p}{\hbar}+o(\hbar^{-2}). \tag{5.51}$$

Lemma 5.10 implies that both $\exp(I_1 p/\hbar)S(qe^{I_1},\hbar)$ and $S'(q,\hbar)$ satisfy the conditions of Lemma 5.9. It follows that

$$\exp\left(\frac{I_1 p}{\hbar}\right)S(qe^{I_1},\hbar) = S'(q,\hbar). \tag{5.52}$$

Multiplying both sides of this identity by $\exp((t_0+pt)/\hbar)$ yields

$$J_V^{\mathrm{eq}}(t_0,t+I_1) = I_V^{\mathrm{eq}}(t_0,t). \tag{5.53}$$

This completes the proof.

Corollary 5.11. *Let $V = (\oplus_{i\in I}\mathbb{O}(k_i))\oplus(\oplus_{j\in J}\mathbb{O}(-l_j))$. For $|J|>1$ or $k+l<s+1$,*

$$e_{1*}\left(\frac{\mathbb{E}'_d\mathbb{E}^-_d}{\hbar(\hbar-c)}\right) = \frac{\prod_{i\in I}\prod_{m=1}^{k_id}(k_iH+m\hbar)\prod_{j\in J}\prod_{m=1}^{l_jd-1}(-l_jH-m\hbar)}{\prod_{m=1}^{d}(H+m\hbar)^{s+1}}. \tag{5.54}$$

Proof. As mentioned above, in this case we have $J_V^{\mathrm{eq}} = I_V^{\mathrm{eq}}$. Recall that

$$J_V^{\mathrm{eq}} = \exp\left(\frac{t_0+pt_1}{\hbar}\right)\left(1+\sum_{d>0}q^d e_{1*}\left(\frac{E'_dE^-_d}{\hbar(\hbar-c)}\right)\cup\prod_{j\in J}(-l_jp)\right). \tag{5.55}$$

We obtain the equivariant identity

$$e_{1*}\left(\frac{E'_dE^-_d}{\hbar(\hbar-c)}\right)\cup\prod_{j\in J}(-l_jp) = \frac{\prod_{i\in I}\prod_{m=1}^{k_id}(k_ip+m\hbar)\prod_{j\in J}\prod_{m=0}^{l_jd-1}(-l_jp-m\hbar)}{\prod_{m=1}^{d}\prod_{k=0}^{s}(p-\lambda_k+m\hbar)}. \tag{5.56}$$

The restriction of p to any fixed point p_i is nonzero. This implies that p is invertible. Therefore, we obtain

$$e_{1*}\left(\frac{E'_dE^-_d}{\hbar(\hbar-c)}\right)=\frac{\prod_{i\in I}\prod_{m=1}^{k_id}(k_ip+m\hbar)\prod_{j\in J}\prod_{m=1}^{l_jd-1}(-l_jp-m\hbar)}{\prod_{m=1}^{d}\prod_{k=1}^{s+1}(p-\lambda_k+m\hbar)}. \tag{5.57}$$

The nonequivariant limit of this identity reads

$$e_{1*}\left(\frac{\mathbb{E}'_d\mathbb{E}^-_d}{\hbar(\hbar-c)}\right)=\frac{\prod_{i\in I}\prod_{m=1}^{k_id}(k_iH+m\hbar)\prod_{j\in J}\prod_{m=1}^{l_jd-1}(-l_jH-m\hbar)}{\prod_{m=1}^{d}(H+m\hbar)^{s+1}}. \tag{5.58}$$

The corollary is proven. □

This corollary is particularly useful when $\operatorname{Euler}(V^-)=0$ in $\mathbb{P}^s$. In this case, $J_V=I_V=\exp((t_0+Ht_1)/\hbar)$, hence, the mirror theorem is true trivially. An example of such a situation is $V=\mathbb{O}_{\mathbb{P}^1}(-1)\oplus\mathbb{O}_{\mathbb{P}^1}(-1)$ which is treated in the next section.

