IJMMS 2003:28, 1755–1770
PII. S0161171203012250
http://ijmms.hindawi.com

ERGODICITY AND EXTREMALITY OF AMS SOURCES AND CHANNELS

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Received 3 March 2001 and in revised form 5 September 2001

Asymptotically mean stationary (AMS) sources (probability measures) and channels are considered as an extension of stationary sources and channels. It is shown that each extreme point of the set of all AMS sources is ergodic, but not vice versa, and that each extreme point in the set of all AMS channels is ergodic, but not vice versa.

2000 Mathematics Subject Classification: 28D05, 94A40.

1. Introduction. The purpose of this paper is to show that every extreme point of the set of all AMS (asymptotically mean stationary) sources (probability measures) is ergodic, but not vice versa, and to show that every extreme point of the set of all AMS channels is ergodic, but not vice versa. Also, this paper serves as a review of ergodicity of stationary sources, AMS sources, stationary channels, and AMS channels. So, many known results on this subject are collected and some new results are obtained as well. We refer to Kakihara [13] as a general reference.

Ergodicity of a stationary (transformation invariant) source has been considered and many equivalence conditions for that are known, among which is the extremality in the set of all stationary sources (cf. Blum and Hanson [1], Breiman [2], and Halmos [9]). As a generalization of stationarity, Dowker [3] introduced asymptotic mean stationarity, which is necessary and sufficient for that the pointwise ergodic theorem holds (cf. Rechard [15]). Gray and Kieffer [7] studied AMS sources extensively. Kakihara [12] used a functional approach to AMS sources and gave some necessary and sufficient conditions for an AMS source to be ergodic.

On the other hand, ergodicity of a stationary channel was first characterized by Shen [16, 17]. Then, Umegaki [18] and Nakamura [14] independently showed some equivalence conditions for ergodicity of a stationary channel. Extremality in the set of all stationary channels is one of these conditions. AMS channels were defined by Fontana et al. [5], where they obtained some equivalence conditions for a channel to be AMS (see also Gray and Saadat [8]). Kakihara [12] gave characterizations of ergodic AMS channels. Also, Jacobs [11] defined almost periodic channels, a special class of AMS channels, and Hu and Shen [10] considered their ergodicity.

In Section 2, ergodicity of stationary and AMS sources is discussed. In Section 3, ergodicity of stationary channels is considered, where we introduce channel operators applied to obtain equivalence conditions for that. Finally in Section 4, ergodicity of AMS channels is treated. We recognize that equivalence conditions for ergodicity in all cases are quite similar to each other.

2. Ergodicity of stationary and AMS sources. Let $(X,\mathfrak{X})$ be a measurable space and let $S : X \to X$ be a measurable transformation. Let $P(X)$ denote the set of all probability measures on X and $P_s(X)$ the set of all S-invariant measures in $P(X)$. A measure $\mu \in P_s(X)$ is called a *stationary source*. Moreover, $P_{se}(X)$ stands for the set of all ergodic measures in $P_s(X)$ and $B(X)$ for the space of all bounded $\mathbb{C}$-valued measurable functions on X, where $\mathbb{C}$ is the complex number field. The transformation S induces an operator $\mathbf{S}$ on $B(X)$:

$$(\mathbf{S}f)(x) = f(Sx), \quad f \in B(X),\ x \in X. \tag{2.1}$$

For an integer $n \geq 1$, we denote by $\mathbf{S}_n$ the operator defined by

$$(\mathbf{S}_n f)(x) = \frac{1}{n}\sum_{j=0}^{n-1}(\mathbf{S}^j f)(x) = \frac{1}{n}\sum_{j=0}^{n-1} f(S^j x), \quad f \in B(X),\ x \in X. \tag{2.2}$$

L^p-spaces are denoted by $L^p(X,\mu)$ with the norm $\|\cdot\|_{p,\mu}$ for $\mu \in P(X)$ and $p \geq 1$. For $p = 2$, denote by $(\cdot,\cdot)_\mu$ the inner product. For $\mu \in P(X)$, we use a notation

$$\mu(f) = \int_X f\,d\mu, \quad f \in L^1(X,\mu). \tag{2.3}$$

The characterizations of stationary ergodic sources, which are collected in the following theorem, are known (cf. Blum and Hanson [1], Breiman [2], and Halmos [9]).

Theorem 2.1. *For a stationary source $\mu \in P_s(X)$, the following conditions are equivalent to each other:*

(1) *$\mu \in P_{se}(X)$, that is, μ is ergodic;*
(2) *there exists some $\eta \in P_{se}(X)$ such that $\mu \ll \eta$;*
(3) *if $\xi \in P_s(X)$ and $\xi \ll \mu$, then $\xi = \mu$;*
(4) *$\mu \in \mathrm{ex}\,P_s(X)$, where $\mathrm{ex}\{\cdot\}$ is the set of all extreme points;*
(5) *if $f \in B(X)$ is S-invariant μ-a.e., then $f = \mathrm{const}$ μ-a.e;*
(6) *$f_S(x) \equiv \lim_{n\to\infty}(\mathbf{S}_n f)(x) = \mu(f)$ μ-a.e. for every $f \in L^1(X,\mu)$;*
(7) *$\lim_{n\to\infty}(\mathbf{S}_n f,g)_\mu = (f,1)_\mu(1,g)_\mu$ for every $f,g \in L^2(X,\mu)$;*
(8) *$\lim_{n\to\infty}\mu((\mathbf{S}_n f)g) = \mu(f)\mu(g)$ for every $f,g \in B(X)$;*
(9) *$\lim_{n\to\infty}(1/n)\sum_{j=0}^{n-1}\mu(S^{-j}A\cap B) = \mu(A)\mu(B)$ for every $A,B \in \mathfrak{X}$.*

To relax stationarity, asymptotic mean stationarity is introduced. A source $\mu \in P(X)$ is said to be *AMS* with respect to S if, for any $A \in \mathfrak{X}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mu(S^{-j}A) \equiv \overline{\mu}(A) \tag{2.4}$$

exists. According to the Vitali-Hahn-Saks theorem (cf. [4, Section III.7]), $\overline{\mu}$ is a source and moreover it is stationary, that is, $\overline{\mu} \in P_s(X)$. The stationary source $\overline{\mu}$ is called the *stationary mean* of μ. Let $P_a(X)$ denote the set of all AMS sources in $P(X)$.

Let $\mu, \eta \in P(X)$. That η *asymptotically dominates* μ, denoted $\mu \overset{a}{\ll} \eta$, means that $\eta(A) = 0$ implies $\lim_{n\to\infty} \mu(S^{-n}A) = 0$. The usual dominance implies the asymptotic dominance in the sense that if $\mu \in P(X)$, $\eta \in P_s(X)$, and $\mu \ll \eta$, then $\mu \overset{a}{\ll} \eta$. Although $\overset{a}{\ll}$ is not transitive, one has that if $\mu \ll \xi \overset{a}{\ll} \eta$ or $\mu \overset{a}{\ll} \xi \ll \eta$, then $\mu \overset{a}{\ll} \eta$. Then AMS sources are characterized as in the following theorem (cf. [3, 7, 15]).

