IJMMS 2003:26, 1683–1692
PII. S0161171203208097
http://ijmms.hindawi.com

TIME-SYMMETRIC CYCLES

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Received 18 August 2002

The main result of this note is that given any two time-symmetric cycles, one can find a time-symmetric extension of one by the other. This means that given a time-symmetric cycle, both time-symmetric doubles and square roots can be found.

2000 Mathematics Subject Classification: 37E05.

1. Introduction. In [4], the idea of time-symmetric cycles was introduced. The basic idea underlying the definition of these cycles is that the dynamics associated to both the cycle and its inverse are isomorphic. This is a property that one would expect of physical data. Such data should not look more complicated if time is reversed. In [4], the classification of unimodal time-symmetric cycles was given. In this note, we drop the unimodal restriction and consider cycles in general.

Section 2 gives the basic definitions. It is followed by some basic results concerning time-symmetric cycles; the numbers of time-symmetric cycles of length n are given as the number of diagonal points in a given cycle. It is then shown that given any two time-symmetric cycles, one can find a time-symmetric extension of one by the other. The note concludes by looking at the process of doubling and finding square roots.

2. Basic ideas. Continuous maps of the interval induce a partial order on the set of cyclic permutations (cycles).

Suppose that a map of the interval has a periodic orbit $P = \{p_1,\dots,p_n\}$ labelled so that $p_1 < \cdots < p_n$. Then P has *cycle type* π if π is a cycle with the property that $f(p_i) = p_j$ if and only if $\pi(i) = j$. A cycle π *forces* a cycle μ, written $\pi \geq \mu$, if every continuous map of the interval that has a periodic point of cycle type π also has a periodic point with cycle type μ. Baldwin [2] showed that the forcing relation was a partial order on the set of cycles.

Given β, a cyclic permutation of $\{1,\dots,m\}$, let $L_\beta : [1,m] \to [1,m]$ denote the map defined by $L_\beta = \beta$ on $\{1,\dots,m\}$ and L_β is linear on each $[i,i+1]$. The nondegenerate intervals with respect to "L_β is strictly monotone on I" are called *laps* of L_β. The cycle β is *n-modal* if L_β has $n+1$ laps. We will call the critical points of L_β the *critical points of* β. Since it is standard to use maps from the unit interval to itself, we will rescale L_β to form $I_\beta : [0,1] \to [0,1]$ by defining $I_\beta = f^{-1}L_\beta f$, where $f(x) = (m-1)x+1$.

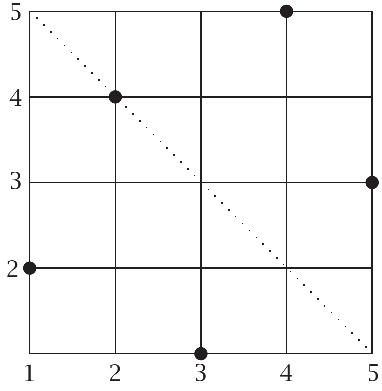

FIGURE 2.1

Later, we will make some comparisons of what happens in the general case to the unimodal case. We note that unimodal cycles are divided into two classes: those with the first lap increasing, and those with the first lap decreasing (the exceptional cycles (1) and (12) will be considered as belonging to both classes). We will denote the first class as *unimax cycles* and the second class as *unimin cycles*. It is well known that each class is linearly ordered by the forcing relation (see, e.g., [1]).

We say that two cycles γ and δ are *dynamically equivalent* if there exists a homeomorphism h of the interval such that $I_\gamma = h^{-1} \circ I_\delta \circ h$.

Let σ_n denote the cycle of length n defined by $\sigma_n(i) = n-i+1$ for $i = 1,\dots,n$. If there is no ambiguity in the value of n, we will write σ instead of σ_n.

In [4], it was shown that the only two cycles that are dynamically equivalent to α are α and $\sigma\alpha\sigma$. (Clearly, from a physical viewpoint, α and $\sigma\alpha\sigma$ correspond to interchanging right and left, and we would expect them to be dynamically equivalent.)

A cycle α is *time-symmetric* if α^{-1} is dynamically equivalent to α. Since the only cycles that have the property that they are self-inverse are (1) and (12), the definition can be restated as α is *time-symmetric* if $\alpha^{-1} = \sigma\alpha\sigma$.