6. Examples

6.1. Multiple covers. Let C be a smooth rational curve in a Calabi-Yau threefold X with normal bundle $N=\mathbb{O}(-1)\oplus\mathbb{O}(-1)$ and $\beta=[C]\in H_2(X,\mathbb{Z})$. Since $K_X=\mathbb{O}_X$, the expected dimension of the moduli space $\overline{M}_{0,0}(X,d\beta)$ is zero. However, this moduli space contains a component of positive dimension, namely, $\overline{M}_{0,0}(\mathbb{P}^1,d)$. Indeed, let $f:\mathbb{P}^1\to C$ be an isomorphism, and $g:\mathbb{P}^1\to\mathbb{P}^1$ a degree d multiple cover. Then, $f\circ g$ is a stable map that belongs to $\overline{M}_{0,0}(X,d\beta)$. For a proof of the fact that $\overline{M}_{0,0}(\mathbb{P}^1,d)$ is a component of $\overline{M}_{0,0}(X,d\beta)$, see [7, Section 7.4.4]. Let N_d be the degree of $[\overline{M}_{0,0}(X,d\beta)]^{\text{virt}}$. We want to compute the contribution n_d of $\overline{M}_{0,0}(\mathbb{P}^1,d)$ to N_d. Kontsevich asserted in [17] and Behrend proved in [1] that the restriction of $[\overline{M}_{0,0}(X,d\beta)]^{\text{virt}}$ to $\overline{M}_{0,0}(\mathbb{P}^1,d)$ is precisely $\mathbb{E}_d$ for $V=\mathbb{O}(-1)\oplus\mathbb{O}(-1)$. Therefore,

$$n_d=\int_{\overline{M}_{0,0}(\mathbb{P}^1,d)}\mathbb{E}_d. \tag{6.1}$$

Note that $\dim\overline{M}_{0,0}(\mathbb{P}^1,d)=2d-2$, and the rank of the bundle V_d is also $2d-2$. We use the mirror theorem to compute numbers n_d. Since V contains two negative line bundles, we can apply Corollary 5.11

$$e_{1*}\left(\frac{\mathbb{E}_d}{\hbar(\hbar-c)}\right)=\frac{\prod_{m=1}^{d-1}(-H-m\hbar)^2}{\prod_{m=1}^{d}(H+m\hbar)^2}=\frac{1}{(H+d\hbar)^2}. \tag{6.2}$$

An expansion of the left-hand side using the divisor property for the modified gravitational descendants yields

$$e_{1*}\left(\frac{\mathbb{E}_d}{\hbar(\hbar-c)}\right)=\frac{dn_d}{\hbar^2}+\frac{H}{\hbar^3}\int_{\overline{M}_{0,1}(\mathbb{P}^1,d)}c\cup\mathbb{E}_d, \tag{6.3}$$

where c is the Chern class of the cotangent line bundle at the marked point. On the other hand,

$$\frac{1}{(H+d\hbar)^2}=\frac{1}{d^2\hbar^2}-\frac{2H}{d^3\hbar^3}. \tag{6.4}$$

We obtain the Aspinwall-Morrison formula

$$n_d=\frac{1}{d^3}, \tag{6.5}$$

which has been proved by several different methods [17, 21, 23]. We also obtain

$$\int_{\overline{M}_{0,1}(\mathbb{P}^1,d)}c\cup\mathbb{E}_d=-\frac{2}{d^3}. \tag{6.6}$$

6.2. Virtual numbers of plane curves. Let X be a Calabi-Yau threefold containing a $\mathbb{P}^2$. As we saw in Remark 4.8, the normal bundle of $\mathbb{P}^2$ in X is $K_{\mathbb{P}^2}=\mathbb{O}(-3)$. Let C be a rational curve of degree d in $\mathbb{P}^2$. Since $K_X=\mathbb{O}_X$, the expected dimension of the moduli space $\overline{M}_{0,0}(X,[C])$ is zero. Lemma 4.4 says that $\overline{M}_{0,0}(\mathbb{P}^2,d)=\overline{M}_{0,0}(X,[C])$; hence, the dimension of this moduli stack is $3d-1$. Recall the diagram

$$\begin{array}{ccc}\overline{M}_{0,1}(\mathbb{P}^2,d) & \xrightarrow{e_1} & \mathbb{P}^2\\ \Big\downarrow{\scriptstyle \pi_1} & & \\ \overline{M}_{0,0}(\mathbb{P}^2,d). & & \end{array} \tag{6.7}$$