Theorem 2.2. *For $\mu \in P(X)$, the following conditions are equivalent:*

(1) *$\mu \in P_a(X)$, that is, μ is AMS;*
(2) *there exists some stationary $\eta \in P_s(X)$ such that $\mu \overset{a}{\ll} \eta$;*
(3) *there exists some stationary $\eta \in P_s(X)$ such that $\eta(A) = 0$ and $A \in \cap_{n=0}^{\infty} S^{-n}\mathfrak{X}$ imply $\mu(A) = 0$;*
(4) *there exists some $\eta \in P_s(X)$ such that $\eta(A) = 0$ and $S^{-1}A = A$ imply $\mu(A) = 0$;*
(5) *$\lim_{n\to\infty} \mathbf{S}_n f = f_S$ μ-a.e. for $f \in B(X)$, where f_S is an S-invariant function;*
(6) *$\lim_{n\to\infty} \mu(\mathbf{S}_n f)$ exists for every $f \in B(X)$.*

If one (and hence all) of the above conditions holds, then the stationary mean $\overline{\mu}$ of μ satisfies

$$\overline{\mu}(f) = \lim_{n\to\infty} \mu(\mathbf{S}_n f) = \mu(f_S), \quad f \in B(X). \tag{2.5}$$

Remark 2.3. (1) We have that $\mu \overset{a}{\ll} \overline{\mu}$ holds for $\mu \in P_a(X)$. For if we assume $\overline{\mu}(A) = 0$ and let $B = \limsup_{n\to\infty} S^{-n}A = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} S^{-k}A$, then

$$\overline{\mu}(B) = \lim_{n\to\infty} \overline{\mu}\Big(\cup_{k=n}^{\infty} S^{-k}A\Big) \le \overline{\mu}\Big(\cup_{k=1}^{\infty} S^{-k}A\Big) \le \sum_{k=1}^{\infty} \overline{\mu}(S^{-k}A) = 0 \tag{2.6}$$

since $\overline{\mu}(S^{-k}A) = \overline{\mu}(A) = 0$ for every $k \ge 1$. This implies $\mu(B) = 0$ since B is clearly S-invariant. Now we see that

$$\limsup_{n\to\infty} \mu(S^{-n}A) \le \mu\Big(\limsup_{n\to\infty} S^{-n}A\Big) = \mu(B) = 0 \tag{2.7}$$

by Fatou's lemma. Thus, $\lim_{n\to\infty} \mu(S^{-n}A) = 0$ and therefore $\mu \overset{a}{\ll} \overline{\mu}$. Note that the above proof constitutes that of (1)⇒(2) of Theorem 2.2.

(2) If $\mu \in P(X)$ and $\mu \ll \eta$ for some stationary $\eta \in P_s(X)$, then $\mu \in P_a(X)$, that is, μ is AMS since $\mu \ll \eta$ implies $\mu \overset{a}{\ll} \eta$ and Theorem 2.2(2) is applicable. Hence, if $\eta \in P_s(X)$ and $f \in L^1(X,\eta)$ is nonnegative with norm 1, then μ defined by

$$\mu(A) = \int_A f\,d\eta, \quad A \in \mathfrak{X}, \tag{2.8}$$

is AMS. In this case, the stationary mean $\overline{\mu}$ of μ is given by

$$\begin{aligned}\overline{\mu}(A) &= \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1}\mu(S^{-j}A) = \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\int_{S^{-j}A} f\,d\eta \\ &= \lim_{n\to\infty}\int_A \frac{1}{n}\sum_{j=0}^{n-1} f(S^j x)\eta(dx) \\ &= \int_A \lim_{n\to\infty} \mathbf{S}_n f\,d\eta = \int_A f_S\,d\eta, \quad A \in \mathfrak{X},\end{aligned} \tag{2.9}$$

where $f_S = \lim_{n\to\infty}\mathbf{S}_n f$, by the L^1-Ergodic theorem since η is stationary.

(3) The pointwise ergodic theorem holds for $\mu \in P(X)$ if and only if $\mu \in P_a(X)$. That is, μ is AMS if and only if for any $f \in B(X)$, there exists some S-invariant function $f_S \in B(X)$ such that $\mathbf{S}_n f \to f_S$ μ-a.e. In this case, for $f \in L^1(X,\overline{\mu})$, the convergence $\mathbf{S}_n f \to f_S$ is in the senses of μ-a.e., $\overline{\mu}$-a.e., and $L^1(X,\overline{\mu})$. Moreover, the limit function f_S is equal to the conditional expectation of f with respect to the σ-algebra $\mathfrak{I}$ of all S-invariant sets under the measure μ or $\overline{\mu}$.

When S is invertible, we can have some more characterizations of AMS sources (cf. [5, 12]).

Proposition 2.4. *Suppose that S is invertible. Then for $\mu \in P(X)$, the following conditions are equivalent:*

(1) $\mu \in P_a(X)$;
(2) *there exists some stationary $\eta \in P_s(X)$ such that $\mu \ll \eta$;*
(3) *there exists some AMS $\eta \in P_a(X)$ such that $\mu \ll \eta$.*

Remark 2.5. In the case where S is invertible, note that the asymptotic dominance implies the usual dominance in the sense that if $\mu \in P(X)$, $\eta \in P_s(X)$, and $\mu \overset{a}{\ll} \eta$, then $\mu \ll \eta$. Hence, $\mu \ll \overline{\mu}$ for $\mu \in P_a(X)$.

An AMS source μ is said to be *ergodic* if $\mu(A) = 0$ or 1 for every S-invariant set $A \in \mathfrak{X}$. Let $P_{ae}(X)$ denote the set of all AMS ergodic sources. We have several equivalence conditions for ergodicity of an AMS source, most of which are similar to those in Theorem 2.1.

Theorem 2.6. *For $\mu \in P_a(X)$ with the stationary mean $\overline{\mu} \in P_s(X)$, the following conditions are equivalent:*

(1) $\mu \in P_{ae}(X)$;

(2) $\overline{\mu} \in P_{se}(X)$;
(3) *there exists some stationary ergodic* $\eta \in P_{se}(X)$ *such that* $\mu \overset{a}{\ll} \eta$;
(4) $f_S(x) \equiv \lim_{n\to\infty}(\mathbf{S}_n f)(x) = \overline{\mu}(f)$ $\overline{\mu}$*-a.e.* x *and* μ*-a.e.* x *for* $f \in L^1(X,\overline{\mu})$;
(5) $\lim_{n\to\infty}(\mathbf{S}_n f, g)_\mu = (f,1)_{\overline{\mu}}(1,g)_\mu$ *for* $f,g \in L^2(X,\mu)\cap L^2(X,\overline{\mu})$;
(6) $\lim_{n\to\infty}\mu((\mathbf{S}_n f)g) = \overline{\mu}(f)\mu(g)$ *for* $f,g \in B(X)$;
(7) $\lim_{n\to\infty}(1/n)\sum_{j=0}^{n-1}\mu(S^{-j}A\cap B) = \overline{\mu}(A)\mu(B)$ *for* $A,B \in \mathfrak{X}$.