Time-symmetric cycles are characterized by the following property.

LEMMA 2.1. *Let θ denote a cycle of length n. Then θ is time-symmetric if and only if it has the property that if $\theta(i) = j$, then $\theta(n+1-j) = n+1-i$.*

This property means that if the cycle is graphed, the points will be symmetrically placed about the diagonal line $y = n-x$.

EXAMPLE 2.2. The cycle (12453) graphed in Figure 2.1 is time-symmetric.

Given a cycle θ of length n, we say that an integer k satisfying $1 \le k \le n$ is a *diagonal point* if $\theta(k) = n+1-k$. Thus, in the above example, 2 is a diagonal point. In Section 3, it will be shown that time-symmetric cycles always have diagonal points.

3. Some counting arguments. We first prove two lemmas that will show the following theorem.

Theorem 3.1. *Let θ be a time-symmetric cycle. If n is odd, then θ has exactly one diagonal point. If n is even, then θ has exactly two diagonal points.*

Proof. The symmetry implies that an odd time-symmetric cycle must have an odd number of diagonal points and that an even time-symmetric cycle must have an even number. The proof will be completed by the following two lemmas which show that a time-symmetric cycle must have either exactly one or two diagonal points. □

Lemma 3.2. *If θ is a time-symmetric cycle, then it has at least one diagonal point.*

Proof. Let θ denote a time-symmetric cycle of length n. The lemma is obviously true if n is odd, so in what follows, we will only consider the case that n is even.

The proof looks at the forward orbit of 1 and the backward orbit of n. For ease of notation, let k_j denote $\theta^j(1)$ for $0 \le j < n$. Let $F_i = \{k_j \mid 0 \le j \le i\}$. Let $B_i = \{n+1-k_j \mid 0 \le j \le i\}$. Clearly, $\{1\} = F_0 \ne B_0 = \{n\}$ and $F_{n-1} = B_{n-1} = \{1,\dots,n\}$. Let r denote the smallest integer such that $k_r \in B_r$. Clearly, $k_r = n+1-k_s$ for some $s \le r$. If $s = r$, we obtain $2k_r = n+1$, which contradicts the fact that n is even, so $s < r$.

Now, $k_r = n+1-k_s$ means that $k_s = n+1-k_r$. By definition, we know that $\theta(k_s) = k_{s+1}$. However, we also know that by time-symmetry, $\theta(n+1-k_r) = n+1-k_{r-1}$. Thus, we have $k_{s+1} = n+1-k_{r-1}$ which can be rewritten as $k_{r-1} = n+1-k_{s+1}$, which shows that $k_{r-1} \in B_r$. Since r is the smallest integer such that $k_r \in B_r$, we know that $k_{r-1} \notin B_{r-1}$. So, we must have $k_{r-1} = n+1-k_r$ which means that $\theta(k_{r-1}) = n+1-k_{r-1}$ and that k_{r-1} is a diagonal point. □

Lemma 3.3. *A time-symmetric cycle can have at most two diagonal points.*

Proof. Suppose that θ is a time-symmetric cycle of length n. Suppose that p is a diagonal point. Let r denote the smallest positive integer such that $\theta^r(p)$ is diagonal. As in the previous lemma, we will look at the forward and backward orbit of a point; only in this case we take p instead of 1.

Let k_j denote $\theta^j(p)$ for $0 \le j$, and let $a \to b$ mean that $\theta(a) = b$.

Then for each j, we have $k_j \to k_{j+1}$ and $n+1-k_j \to n+1-k_{j-1}$. Since k_0 and k_r are diagonal, we know that $k_0 \to n+1-k_0 = k_1$ and $k_r \to n+1-k_r$. Combining this information gives the following path:

$$\begin{aligned} p = k_0 \longrightarrow k_1 \longrightarrow \cdots \longrightarrow k_r \longrightarrow n+1-k_r \longrightarrow n+1-k_{r-1} \longrightarrow \cdots \longrightarrow n+1-k_1 \\ = k_0 = p. \end{aligned} \tag{3.1}$$

None of the points $n+1-k_i$, for $i=2,\dots,r$, are diagonal for if $\theta(n+1-k_i)=k_i$, then $\theta(k_{i-1})=k_i=\theta(n+1-k_i)=n+1-k_{i-1}$, and so k_{i-1} would be diagonal. □

The above proof gives a little more information that will be used for the following proposition.

