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AN APPLICATION OF A SUBORDINATION CHAIN

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Let K denote the class of functions $g(z) = z + a_2 z^2 + \cdots$ which are regular and univalently convex in the unit disc E. In the present note, we prove that if f is regular in E, $f(0) = 0$, then for $g \in K$, $f(z) + \alpha z f'(z) \prec g(z) + \alpha z g'(z)$ in E implies that $f(z) \prec g(z)$ in E, where $\alpha > 0$ is a real number and the symbol "$\prec$" stands for subordination.

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1. Introduction. Let S denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are regular and univalent in the unit disc $E = \{z : |z| < 1\}$. A function $f \in S$ is said to be convex of order β, $0 \le \beta < 1$, if and only if

$$\operatorname{Re}\left[1 + \frac{z f''(z)}{f'(z)}\right] > \beta, \quad z \in E. \tag{1.2}$$

For a given β, $0 \le \beta < 1$, let $K(\beta)$ denote the subclass of S consisting of convex functions of order β and let $K = K(0)$ be the usual class of convex functions.

A function f given by (1.1) is said to be close-to-convex in E if f is regular in E and if there exists a function $g \in K$ such that

$$\operatorname{Re}\left[\frac{f'(z)}{g'(z)}\right] > 0, \quad z \in E. \tag{1.3}$$

It is well known that if a function is close-to-convex in E, then it is univalent in E.

Suppose that f and g are regular in $|z| < \rho$ and $f(0) = g(0)$. In addition, suppose that g is also univalent in $|z| < \rho$. We say that f is subordinate to g in $|z| < \rho$ (in symbols, $f(z)g \prec (z)$ in $|z| < \rho$) if $f(|z| < \rho) \subset g(|z| < \rho)$.

In 1947, Robinson [4] proved that if $g(z) + z g'(z)$ is in S and $f(z) + z f'(z) \prec g(z) + z g'(z)$ in $|z| < 1$, then $f(z) \prec g(z)$ at least in $|z| < r_0 = 1/5$. S. Singh and R. Singh [6], in 1981, increased the constant r_0 to $2 - \sqrt{3} = 0.268\ldots$. Subsequently, in 1984, Miller et al. [2] further increased this constant to $4 - \sqrt{13} = 0.3944\ldots$.

Recently, R. Singh and S. Singh [5] pursued the problem initiated by Robinson when $g \in K(\beta)$. In fact, they considered the cases when $\beta = 0$ and $\beta = 1/2$ and proved the following results.

Theorem 1.1. *Let f be regular in E with $f(0) = 0$ and let $g \in K$. Suppose that*

$$f(z) + zf'(z) \prec g(z) + zg'(z) \tag{1.4}$$

in E. Then,

$$f(z) \prec g(z) \tag{1.5}$$

at least in $|z| < r_0$, where $r_0 = \sqrt{5}/3 = 0.745\ldots$.

Theorem 1.2. *Let f be regular in E, $f(0) = 0$, and let $g \in K(1/2)$. Then*

$$f(z) + zf'(z) \prec g(z) + zg'(z) \tag{1.6}$$

in E implies that

$$f(z) \prec g(z) \tag{1.7}$$

at least in $|z| < r_1$, where $r_1 = ((51 - 24\sqrt{2})/23)^{1/2} = 0.8612\ldots$.

In the present note, we consider the subordination $f(z) + \alpha zf'(z) \prec g(z) + \alpha zg'(z)$ in $|z| < 1$, $g \in K$ and $\alpha > 0$, and show that the subordination $f(z) \prec g(z)$ holds in the entire disc $|z| < 1$ and does not depend upon the order of convexity of g as claimed by R. Singh and S. Singh in [5].

2. Preliminaries.

We will need the following definition and results to prove our theorem.

Definition 2.1. A function $L(z,t)$, $z \in E$ and $t \geq 0$, is said to be a subordination chain if $L(\cdot,t)$ is analytic and univalent in E for all $t \geq 0$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all z in E, and $L(z,t_1) \prec L(z,t_2)$ for $0 \leq t_1 \leq t_2$.

Lemma 2.2 [3, page 159]. *The function $L(z,t) = a_1(t)z + \cdots$, with $a_1(t) \neq 0$ for $t \geq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$, is a subordination chain if and only if*

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] > 0, \quad z \in E,\ t \geq 0. \tag{2.1}$$

Lemma 2.3. *Let p be analytic in E and q analytic and univalent in $\overline{E}$ except for points where $\lim_{z\to\varsigma} p(z) = \infty$ with $p(0) = q(0)$. If p is not subordinate to q, then there is a point $z_0 \in E$ and $\zeta_0 \in \partial E$ (boundary of E) such that $p(|z| < |z_0|) \subset q(E)$, $p(z_0) = q(\zeta_0)$, and $z_0p'(z_0) = m\zeta_0q'(\zeta_0)$ for $m \geq 1$.*

Lemma 2.3 is due to Miller and Mocanu [1].