Proof. (1)⇔(2) is noted by Gray [6]. {(1),(2)}⇒(3) follows from Remark 2.3(1) by taking $\eta = \overline{\mu}$.

(3)⇒(1). Let $\eta \in P_{se}(X)$ be such that $\mu \overset{a}{\ll} \eta$. If $A \in \mathfrak{X}$ is S-invariant, then $\eta(A) = 0$ or 1. If $\eta(A) = 0$, then $\mu(A) = \mu(S^{-n}A) \to 0$ by $\mu \overset{a}{\ll} \eta$, that is, $\mu(A) = 0$. Similarly, if $\eta(A) = 1$, then we have $\mu(A) = 1$. Thus, $\mu \in P_{ae}(X)$.

Conditions (4), (5), (6), and (7) are analogy of those in Theorem 2.1 and (1)⇒(4)⇒(5)⇒(6)⇒(7)⇒(1) is easily verified. Conditions (6) and (7) are included in Gray [6] and (7) was noted by Hu and Shen [10] when μ is an almost periodic source, which is a slightly special case of an AMS source. □

Remark 2.7. Two distinct stationary ergodic sources are singular. Similarly, this is true for AMS ergodic sources, which is seen from the Pointwise Ergodic theorem.

For a stationary source, it is ergodic if and only if it is extremal in the set of all stationary sources. However, this is not the case for an AMS source as seen in the following theorem.

Theorem 2.8. (1) *If* $\mu \in \mathrm{ex}\,P_a(X)$*, then* $\mu \in P_{ae}(X)$*. That is,* $\mathrm{ex}\,P_a(X) \subseteq P_{ae}(X)$.

(2) *If* $(X,\mathfrak{X})$ *is not trivial and* $P_{se}(X) \neq \emptyset$*, then the above set inclusion is proper. That is, there exists an* $\eta \in P_{ae}(X)$ *such that* $\eta \notin \mathrm{ex}\,P_a(X)$.

Proof. (1) is given in Kakihara [12].

(2) Let $\mu \in P_{ae}(X)$ be such that $\mu \neq \overline{\mu}$. The existence of such a μ is seen as follows. Take any stationary ergodic source $\xi \in P_{se}(X)$ $(\neq \emptyset)$ and any nonnegative $f \in L^1(X,\xi)$ with norm 1 which is not S-invariant on a set of positive ξ measure. Define μ by

$$\mu(A) = \int_A f\,d\xi, \quad A \in \mathfrak{X}. \tag{2.10}$$

We see that μ is AMS by Remark 2.3(2) and ergodic because ξ is so. Clearly, μ is not stationary. Hence $\mu \neq \overline{\mu}$. Also note that $\overline{\mu} = \xi$ since for $A \in \mathfrak{X}$,

$$\overline{\mu}(A) = \int_A f_S\,d\xi = \xi(A) \tag{2.11}$$

by (2.9) and $f_S = 1$ ξ-a.e. because of the ergodicity of ξ. Then $\eta = (1/2)(\mu+\overline{\mu})$ is a proper convex combination of two distinct AMS sources and $\eta(A) = 0$ or 1 for S-invariant $A \in \mathfrak{X}$. Thus, $\eta \notin \mathrm{ex}\,P_a(X)$ and $\eta \in P_{ae}(X)$. □

Again, if S is invertible, ergodicity of AMS sources is characterized as in the following proposition.

Proposition 2.9. *If S is invertible, then the following conditions are equivalent for $\mu \in P_a(X)$:*

(1) $\mu \in P_{ae}(X)$;

(2) *there exists some $\eta \in P_{se}(X)$ such that $\mu \ll \eta$;*

(3) *there exists some $\eta \in P_{ae}(X)$ such that $\mu \ll \eta$;*

(4) *there exists some $\eta \in P_{ae}(X)$ such that $\mu \overset{a}{\ll} \eta$.*

Proof. (1)⇔(3) was given in Kakihara [12].

(1)⇒(2). Take $\eta = \overline{\mu} \in P_{se}(X)$, then $\mu \ll \overline{\mu} = \eta$ by Remark 2.5.

(2)⇒(3) is clear.

(3)⇒(4). Let $\eta \in P_{ae}(X)$ be such that $\mu \ll \eta$. Then $\overline{\eta} \in P_{se}(X)$ and $\eta \ll \overline{\eta}$. Hence, $\mu \ll \overline{\eta}$ and $\mu \overset{a}{\ll} \overline{\eta}$ since $\overline{\eta}$ is stationary.

(4)⇒(1). Let $\eta \in P_{ae}(X)$ be such that $\mu \overset{a}{\ll} \eta$. Then $\eta \ll \overline{\eta}$ and $\mu \overset{a}{\ll} \overline{\eta}$. Since $\overline{\eta} \in P_{se}(X)$, Theorem 2.6 concludes the proof. □

3. Ergodicity of stationary channels. In this section, ergodicity of stationary channels are considered. Our setting is as follows. Let X and Y be a pair of compact Hausdorff spaces with the Baire σ-algebras $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. Let $S : X \to X$ and $T : Y \to Y$ be continuous Baire measurable transformations. The symbol $\mathfrak{X} \otimes \mathfrak{Y}$ stands for the σ-algebra generated by the set $\mathfrak{X} \times \mathfrak{Y} = \{A \times C : A \in \mathfrak{X},\ C \in \mathfrak{Y}\}$ of all rectangles, which is the Baire σ-algebra of $X \times Y$. Let $C(\Omega)$ denote the Banach space of all $\mathbb{C}$-valued continuous functions on Ω with the sup-norm, while $B(\Omega)$ denotes the Banach space of all $\mathbb{C}$-valued bounded Baire functions on Ω with the sup-norm, where $\Omega = X, Y$ or $X \times Y$. We use symbols $P(\Omega)$, $P_s(\Omega)$, and so forth for $\Omega = X, Y$ and $X \times Y$. Operators $\mathbf{T}$ and $\mathbf{T}_n$ on $B(Y)$ and $\mathbf{S} \otimes \mathbf{T}$, and $(\mathbf{S} \otimes \mathbf{T})_n$ on $B(X \times Y)$ are defined in a similar manner as in (2.1) and (2.2).

We need several definitions. A *channel with input X and output Y* is a triple $[X, \nu, Y]$ for which the function $\nu : X \times \mathfrak{Y} \to [0,1]$ satisfies

(c1) $\nu(x, \cdot) \in P(Y)$ for every $x \in X$;

(c2) $\nu(\cdot, C) \in B(X)$ for every $C \in \mathfrak{Y}$.

In this case, ν is called a *channel distribution* or simply a *channel*. Let $\mathscr{C}(X,Y)$ denote the set of all channels with input X and output Y. A channel $\nu \in \mathscr{C}(X,Y)$ is said to be *stationary* if

(c3) $\nu(Sx, C) = \nu(x, T^{-1}C)$ for every $x \in X$ and $C \in \mathfrak{Y}$,

which is equivalent to

(c3′) $\nu(Sx, E_x) = \nu(x, T^{-1}E_x)$ for every $x \in X$ and $E \in \mathfrak{X} \otimes \mathfrak{Y}$, where $E_x = \{y \in Y : (x,y) \in E\}$, the *$x$-section* of E.