Proposition 3.4. *If θ is a time-symmetric cycle of even length n and if p is a diagonal point, then the other diagonal point is $\theta^{n/2}(p)$. If θ is a time-symmetric cycle of odd length n and if p is a diagonal point, then $\theta^{(n+1)/2}(p)=(n+1)/2$.*

Proof. The proof in the even case comes immediately from the proof of the above lemma. We will give the proof for the case when n is odd. As above, let k_j denote $\theta^j(p)$ for $0\le j$ and let $a\to b$ mean that $\theta(a)=b$. Let r denote the smallest positive integer such that $\theta^r(p)=(n+1)/2$. By time-symmetry, $k_{r-1}\to k_r=(n+1)/2$ tells us that $n+1-(n+1)/2=(n+1)/2\to n+1-k_{r-1}$. Thus, we have the path

$$\begin{aligned} p=k_0\longrightarrow k_1\longrightarrow\cdots\longrightarrow k_{r-1}\longrightarrow \frac{n+1}{2}\longrightarrow n+1-k_{r-1}\longrightarrow\cdots\longrightarrow n+1-k_1\\ =k_0=p, \end{aligned}\tag{3.2}$$

and r must equal $(n+1)/2$. □

In [4], it was shown that there are exactly four unimodal time-symmetric cycles of period n if n is even and exactly two if n is odd. Here, we give the total number of time-symmetric cycles when the unimodal restriction is dropped.

Theorem 3.5. *There are $((n-1)/2)!2^{(n-1)/2}$ time-symmetric cycles of length n if n is odd. There are $(n/2)!2^{(n-2)/2}$ time-symmetric cycles of length n if n is even.*

Proof. First, we consider the case when n is odd. The previous proposition shows that if p is the diagonal point, then $\theta^{(n+1)/2}(p)=(n+1)/2$. This means that the point $(n+1)/2$ cannot be the diagonal point. Thus, there are $n-1$ choices for the diagonal point p, and the following construction shows that each of these choices is allowed. Once p is chosen, $\theta(p)=n+1-p$, and there are $n-3$ choices for $\theta^2(p)$ (the points $(n+1)/2$, p, and $n+1-p$ are not allowed). Proceeding inductively for $0<i<(n-1)/2$, we see that there are $n+1-2i$ choices for $\theta^i(p)$ once $\theta^{i-1}(p)$ has been chosen. Once $\theta^{(n-1)/2}(p)$ has been chosen, everything else is forced. Thus, the total number of ways of choosing a time-symmetric cycle is $(n-1)(n-3)\cdots 1=2^{(n-1)/2}((n-1)/2)!$.

In the case when n is even, there are n choices for a diagonal point. Once the diagonal point p has been chosen, there are $n-2$ choices for $\theta^2(p)$. Similarly, for $0<i\le n/2$, there are $n+2-2i$ choices for $\theta^i(p)$ once $\theta^{i-1}(p)$ has been chosen. Once $\theta^{n/2}(p)$ has been chosen, everything else is forced. This gives

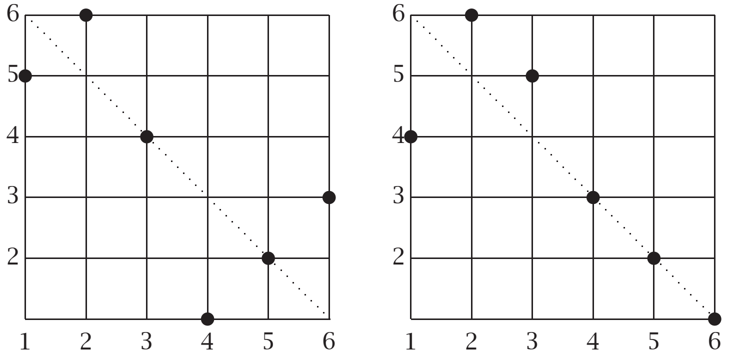

FIGURE 4.1

$n(n-2)\cdots 2 = 2^{n/2}(n/2)!$ choices, but since each of the choices has two diagonal points, we have double counted and the total number of time-symmetric cycles is $(n/2)!2^{(n-2)/2}$. $\square$

4. Extensions. In combinatorial dynamics, the idea of extension is important. (For basic results on extensions, see [1, 5].) In this section, we show that if α and β are time-symmetric cycles, we can always form a time-symmetric extension of α by β. In Section 5, we study some applications of this result.