From Lemma 4.5, the virtual fundamental class of $\overline{M}_{0,0}(X,[C])$ is the refined top Chern class of the bundle $V_d=R^1\pi_{1*}(e_1^*(K_{\mathbb{P}^2}))$ over $\overline{M}_{0,0}(\mathbb{P}^2,d)$. The zero-pointed Gromov-Witten invariant

$$N_d:=\deg\left[\overline{M}_{0,0}(X,[C])\right]^{\text{virt}}=\int_{\overline{M}_{0,0}(\mathbb{P}^2,d)}\mathbb{E}_d \tag{6.8}$$

is called *the virtual number of degree d rational curves in X*. As promised in Remark 3.10, we show that the modified equivariant quantum product in this case has a nonequivariant limit. We also use the mirror theorem to calculate these numbers N_d.

The modified pairing on $\mathbb{P}^2_{\mathbb{C}^*}$ corresponding to $V = \mathbb{O}_{\mathbb{P}^2}(-3)$ is

$$\langle a,b\rangle := \int_{(\mathbb{P}^2)_{\mathbb{C}^*}} a\cup b\cup\left(\frac{1}{-3p-\lambda}\right). \tag{6.9}$$

Recall that p denotes the equivariant hyperplane class in $\mathbb{P}^2$. Then, $1,p,p^2$ is a basis for $\mathcal{R}$ as a $\mathbb{Q}(\lambda)$-module. A simple calculation shows that $-\lambda p^2, -3p^2 - \lambda H, -3p-\lambda$ is its dual basis with respect to the above pairing. Since both bases and E_d are polynomials in λ, we can restrict $\tilde{I}_d$ and $*_V$ in $\mathcal{P} = H^*(\mathbb{P}^2,\mathbb{Q}[\lambda])$ and take the nonequivariant limit. Denote by H the nonequivariant limit of p. We obtain the following nonequivariant quantum product on $H^*\mathbb{P}^2\otimes\mathbb{Q}[[q]]$:

$$a *_V b := a\cup b + \sum_{d=1}^{\infty} q^d T^k I_d(a,b,-3HT_k), \tag{6.10}$$

where $T^0 = 1$, $T^1 = H$, $T^2 = H^2$, and, for $\gamma_1,\gamma_2,\dots,\gamma_n \in H^*\mathbb{P}^2$,

$$I_d(\gamma_1,\gamma_2,\dots,\gamma_n) = \int_{\overline{M}_{0,n}(\mathbb{P}^2,d)} e_1^*\gamma_1\cup e_2^*\gamma_2\cup\cdots\cup e_n^*\gamma_n\cup\mathbb{E}_d. \tag{6.11}$$

For example, using the divisor axiom, we obtain

$$H *_V H = H^2\left(1-3\sum_{d>0} q^d d^3 N_d\right). \tag{6.12}$$

Theorem 3.9 implies the following theorem.

Theorem 6.1. *The ring $(H^*\mathbb{P}^2\otimes\mathbb{Q}[[q]], *_V)$ is an associative, commutative, and unital ring with unity $1 = [\mathbb{P}^2]$.*