Let $\mathscr{C}_s(X,Y)$ denote the set of all stationary channels in $\mathscr{C}(X,Y)$. A channel $\nu \in \mathscr{C}(X,Y)$ is said to be *dominated* if

(c4) there exists some $\eta \in P(Y)$ such that $\nu(x, \cdot) \ll \eta$ for every $x \in X$.

Now, let $\nu \in \mathscr{C}(X,Y)$ and $\mu \in P(X)$, which is called an *input source*. Then the *output source* $\mu\nu \in P(Y)$ and the *compound source* $\mu \otimes \nu \in P(X \times Y)$ are, respectively, defined by

$$\mu\nu(C) = \int_X \nu(x,C)\mu(dx), \quad C \in \mathfrak{Y}, \tag{3.1}$$

$$\mu \otimes \nu(E) = \int_X \nu(x,E_x)\mu(dx), \quad E \in \mathfrak{X} \otimes \mathfrak{Y}. \tag{3.2}$$

Equation (3.2) can also be written as

$$\mu \otimes \nu(A \times C) = \int_A \nu(x,C)\mu(dx), \quad A \in \mathfrak{X},\ C \in \mathfrak{Y}. \tag{3.3}$$

It is well known that, for a stationary channel, if an input source is stationary, then the output and compound sources are also stationary. Note that any output source $\eta \in P(Y)$ can be regarded as a "constant" channel by letting

$$\nu_\eta(x,C) = \eta(C), \quad x \in X,\ C \in \mathfrak{Y}. \tag{3.4}$$

So, we may write $P(Y) \subset \mathscr{C}(X,Y)$. In this case,

$$\mu \otimes \nu_\eta = \mu \times \eta, \quad \mu\nu_\eta = \eta, \quad \mu \in P(X). \tag{3.5}$$

Thus, if η is stationary, the channel ν_η is stationary. Consequently, we may write $P_s(Y) \subset \mathscr{C}_s(X,Y)$ as well.

We need the following lemma which is viewed as an ergodic theorem for a stationary channel (cf. [16]).

Lemma 3.1. *If $\nu \in \mathscr{C}_s(X,Y)$ and $\mu \in P_s(X)$, then for every $E,F \in \mathfrak{X} \otimes \mathfrak{Y}$, the following limit exists μ-a.e. x:*

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \nu\Big(x,[(S \times T)^{-j}E \cap F]_x\Big) \\ = \int_Y 1_F(x,y)E_{\mu\otimes\nu}(1_E|\mathfrak{I})(x,y)\nu(x,dy),\end{aligned} \tag{3.6}$$

where $E_{\mu\otimes\nu}(\cdot|\mathfrak{I})$ is the conditional expectation with respect to the σ-algebra $\mathfrak{I} = \{E \in \mathfrak{X} \otimes \mathfrak{Y} : (S \times T)^{-1}E = E\}$ under the measure $\mu \otimes \nu$. In particular, for every $C,D \in \mathfrak{Y}$, the following limit exists μ-a.e. x:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \nu(x,T^{-j}C \cap D) = \int_Y 1_D(y)E_{\mu\nu}(1_C|\mathfrak{I}_Y)(y)\nu(x,dy), \tag{3.7}$$

where $\mathfrak{I}_Y = \{C \in \mathfrak{Y} : T^{-1}C = C\}$.

Proof. Apply the pointwise ergodic theorem to the function 1_E and use the bounded convergence theorem. □

A stationary channel $\nu \in \mathscr{C}_s(X,Y)$ is said to be *ergodic* if

(c5) $\mu \in P_{se}(X) \Rightarrow \mu \otimes \nu \in P_{se}(X\times Y)$.

Let $\mathscr{C}_{se}(X,Y)$ denote the set of all stationary ergodic channels. Let $\mathscr{P} \subseteq P(X)$. Two channels $\nu_1,\nu_2 \in \mathscr{C}(X,Y)$ are said to be *equivalent* $(\operatorname{mod}\mathscr{P})$, denoted $\nu_1 \equiv \nu_2 \pmod{\mathscr{P}}$, if

$$\nu_1(x,\cdot) = \nu_2(x,\cdot) \quad \mu\text{-a.e. } x \in X \tag{3.8}$$

for every $\mu \in \mathscr{P}$, which is equivalent to that $\mu\otimes\nu_1 = \mu\otimes\nu_2$ for every $\mu\in\mathscr{P}$ (cf. [18]). In this case, we write $\nu_1(x,\cdot) = \nu_2(x,\cdot)$ $\mathscr{P}$-a.e. x. A stationary channel $\nu \in \mathscr{C}_s(X,Y)$ is said to be *extremal* in $\mathscr{C}_s(X,Y)$ $\operatorname{mod}\mathscr{P}$, denoted $\nu \in \operatorname{ex}\mathscr{C}_s(X,Y)$ $(\operatorname{mod}\mathscr{P})$, provided that if $\nu_1,\nu_2 \in \mathscr{C}_s(X,Y)$ and $\alpha \in (0,1)$ are such that $\nu \equiv \alpha\nu_1 + (1-\alpha)\nu_2 \pmod{\mathscr{P}}$, then $\nu_1 \equiv \nu_2 \pmod{\mathscr{P}}$.

Equivalence conditions for ergodicity of a stationary channel are known (cf. [14, 16, 17, 18]).

Theorem 3.2. *For a stationary channel $\nu \in \mathscr{C}_s(X,Y)$, the following conditions are equivalent:*

(1) *$\nu \in \mathscr{C}_{se}(X,Y)$, that is, ν is ergodic;*

(2) *if $E \in \mathfrak{X}\otimes\mathfrak{Y}$ is $S\times T$-invariant, then for each $\mu \in P_{se}(X)$, $\nu(x,E_x) = 0$ μ-a.e. x or $\nu(x,E_x) = 1$ μ-a.e. x;*

(3) *there exists some ergodic channel $\nu_1 \in \mathscr{C}_{se}(X,Y)$ such that $\nu(x,\cdot) \ll \nu_1(x,\cdot)$ $P_{se}(X)$-a.e. x;*

(4) *if a stationary channel $\nu_0 \in \mathscr{C}_s(X,Y)$ is such that $\nu_0(x,\cdot) \ll \nu(x,\cdot)$ $P_{se}(X)$-a.e. x, then $\nu_0 \equiv \nu \pmod{P_{se}(X)}$;*

(5) *$\nu \in \operatorname{ex}\mathscr{C}_s(X,Y) \pmod{P_{se}(X)}$;*

(6) *for $E,F \in \mathfrak{X}\otimes\mathfrak{Y}$ and $\mu \in P_{se}(X)$, it holds that*

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\nu\Big(x,\big[(S\times T)^{-j}E\cap F\big]_x\Big)\\
&\quad=\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\nu\Big(x,\big[(S\times T)^{-j}E\big]_x\Big)\nu(x,F_x)\\
&\quad=\mu\otimes\nu(E)\nu(x,F_x)\quad \mu\text{-a.e. } x;
\end{aligned}\tag{3.9}$$

(7) *for $A,B \in \mathfrak{X}$, $C,D \in \mathfrak{Y}$, and $\mu \in P_{se}(X)$, it holds that*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\int_{S^{-j}A\cap B}\big\{\nu(x,T^{-j}C\cap D)-\nu(x,T^{-j}C)\nu(x,D)\big\}\mu(dx)=0. \tag{3.10}$$

Proof. Equivalences between (1), (2), (3), (4), (5), and (7) are given by Nakamura [14] and equivalences between (1), (2), and (6) by Shen [16]. Since [16] is written in Chinese, we sketch the proof of (1)⇒(6).