DEFINITION 4.1. Let α be a cycle of length n and β a cycle of length m. Let $P_i = \{(i-1)m+1, (i-1)m+2, \dots, im\}$ for $1 \le i \le n$. If a cycle θ of length $n+m$ exists with the following properties:
(1) θ sends P_i onto $P_{\alpha(i)}$ for $1 \le i \le n$,
(2) θ is monotone on at least $n-1$ of the P_i,
(3) θ^n restricted to $\{1,2,\dots,m\}$ is either β or $\sigma\beta\sigma$,
then θ is called an *extension of* α *by* β.

We will consider extensions in general before restricting to the time-symmetric case.

EXAMPLE 4.2. First, we give a couple of examples of an extension of (12) by (123). These are the cycles $\pi_1 = (152634)$ and $\pi_2 = (143526)$ that are graphed in Figure 4.1. Clearly, π_1 is time-symmetric and π_2 is not.

We will call the P_i in the definition the *blocks* of the extension, and the one P_i on which θ is not monotone, if there is one, will be denoted as the *nm-block*. The other blocks, on which θ is monotone, will be called the *m-blocks*. In both of the above examples, the nm-block is $P_1 = \{1,2,3\}$, and there is just one m-block $\{4,5,6\}$. Notice that the graph of π_1 restricted to P_1 is a translation of the graph of the cycle (123) and that the graph of π_2 restricted to P_1 is a translation of the graph of $(1)(23) = (123)\sigma$. Notice also that π_2^2 restricted to P_1 corresponds to (132), but π_2^2 restricted to P_2 corresponds to (123).

In general, suppose that α is a cycle of length n and β is a cycle of length m, where $m > 2$. We will show how to construct all extensions of α by β.

Any of the P_i can be chosen to be the nm-block. Once the nm-block has been chosen, orientations for extension on the m-blocks can be chosen. Any of the 2^{n-1} choices is allowed. We then count the number of m-blocks that are orientation reversing. There are two cases to consider: the case when the number of orientation reversing m-blocks is even and the case when this number is odd. If the number of orientation reversing m-blocks is even, then either a translation of β or $\sigma\beta\sigma$ can be used for the extension restricted to the nm-block and these are the only choices. If the number of orientation reversing m-blocks is odd, then either a translation of $\sigma\beta$ or $\beta\sigma$ can be used for the extension restricted to the nm-block and these are the only choices. Notice that every distinct choice gives rise to a distinct cycle unless $\beta = \sigma\beta\sigma$ or, equivalently, $\sigma\beta = \beta\sigma$. (Such cycles do exist, e.g., $\sigma(135642)\sigma = (135642)$.)

If the cycle β has length 2, then the extension restricted to every block will be monotone. In order to get a cycle of length $2n$, the total number of blocks that are orientation reversing must be odd. So, once the orientations are chosen for $n-1$ blocks, the orientation for the last block is forced. (This was noted in [3].) We summarize this below.

Lemma 4.3. *Suppose that α is a cycle of length n and β is a cycle of length m, where $m > 2$, then there are exactly $n2^n$ extensions of α by β if $\sigma\beta\sigma \neq \beta$ and exactly $n2^{n-1}$ extensions if $\sigma\beta\sigma = \beta$. If α is a cycle of length n and $\beta = (12)$, then there are 2^{n-1} extensions of α by β.*

We state a couple of easy results about extensions.