3. Main theorem

Theorem 3.1. *Let f be regular in E with $f(0) = 0$ and let $g \in K$. For any real number α, $\alpha > 0$, suppose that*

$$f(z)+\alpha z f'(z) \prec g(z)+\alpha z g'(z) \tag{3.1}$$

in E. Then,

$$f(z) \prec g(z) \tag{3.2}$$

in E.

Proof. First, we observe that $g(z)+\alpha z g'(z) = h(z)$, say, is close-to-convex and hence univalent in E whenever $g \in K$. Without any loss of generality, we can assume that g is regular and univalent in the closed disc $\overline{E}$. If possible, suppose that $f(z)$ is not subordinate to $g(z)$ whenever (3.1) holds. Then by Lemma 2.3, there exist points $z_0 \in E$, $\zeta_0 \in \partial E$, and $m \geq 1$ such that $f(|z| < |z_0|) \subset g(E)$, $f(z_0) = g(\zeta_0)$, and $z_0 f'(z_0) = m\zeta_0 g'(\zeta_0)$. This gives

$$f(z_0)+\alpha z_0 f'(z_0) = g(\zeta_0)+m\alpha\zeta_0 g'(\zeta_0). \tag{3.3}$$

Define a function

$$L(z,t) = h(z)+\alpha t z g'(z) = a_1(t)z+\cdots. \tag{3.4}$$

Since $h(z)$ and $zg'(z)$ are analytic in E, $L(z,t)$ is also analytic in E for all $t \geq 0$, and is continuously differentiable on $[0,\infty)$ for all $z \in E$. Now, from (3.4), we get

$$a_1(t) = \frac{\partial L}{\partial z}(0,t) = 1+\alpha(1+t) \neq 0 \tag{3.5}$$

for all $t \geq 0$ and $\alpha > 0$. Also

$$\lim_{t\to\infty} |a_1(t)| = \infty. \tag{3.6}$$

As $g \in K$, a simple calculation yields

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] = \operatorname{Re}\left[\frac{1}{\alpha}+(1+t)\left(1+\frac{zg''(z)}{g'(z)}\right)\right] > 0 \tag{3.7}$$

for $z \in E$, $t \geq 0$, and $\alpha > 0$. Hence, by Lemma 2.2, $L(z,t)$ is a subordination chain. Therefore, in view of Definition 2.1, we have $L(z,t_1) \prec L(z,t_2)$ for $0 \leq t_1 \leq t_2$. Since, from (3.4), $L(z,0) = h(z)$, we deduce that

$$L(\zeta_0,t) \notin h(E) \tag{3.8}$$

for $|\zeta_0| = 1$ and $t \geq 0$.

Now, in view of (3.4) and (3.3), we can write

$$f(z_0) + \alpha z_0 f'(z_0) = L(\zeta_0, m-1), \tag{3.9}$$

where $z_0 \in E$, $|\zeta_0| = 1$, and $m \geq 1$. Formula (3.9), when combined with (3.8), contradicts (3.1). Hence, we must have $f(z) \prec g(z)$ in E. This completes the proof of our theorem. □

Letting α approach infinity, we arrive at the following well-known result of Suffridge [7].

Corollary 3.2. *Let f be regular in E with $f(0) = 0$ and let $g \in K$. If $zf'(z) \prec zg'(z)$ in E, then $f(z) \prec g(z)$ in E.*

We now present some interesting examples choosing g as some distinguished member of the class K.

Let f be regular in E, $f(0) = 0$, and let $\alpha > 0$. Then

(a) $f(z) + \alpha z f'(z) \prec z/(1-z) + \alpha z/(1-z)^2$ in $E \Rightarrow f(z) \prec z/(1-z)$ in E;

(b) $f(z) + \alpha z f'(z) \prec e^z(1+\alpha z) - 1$ in $E \Rightarrow f(z) \prec e^z - 1$ in E;

(c) $f(z) + \alpha z f'(z) \prec -\log(1-z) + \alpha z/(1-z)$ in $E \Rightarrow f(z) \prec -\log(1-z)$ in E.

Note that the function $-\log(1-z)$ is convex of order $1/2$ in E.

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