(1)⇒(6). Suppose that ν is ergodic and let $\mu \in P_{se}(X)$. Then $\mu\otimes\nu \in P_{se}(X\times Y)$, and hence for every $E, F' \in \mathfrak{X}\otimes\mathfrak{Y}$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\mu\otimes\nu\big((S\times T)^{-j}E\cap F'\big) = \mu\otimes\nu(E)\mu\otimes\nu(F'). \tag{3.11}$$

If we take $F' = F\cap(A\times Y)$ for $F\in\mathfrak{X}\otimes\mathfrak{Y}$ and $A\in\mathfrak{X}$, then on the one hand

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}&\sum_{j=0}^{n-1}\int_X \nu\Big(x,\big[(S\times T)^{-j}E\cap F\cap(A\times Y)\big]_x\Big)\mu(dx)\\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{k-1}\mu\otimes\nu\big((S\times T)^{-j}E\cap F\cap(A\times Y)\big)\\ &= \mu\otimes\nu(E)\mu\otimes\nu\big(F\cap(A\times Y)\big)\\ &= \mu\otimes\nu(E)\int_A \nu(x,F_x)\mu(dx),\end{aligned} \tag{3.12}$$

and on the other hand, by Lemma 3.1,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\mu\otimes\nu\big((S\times T)^{-j}E\cap F'\big) &= \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\int_A \nu\Big(x,\big[(S\times T)^{-j}E\cap F\big]_x\Big)\mu(dx)\\ &= \int_A \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\nu\Big(x,\big[(S\times T)^{-j}E\cap F\big]_x\Big)\mu(dx).\end{aligned} \tag{3.13}$$

Since (3.12) and (3.13) are equal for every $A\in\mathfrak{X}$, one has the equation in (6). □

Note that Theorem 3.2 holds for a general pair of measurable spaces $(X,\mathfrak{X},S)$ and $(Y,\mathfrak{Y},T)$ with measurable transformations S and T, respectively, if $P_s(X)$ is *complete for ergodicity*. That is, if $\mu\in P_s(X)$ and $A\in\mathfrak{X}$ are such that $\mu(A)>0$, then there exists some ergodic $\eta\in P_{se}(X)$ such that $\eta(A)>0$. In our setting here, since S is assumed to be continuous, $P_s(X)$ is necessarily complete for ergodicity (cf. [18]). Also, note that conditions for a stationary channel to be ergodic are very similar to those for a stationary source to be ergodic (see Theorem 2.1).

Now, Umegaki [18] considered channel operators to obtain further characterizations of stationary ergodic channels. A linear operator $\mathbf{A}$ from $B(X\times Y)$ onto $B(X)$ is called an *averaging operator* if for $f,g\in B(X\times Y)$,

(a1) $\mathbf{A}1 = 1$, $\mathbf{A}(f\mathbf{A}g) = (\mathbf{A}f)(\mathbf{A}g)$;

(a2) $f\ge 0 \Rightarrow \mathbf{A}f\ge 0$, where 1 is the identity function on X, Y or $X \times Y$. Denote by $\mathcal{A}(X,Y)$ the set of all averaging operators $\mathbf{A}$ from $B(X \times Y)$ onto $B(X)$ such that

(a3) $f_n \downarrow 0 \Rightarrow \mathbf{A} f_n \downarrow 0$.

An averaging operator $\mathbf{A} \in \mathcal{A}(X,Y)$ is said to be *stationary* if

(a4) $\mathbf{SA} = \mathbf{A}(\mathbf{S} \otimes \mathbf{T})$.

Let $\mathcal{A}_s(X,Y)$ denote the set of all stationary operators in $\mathcal{A}(X,Y)$. Let $\mathcal{K}(X,Y)$ denote the set of all bounded linear operators $\mathbf{K}: B(Y) \to B(X)$ such that

(k1) $\mathbf{K}1 = 1$, $\mathbf{K}b \geq 0$ if $b \geq 0$;

(k2) $b_n \downarrow 0 \Rightarrow \mathbf{K} b_n \downarrow 0$.

An operator $\mathbf{K} \in \mathcal{K}(X,Y)$ is said to be *stationary* if

(k3) $\mathbf{KT} = \mathbf{SK}$.

Let $\mathcal{K}_s(X,Y)$ stand for the set of all stationary $\mathbf{K} \in \mathcal{K}(X,Y)$.

Let $\nu \in \mathcal{C}(X,Y)$ be a channel and define operators $\mathbf{A}: B(X \times Y) \to B(X)$ and $\mathbf{K}: B(Y) \to B(X)$, respectively, by

$$(\mathbf{A}f)(x) = \int_Y f(x,y)\nu(x,dy), \quad f \in B(X \times Y), \tag{3.14}$$

$$(\mathbf{K}b)(x) = \int_Y b(y)\nu(x,dy), \quad b \in B(Y). \tag{3.15}$$

Sometimes, $\mathbf{A}$ and $\mathbf{K}$ are denoted by $\mathbf{A}_\nu$ and $\mathbf{K}_\nu$, respectively. Then, Umegaki [18] proved the following theorem.

Theorem 3.3. *There exist one-to-one, onto, and affine correspondences among $\mathcal{C}(X,Y)$ (resp., $\mathcal{C}_s(X,Y)$), $\mathcal{A}(X,Y)$ (resp., $\mathcal{A}_s(X,Y)$), and $\mathcal{K}(X,Y)$ (resp., $\mathcal{K}_s(X,Y)$) given by (3.14) and (3.15).*