Denote by i the embedding $i:\mathbb{P}^2\hookrightarrow X$. Since the normal bundle of $\mathbb{P}^2$ in X is $\mathbb{O}_{\mathbb{P}^2}(-3)$, it follows that $i^*(-(1/3)[\mathbb{P}^2]) = T^1$ and $i^*(-(1/3)[l]) = T^2$. Therefore, the map $i^* : (H^*X,\mathbb{Q}) \to (H^*\mathbb{P}^2,\mathbb{Q})$ is surjective. Consider the small quantum cohomology rings $SQH^*X = (H^*X\otimes\mathbb{Q}[[\beta]],*)$ and $SQH^*_V\mathbb{P}^2 := (H^*\mathbb{P}^2\otimes\mathbb{Q}[[q]],*_V)$, where the products are given by three-point correlators. Recall from Section 4.2 the extension of $i^* : H^*(X,\mathbb{Q}) \to H^*(\mathbb{P}^2,\mathbb{Q})$ to $\tilde{i}^* : SQH^*X \to SQH^*_V\mathbb{P}^2$. There is a natural relation between the modified quantum product in $\mathbb{P}^2$ and the pure product in X.

Theorem 6.2. *The map $\tilde{i}^*$ is a ring homomorphism.*

Proof. Complete $\tau^0 = [X]$, $\tau^1 = -(1/3)[\mathbb{P}^2]$, and $\tau^2 = -(1/3)H$ into a basis of $(H^*X,\mathbb{Q})$ by adding elements from $\operatorname{Ker}(i^*)$. Let $\tau_0 = [pt], \tau_1 = H, \tau_2 = [\mathbb{P}^2],\dots$ be the dual basis. Let $a,b\in H^*X$. We want to show

$$\tilde{i}^*(a*b) = i^*(a) *_V i^*(b). \tag{6.13}$$

But

$$a * b = \sum_{\beta \in H_2(X,\mathbb{Q})} \sum_r q^\beta \tau^r \int_{[\overline{M}_{0,3}(X,\beta)]^{\mathrm{virt}}} e_1^* a \cup e_2^* b \cup e_3^* \tau_r. \tag{6.14}$$

Note that this formula is true for a $\mathbb{Z}$-basis, but, due to the uniqueness of the quantum product, it is true for any $\mathbb{Q}$-basis as well. Now,

$$\tilde{i^*}(a * b) = \sum_{d \geq 0} \sum_r q^d i^* \tau^r \int_{\overline{M}_{0,3}(\mathbb{P}^2,d)} e_1^*(i^* a) \cup e_2^*(i^* b) \cup e_3^*(i^* \tau_r) \mathbb{E}_d. \tag{6.15}$$

But $i^*(\tau^r) = T^k$ for $r = 0,1,2$ and $i^*(\tau^r) = 0$ for $r \geq 2$. The theorem follows from the readily checked fact that $i^*(\tau_k) = -3HT_k$ for $k = 0,1,2$. □

Using the divisor and fundamental class properties of the modified gravitational descendants, it is easy to show that

$$J_V = \exp\left(\frac{t_0 + t_1 H}{\hbar}\right)\left(1 - 3\frac{H^2}{\hbar^2}\sum_{d=1}^{\infty} q^d d N_d\right). \tag{6.16}$$

The hypergeometric series corresponding to the total space of $V = \mathbb{O}_{\mathbb{P}^2}(-3)$ is

$$I_V := \exp\left(\frac{t_0 + t_1 H}{\hbar}\right)\sum_{d=0}^{\infty} q^d \frac{\prod_{m=0}^{3d-1}(-3H - m\hbar)}{\prod_{m=1}^{d}(H + m\hbar)^3}. \tag{6.17}$$

We expand the function

$$I_V = \exp\left(\frac{t_0 + t_1 H}{\hbar}\right)\left(1 + I_1 \frac{H}{\hbar} + o\left(\frac{1}{\hbar}\right)\right), \tag{6.18}$$

where

$$I_1 = 3\sum_{d=1}^{\infty} q^d (-1)^d \frac{(3d-1)!}{(d!)^3}. \tag{6.19}$$

The mirror theorem for this case says that $J(t_0, t_1 + I_1) = I_V(t_0, t_1)$. This theorem allows us to compute the virtual number of rational plane curves in the Calabi-Yau X. The first few numbers are 3, $-45/8$, $244/9$.

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