Lemma 4.4. *Suppose that θ is an extension of α by β. If α has modality p and β has modality q, then the modality of θ is at least $p+q-1$.*

Proof. First, it is clearly true if either α or β equal either of (1) or (12) (recall that both of these are regarded as being unimodal). We will assume that neither α nor β equal the cycles (1) or (12) and initially restrict the attention to the laps of the piecewise linear map L_θ that lies between the blocks. Notice that L_θ restricted to $\{im, im+1\}$ has the same orientation as L_α on $\{i, i+1\}$, for $1 \le i \le n-1$. So, p of the blocks must contain at least one critical point of L_θ. The modality of L_θ on the nm-block is q. So, L_θ restricted to the nm-block must have q critical points. So, we can deduce that at least $p-1$ blocks contain at least one critical point and that one block must contain q critical points. Thus, θ must have at least $p+q-1$ critical points. □

Lemma 4.5. *Let α be a cycle of length n. If θ is an extension of α by β and at least one of the blocks is orientation reversing, then there will be at least* *one block where θ^n is a translation of β and at least one block where θ^n is a translation of $\sigma\beta\sigma$.*

Proof. Let P_i denote a block on which θ is orientation reversing. Then θ^n restricted to P_i is a translation of $\sigma^p\beta\sigma^q$, where p and q are nonnegative integers that are either both odd or both even. Clearly, the restriction of θ^n to $P_{\alpha(i)}$ is $\sigma^{p-1}\beta\sigma^{q+1}$. $\square$

We now return the attention to the time-symmetric case. Let α be a time-symmetric cycle of length n and β be a time-symmetric cycle of length m, where $m > 2$. We consider all the ways in which we can form an extension of α by β that is time-symmetric. First, it is clear that if P_i is going to be the nm-block, then i must be a diagonal point of α. Theorem 3.1 tells us that if n is odd, there is just one choice for which block can be the nm-block and if n is even, there are two choices. If k is not a diagonal point of α, once an orientation is chosen for the extension restricted to P_k is chosen, the orientation for $P_{n-\alpha(k)}$ is forced to be the same. If n is even, then α has two diagonal points. Suppose that l is the diagonal point and the extension is monotone on P_l. Since the extension can have at most two diagonal points and $m > 2$, the restriction to P_l must be orientation-preserving. Thus, in the time-symmetric case, it is always true that the number of orientation reversing m-blocks is even. So, the nm-block can contain either β or $\sigma\beta\sigma$ (in [4], it is shown that these are distinct when $m > 2$).

In the case when $\beta = (12)$, the above argument applies with the exception that $\beta = \sigma\beta\sigma$. Thus, we have the following theorem.

Theorem 4.6. *Suppose that α and β are time-symmetric cycles of length n and m, respectively.*

(i) *If n is odd and $m > 2$, there exist exactly $2^{(n+1)/2}$ time-symmetric extensions of α by β.*

(ii) *If n is even and $m > 2$, there exist exactly $2^{(n+2)/2}$ time-symmetric extensions of α by β.*

(iii) *If n is odd and $m = 2$, then there are exactly $2^{(n-1)/2}$ time-symmetric extensions of α by β.*

(iv) *If n is even and $m = 2$, then there are exactly $2^{n/2}$ time-symmetric extensions of α by β.*

Corollary 4.7. *Suppose that α and β are time-symmetric cycles. Then time-symmetric extensions of α by β exist.*

5. Doubles and square roots.

Given a cycle α, a *double* of α is an extension of α by (12) and a *square root* of α is an extension of (12) by α. (Example 4.2 gives two examples of square roots of (123).) Both these doubling and square-rooting operations are important. For example, the proof of Sarkovskii's theorem is really an argument about cycles and all the even cycles in this proof are obtained by using these two operations (see [1] for a proof where this is made explicit). Also, it is often claimed that physical experiments exhibit a sequence of period doubling bifurcations.

In [3], it is shown that a double of a cycle α is next to α in the forcing relation in the sense that if $\gamma \neq \alpha$ and γ forces α, then γ must force a double of α, and that doubles of α force α. In the general case, the non-time-symmetric case, it is always possible to find a double of a cycle with the same modality. However, in [4], it was shown that in the time-symmetric unimodal case, it is not possible to have an extended sequence of period doubling. Section 4 of this note shows that time-symmetric doubles of time-symmetric cycles do exist. In what follows, we examine what happens to the modality as we go through a time-symmetric sequence of doubles.