In view of the above theorem, each operator in $\mathcal{A}(X,Y)$ or $\mathcal{K}(X,Y)$ is called a *channel operator*. Let $\nu \in \mathcal{C}(X,Y)$, $\mathbf{K} \in \mathcal{K}(X,Y)$, and $\mathbf{A} \in \mathcal{A}(X,Y)$ correspond to each other. Then for $\mu \in P(X)$, it holds that

$$\mathbf{K}^*\mu = \mu\nu, \qquad \mathbf{A}^*\mu = \mu \otimes \nu, \tag{3.16}$$

where $\mathbf{K}^*$ and $\mathbf{A}^*$ are adjoint operators of $\mathbf{K}$ and $\mathbf{A}$, respectively. Let $\mathcal{P} \subseteq P(X)$. Two operators $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{K}(X,Y)$ are said to be *equivalent* mod $\mathcal{P}$, denoted $\mathbf{K}_1 \equiv \mathbf{K}_2 \pmod{\mathcal{P}}$, if

$$(\mathbf{K}_1 b)(x) = (\mathbf{K}_2 b)(x) \quad \mu\text{-a.e. } x \in X \tag{3.17}$$

for every $b \in B(Y)$ and $\mu \in \mathcal{P}$. Two operators $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{A}(X,Y)$ are said to be *equivalent* mod $\mathcal{P}$, denoted $\mathbf{A}_1 \equiv \mathbf{A}_2 \pmod{\mathcal{P}}$, if

$$\mu(\mathbf{A}_1 f) = \mu(\mathbf{A}_2 f), \quad f \in C(X \times Y),\ \mu \in \mathcal{P}. \tag{3.18}$$

Also, we say that a stationary channel operator $\mathbf{K} \in \mathcal{K}_s(X,Y)$ is *extremal* in $\mathcal{K}_s(X,Y)$ mod $\mathcal{P}$, denoted $\mathbf{K} \in \operatorname{ex}\mathcal{K}_s(X,Y)$ (mod $\mathcal{P}$), if $\mathbf{K} \equiv \alpha\mathbf{K}_1 + (1-\alpha)\mathbf{K}_2$ (mod $\mathcal{P}$) for some $\alpha \in (0,1)$ and $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{K}_s(X,Y)$ imply $\mathbf{K}_1 \equiv \mathbf{K}_2$ (mod $\mathcal{P}$). Similarly, a stationary channel operator $\mathbf{A} \in \mathcal{A}_s(X,Y)$ is said to be *extremal* in $\mathcal{A}_s(X,Y)$ mod $\mathcal{P}$, denoted $\mathbf{A} \in \operatorname{ex}\mathcal{A}_s(X,Y)$ (mod $\mathcal{P}$), if $\mathbf{A} \equiv \alpha\mathbf{A}_1 + (1-\alpha)\mathbf{A}_2$ (mod $\mathcal{P}$) for some $\alpha \in (0,1)$ and $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{A}_s(X,Y)$ imply $\mathbf{A}_1 \equiv \mathbf{A}_2$ (mod $\mathcal{P}$).

Under these preparations, we have the following theorem (cf. [18]).

Theorem 3.4. *For a stationary channel $\nu \in \mathcal{C}_s(X,Y)$, let $\mathbf{A} \in \mathcal{A}_s(X,Y)$ and $\mathbf{K} \in \mathcal{K}_s(X,Y)$ be corresponding channel operators. Then, the following conditions are equivalent:*

(1) *ν is ergodic;*

(2) *$\nu \in \operatorname{ex}\mathcal{C}_s(X,Y)$ (mod $P_{se}(X)$);*

(3) *$\mathbf{A} \in \operatorname{ex}\mathcal{A}_s(X,Y)$ (mod $P_{se}(X)$);*

(4) *$\mathbf{K} \in \operatorname{ex}\mathcal{K}_s(X,Y)$ (mod $P_{se}(X)$);*

(5) *for $f, g \in C(X \times Y)$, it holds that*

$$\lim_{n\to\infty} \mathbf{A}(f_n g)(x) = \lim_{n\to\infty} \mathbf{A} f_n(x) \mathbf{A} g(x) \quad P_{se}(X)\text{-a.e. } x, \tag{3.19}$$

where $f_n = (\mathbf{S}\otimes\mathbf{T})_n f$, $n \geq 1$;

(6) *for $f, g \in B(X \times Y)$, (3.19) holds.*

4. Ergodicity of AMS channels. In this section, $(X, \mathfrak{X}, S)$ and $(Y, \mathfrak{Y}, T)$ are as in Section 3. We assume that S and T are homeomorphisms and $\mathfrak{Y}$ has a countable generator $\mathfrak{Y}_0$.

A channel $\nu \in \mathcal{C}(X,Y)$ is said to be *AMS* if

(c6) $\mu \in P_a(X) \Rightarrow \mu \otimes \nu \in P_a(X \times Y)$.

That is, if the input source is AMS, then the compound source is also AMS. Let $\mathcal{C}_a(X,Y)$ denote the set of all AMS channels. Then we have the following theorem (cf. [5, 12]).

Theorem 4.1. *For a channel $\nu \in \mathcal{C}(X,Y)$, the following conditions are equivalent:*

(1) *$\nu \in \mathcal{C}_a(X,Y)$, that is, ν is AMS;*

(2) *$\mu \in P_s(X) \Rightarrow \mu \otimes \nu \in P_a(X \times Y)$;*

(3) *there exists a stationary channel $\nu_1 \in \mathcal{C}_s(X,Y)$ such that $\nu(x,\cdot) \ll \nu_1(x,\cdot)$ $P_s(X)$-a.e. x;*

(4) *there exists an AMS channel $\nu_1 \in \mathcal{C}_a(X,Y)$ such that $\nu(x,\cdot) \ll \nu_1(x,\cdot)$ $P_s(X)$-a.e. x;*

(5) *there exists a stationary channel $\overline{\nu} \in \mathcal{C}_s(X,Y)$ such that for $C \in \mathfrak{Y}$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \nu(S^{-j}x, T^{-j}C) = \overline{\nu}(x,C) \quad P_s(X)\text{-a.e. } x; \tag{4.1}$$

(6) *for* $f \in B(X \times Y)$ *and* $\mu \in P_s(X)$*, the following limit exists:*

$$\lim_{n\to\infty} \int_X [\mathbf{A}_\nu (\mathbf{S}\otimes\mathbf{T})_n f](x)\mu(dx), \tag{4.2}$$

where $\mathbf{A}_\nu \in \mathscr{A}(X,Y)$ *corresponds to* ν.

If one (and hence all) of the above is true, then it holds that

(7) $\overline{\mu\otimes\nu} = \mu\otimes\overline{\nu}$ *for* $\mu \in P_s(X)$;

(8) $\nu(x,\cdot) \ll \overline{\nu}(x,\cdot)$ $P_s(X)$*-a.e.* x;

(9) *for* $\mu \in P_s(X)$ *and* $f \in B(X\times Y)$,

$$\lim_{n\to\infty} \int_X [\mathbf{A}_\nu (\mathbf{S}\otimes\mathbf{T})_n f](x)\mu(dx) = \int_X (\mathbf{A}_{\overline{\nu}} f)(x)\mu(dx). \tag{4.3}$$

Note that we need invertibility assumption on S for (5) in the above theorem. For an AMS channel $\nu \in \mathscr{C}_a(X,Y)$, the stationary channel $\overline{\nu} \in \mathscr{C}_s(X,Y)$, obtained in Theorem 4.1(5), is called a *stationary mean* of ν. The stationary channel $\overline{\nu}$ is unique in the $P_s(X)$-a.e. sense.

Example 4.2. (1) As was mentioned before, each output source $\eta \in P(Y)$ can be viewed as a channel by letting $\nu_\eta(x,C) = \eta(C)$ for $x \in X$ and $C \in \mathfrak{Y}$. If $\eta \in P_a(Y)$, then ν_η is AMS. In fact, $\eta \ll \overline{\eta}$ since T is invertible so that $\nu_\eta(x,\cdot) = \eta \ll \overline{\eta} = \nu_{\overline{\eta}}(x,\cdot)$ for $x \in X$. Moreover, $\nu_{\overline{\eta}} \in \mathscr{C}_s(X,Y)$ implies that $\nu_\eta \in \mathscr{C}_a(X,Y)$ by Theorem 4.1(3). In this case, we have $\overline{\nu}_\eta = \nu_{\overline{\eta}}$.

(2) If a channel $\nu \in \mathscr{C}(X,Y)$ is dominated by some AMS $\eta \in P_a(Y)$, that is,

$$\nu(x,\cdot) \ll \eta \quad P_s(X)\text{-a.e. } x, \tag{4.4}$$

then ν is AMS by Theorem 4.1(4) and (1). Let

$$k(x,y) = \frac{\nu(x,dy)}{\overline{\eta}(dy)}, \quad (x,y) \in X\times Y, \tag{4.5}$$

where $\overline{\eta} \in P_s(Y)$ is the stationary mean of η. Observe that k is $\mathfrak{X}\otimes\mathfrak{Y}$-measurable by Umegaki [19]. Then ν can be written as

$$\nu(x,C) = \int_C k(x,y)\overline{\eta}(dy), \quad x \in X,\ C \in \mathfrak{Y}, \tag{4.6}$$

and its stationary mean $\overline{\nu}$ as

$$\overline{\nu}(x,C) = \int_C k^*(x,y)\overline{\eta}(dy), \quad x \in X,\ C \in \mathfrak{Y}, \tag{4.7}$$

where $k^*(x,y) = \lim_{n\to\infty}(\mathbf{S}^{-1}\otimes\mathbf{T}^{-1})_n k(x,y)$ $\mu\otimes\overline{\nu}$-a.e. (x,y) for $\mu \in P_s(X)$.

An AMS channel $\nu \in \mathscr{C}_a(X,Y)$ is said to be *ergodic* if

(c7) $\mu \in P_{ae}(X) \Rightarrow \mu \otimes \nu \in P_{ae}(X \times Y)$.

Let $\mathscr{C}_{ae}(X,Y)$ denote the set of all ergodic AMS channels.

The following lemma is an AMS version of Lemma 3.1.

Lemma 4.3. *Let $\nu \in \mathscr{C}_a(X,Y)$ be AMS and let $\mu \in P_s(X)$ be stationary. Then for every $E, F \in \mathfrak{X} \otimes \mathfrak{Y}$, the following limit exists μ-a.e. x:*

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \nu\left(x, [(S\times T)^{-j}E \cap F]_x\right) \\ = \int_Y 1_F(x,y) E_{\mu\otimes\overline{\nu}}(1_E|\mathfrak{I})(x,y)\nu(x,dy),\end{aligned} \tag{4.8}$$

where $\mathfrak{I} = \{E \in \mathfrak{X} \otimes \mathfrak{Y} : (S\times T)^{-1}E = E\}$.

Now, ergodic AMS channels are characterized as in the following theorem.

Theorem 4.4. *Let $\nu \in \mathscr{C}_a(X,Y)$ with the stationary mean $\overline{\nu} \in \mathscr{C}_s(X,Y)$. Then the following conditions are equivalent:*

(1) *$\nu \in \mathscr{C}_{ae}(X,Y)$, that is, ν is ergodic;*

(2) *$\mu \in P_{se}(X) \Rightarrow \mu \otimes \nu \in P_{ae}(X\times Y)$;*

(3) *$\overline{\nu} \in \mathscr{C}_{se}(X,Y)$;*

(4) *there exists a stationary ergodic channel $\nu_1 \in \mathscr{C}_{se}(X,Y)$ such that*

$$\nu(x,\cdot) \ll \nu_1(x,\cdot) \quad P_{se}(X)\text{-a.e. } x; \tag{4.9}$$

(5) *there exists an AMS ergodic channel $\nu_1 \in \mathscr{C}_{ae}(X,Y)$ such that (4.9) holds;*

(6) *if $E \in \mathfrak{X} \otimes \mathfrak{Y}$ is $S\times T$-invariant, then for each $\mu \in P_{se}(X)$, $\nu(x,E_x) = 0$ μ-a.e. x or $\nu(x,E_x) = 1$ μ-a.e. x;*

(7) *for $E, F \in \mathfrak{X} \otimes \mathfrak{Y}$ and $\mu \in P_{se}(X)$,*

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \nu\left(x, [(S\times T)^{-j}E \cap F]_x\right) \\ = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \overline{\nu}\left(x, [(S\times T)^{-j}E]_x\right)\nu(x,F_x) \\ = \mu\otimes\overline{\nu}(E)\nu(x,F_x) \quad \mu\text{-a.e. } x.\end{aligned} \tag{4.10}$$

Proof. Conditions (2), (4), and (5) are given in [12] and condition (3) is noted in [5]. Conditions (6) and (7) are obtained for an almost periodic channel by Hu and Shen [10]. $\square$

We noted that $\operatorname{ex} P_a(X) \subseteq P_{ae}(X)$ and the set inclusion is proper (see Theorem 2.8). Similarly, we can prove the following theorem.

Theorem 4.5. (1) *If* $\nu \in \mathrm{ex}\,\mathscr{C}_a(X,Y) \pmod{P_{se}(X)}$*, then* $\nu \in \mathscr{C}_{ae}(X,Y)$*. That is,* $\mathrm{ex}\,\mathscr{C}_a(X,Y) \subseteq \mathscr{C}_{ae}(X,Y)$*.*

(2) *If* $(Y,\mathfrak{Y})$ *is not trivial and there exists a stationary weakly mixing source in* $P_{se}(Y)$*, then the above set inclusion is proper. That is, there exists some AMS ergodic channel* $\nu \in \mathscr{C}_{ae}(X,Y)$ *such that* $\nu \notin \mathrm{ex}\,\mathscr{C}_a(X,Y) \pmod{P_{se}(X)}$*.*