Theorem 5.1. *Let α denote a time-symmetric cycle. Then the modality of any time-symmetric double of a double of a double of α is greater than the modality of α.*

Proof. Suppose that α is a time-symmetric cycle. Let α_1 denote a time-symmetric double of α. There must be a diagonal point k of α such that $2k-1$ and $2k$ are diagonal points of α_1.

Suppose that k is not a critical point of the piecewise linear map L_α. Now, if the modality of α equals the modality of α_1, we must have $\alpha(k-1) > \alpha(k) > \alpha(k+1)$ and $\alpha_1(2(k-1)) > \alpha_1(2k-1) > \alpha_1(2k) > \alpha_1(2(k+1)-1)$. Now, any double of α_1 must have exactly two diagonal points by Theorem 3.1. So, one of the blocks P_{2k-1} or P_{2k} for the double of α_1 must be orientation-preserving and the modality must be increased by at least two.

Suppose that k is a critical point of L_α. Without loss of generality, assume that $\alpha(k) > \alpha(k+1)$ and $\alpha(k) > \alpha(k-1)$. If the modality of α equals the modality of α_1, we must have $2k-1$ and $2k$ as diagonal points and $2k-1$ as a local maximum of L_{α_1}. Let α_2 denote a double of α_1. If α_2 has the same modality as α_1, then the block P_{2k-1} must be orientation-preserving and P_{2k} orientation-reversing. Thus, the diagonal points for α_2 must be $2(2k)-1$ and $2(2k)$ and we have $\alpha_2(2(2k-1)) > \alpha_2(2(2k)-1) > \alpha_2(2(2k))$. The argument in the previous paragraph then shows that any double of α_2 must increase the modality by at least two. □

Example 5.2. The time-symmetric cycle (12453), graphed in Example 2.2, is an example where 2 is both a diagonal point and a critical point. This cycle has modality 4. It is easily checked that $(1,3,8,10,5,2,4,7,9,6)$ followed by $(1,5,15,19,10,3,7,14,18,11,2,6,16,20,9,4,8,13,17,12)$ is the only time-symmetric sequence of doublings that have the same modality. The next time-symmetric double must increase the modality by at least two.

We now turn our attention to square roots. In [1], it is shown how the entropy of an extension is related to the topological entropy of its component cycles. For the special case of square roots, it gives us the following theorem.

Theorem 5.3. *Let α denote any cycle and $h(\alpha)$ denote its topological entropy. The entropy of any square root of α is $(1/2)h(\alpha)$.*

Thus, if α has positive topological entropy, any square root of α will have lower entropy. In the non-time-symmetric unimodal case, this is important because it is always possible to choose a unimax/unimin square root of a unimax/unimin cycle. Since unimax/unimin cycles are linearly ordered, we know that any unimodal cycle with positive entropy will force a square root of this cycle.

In the time-symmetric case, Section 4 shows that we can find time-symmetric square roots of time-symmetric cycles. As above, if the topological entropy of the initial cycle is positive, then the topological entropy of the square root will be half of the entropy of the initial cycle. However, we will show that the modality of the square root is greater than the modality of the initial cycle. So, in the time-symmetric case, a cycle can never force its square root.

Theorem 5.4. *Let α denote a time-symmetric cycle of length $n > 2$ and modality m. Then there are exactly four time-symmetric square roots of α. Two of these have modality $m+1$ and two have modality $m+2$.*

Proof. The cycle (12) has to be extended by α to obtain the square root. Thus, there are two blocks P_1 and P_2. Since the extension will have exactly two diagonal points, the orientation of the extension on the monotone block must be positive. If P_1 is the m-block, then the extension will have a maximum at n. If P_2 is the m-block, then the extension will have a minimum at $n+1$. The number of critical points of the extension restricted to the nm-block is exactly the number of critical points of α. There are two choices of what can be placed in the nm-block: either a translation of α or $\sigma\alpha\sigma$. One of these will force an endpoint of the nm-block to be a critical point of the extension and the other will not. Thus, the extension will have one critical point in the m-block and either m or $m+1$ in the nm-block. □

Corollary 5.5. *Let α be a time-symmetric cycle. Then α cannot force a time-symmetric square root of α.*

Proof. A cycle can never force a cycle of higher modality, so this result follows immediately from the above theorem unless α is (1) or (12). These two special cases are easily checked. □

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