Proof. (1) Let $\nu \in \mathscr{C}_a(X,Y)$, $\overline{\nu} \in \mathscr{C}_s(X,Y)$ its stationary mean, and $\mathbf{A} \in \mathscr{A}(X,Y)$ the corresponding channel operator to ν. Suppose that $\nu \notin \mathscr{C}_{ae}(X,Y)$. Then there exists some $\mu \in P_{se}(X)$ such that

$$\mathbf{A}^*\mu \notin P_{ae}(X\times Y) \tag{4.11}$$

by Theorem 4.4(2) (see (3.16)). Hence, there exists some $S\times T$-invariant set $E \in \mathfrak{X}\otimes\mathfrak{Y}$ such that $0 < \lambda_1 \equiv \mathbf{A}^*\mu(E) < 1$. Letting $\lambda_2 = 1-\lambda_1$, take $\gamma > 0$ so that $0 < \gamma < \min\{\lambda_1,\lambda_2\}$. Let $\alpha_i = \gamma/\lambda_i$ $(i = 1,2)$ and define operators $\mathbf{A}_1$ and $\mathbf{A}_2$ on $B(X\times Y)$ by

$$\begin{aligned} \mathbf{A}_1 f &= \alpha_1\mathbf{A}(f1_E) + (1-\alpha_1\mathbf{A}1_E)\mathbf{A}f, \quad f \in B(X\times Y), \\ \mathbf{A}_2 f &= \alpha_2\mathbf{A}(f1_{E^c}) + (1-\alpha_2\mathbf{A}1_{E^c})\mathbf{A}f, \quad f \in B(X\times Y). \end{aligned} \tag{4.12}$$

Then by Umegaki [18], we see that $\mathbf{A}_1,\mathbf{A}_2 \in \mathscr{A}(X,Y)$, $\mathbf{A}_1 \not\equiv \mathbf{A}_2 \pmod{P_{se}(X)}$, and $\mathbf{A}$ is a proper convex combination of $\mathbf{A}_1$ and $\mathbf{A}_2$:

$$\mathbf{A} = \lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2. \tag{4.13}$$

Let $\nu_i \in \mathscr{C}(X,Y)$ correspond to $\mathbf{A}_i$ for $i = 1,2$. We will show that ν_1 and ν_2 are AMS channels. Observe that for $f \in B(X\times Y)$,

$$\begin{aligned} &\int_X [\mathbf{A}_1(\mathbf{S}\otimes\mathbf{T})_n f](x)\mu(dx) \\ &\quad= \int_X [\alpha_1\mathbf{A}(((\mathbf{S}\otimes\mathbf{T})_n f)1_E) + (1-\alpha_1\mathbf{A}1_E)\mathbf{A}(\mathbf{S}\otimes\mathbf{T})_n f](x)\mu(dx) \\ &\quad= \int_X [\alpha_1\mathbf{A}(\mathbf{S}\otimes\mathbf{T})_n(f1_E) + (1-\alpha_1\mathbf{A}1_E)\mathbf{A}(\mathbf{S}\otimes\mathbf{T})_n f](x)\mu(dx) \end{aligned} \tag{4.14}$$

since E is $S\times T$-invariant. Since ν is AMS, $\lim_{n\to\infty}\int_X \alpha_1\mathbf{A}(\mathbf{S}\otimes\mathbf{T})_n(f1_E)\,d\mu$ exists again by Theorem 4.1. Also, $\lim_{n\to\infty}\int_X (1-\alpha_1\mathbf{A}1_E)\mathbf{A}(\mathbf{S}\otimes\mathbf{T})_n f\,d\mu$ exists by Theorem 4.1 and the bounded convergence theorem. Thus, we proved that

$$\lim_{n\to\infty}\int_X \mathbf{A}_1(\mathbf{S}\otimes\mathbf{T})_n f\,d\mu \tag{4.15}$$

exists for every $f \in B(X\times Y)$, and hence ν_1 is AMS by Theorem 4.1. Similarly, ν_2 is AMS. Consequently, we see that $\nu \notin \mathrm{ex}\,\mathscr{C}_a(X,Y) \pmod{P_{se}(X)}$.

(2) By assumption, we can take an $\eta \in P_{se}(Y)$ that is weakly mixing and define ξ by

$$\xi(C) = \int_C g\,d\eta, \quad C \in \mathfrak{Y}, \tag{4.16}$$

where $g \in L^1(Y,\eta)$ is nonnegative with norm 1 which is not T-invariant on a set of positive η measure. Then, as in the proof of Theorem 2.8, we see that $\xi \in P_{ae}(Y)$, $\xi \neq \eta$, $\overline{\xi} = \eta$, and $\zeta \equiv (1/2)(\xi+\eta) \in P_{ae}(Y)$ is a proper convex combination of two distinct AMS sources. Hence

$$\nu_\zeta \notin \mathrm{ex}\,\mathscr{C}_a(X,Y)\ (\mathrm{mod}\, P_{se}(X)) \tag{4.17}$$

since $\nu_\zeta = (1/2)(\nu_\xi + \nu_\eta)$, $\nu_\xi, \nu_\eta \in \mathscr{C}_a(X,Y)$, and $\nu_\xi \neq \nu_\eta$. We need to show $\nu_\zeta \in \mathscr{C}_{ae}(X,Y)$. Clearly, $\overline{\nu}_\zeta = \nu_{\overline{\zeta}} = \nu_\eta$ and $\mu \otimes \nu_\eta = \mu \times \eta$ for $\mu \in P(X)$. Since ξ is weakly mixing, we have $\mu \otimes \nu_\eta \in P_{se}(X \times Y)$ for $\mu \in P_{se}(X)$. Thus, $\overline{\nu}_\zeta \in \mathscr{C}_{se}(X,Y)$, and therefore $\nu_\zeta \in \mathscr{C}_{ae}(X,Y)$ by Theorem 4.4. □

Example 4.6. Consider an alphabet message space $Y = \{b_1,\dots,b_\ell\}^{\mathbb{Z}}$, the doubly infinite product space of a finite alphabet $Y_0 = \{b_1,\dots,b_n\}$ over the set $\mathbb{Z}$ of all integers. The *shift* T on Y is defined by

$$T : y = (y_k) \longmapsto y' = Ty = (\dots, y'_{-1}, y'_0, y'_1, \dots), \quad y'_k = y_{k+1},\ k \in \mathbb{Z}, \tag{4.18}$$

and the σ-algebra $\mathfrak{Y}$ is generated by the set of all cylinder sets of the form

$$\begin{aligned} [y_i^0 \cdots y_j^0] &= [y_i = y_i^0, \dots, y_j = y_j^0] \\ &= \{y = (y_k) \in Y : y_k = y_k^0,\ i \le k \le j\}, \end{aligned} \tag{4.19}$$

where $y_k^0 \in Y_0$ for $i \le k \le j$. Note that Y is a compact Hausdorff space (with the product topology), T is a homeomorphism, and $\mathfrak{Y}$ has a countable generator. Let $(p_1,\dots,p_\ell)$ be a probability distribution on the alphabet Y_0 and define η_0 by

$$\eta_0([y_i^0 \cdots y_j^0]) = p(y_i^0) \cdots p(y_j^0). \tag{4.20}$$

Then, η_0 can be extended to a T-invariant probability measure η on $\mathfrak{Y}$, that is, $\eta \in P_s(Y)$. The stationary mean η is called a $(p_1,\dots,p_\ell)$-*Bernoulli source*. Now, it is well known that η is strongly mixing, so that the assumption of Theorem 4.5(2) is satisfied. Therefore, we can construct a constant ergodic AMS channel ν_ζ that is not an extreme point of $\mathscr{C}_a(X,Y)$.

Acknowledgment. The author is grateful to the referees for careful reading of the manuscript, helpful comments, and valuable suggestions.

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