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BIHARMONIC CURVES IN MINKOWSKI 3-SPACE

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We give a differential geometric interpretation for the classification of biharmonic curves in semi-Euclidean 3-space due to Chen and Ishikawa (1991).

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1. Introduction. Chen and Ishikawa [1] classified biharmonic curves in semi-Euclidean space $\mathbf{E}^n_\nu$. They showed that every biharmonic curve lies in a 3-dimensional totally geodesic subspace. Thus, it suffices to classify biharmonic curves in semi-Euclidean 3-space.

In this note, we point out that every biharmonic Frenet curve in Minkowski 3-space $\mathbb{E}^3_1$ is a helix whose curvature κ and torsion τ satisfy $\kappa^2 = \tau^2$.

2. Preliminaries. Let (M^3,h) be a time-oriented Lorentz 3-manifold. Let $\gamma : I \to M$ be a unit speed curve. Namely, the velocity vector field γ' satisfies $h(\gamma',\gamma') = \varepsilon_1 = \pm 1$. The constant ε_1 is called the *causal character* of γ. A unit speed curve is said to be *spacelike* or *timelike* if its causal character is 1 or -1, respectively.

A unit speed curve γ is said to be a *geodesic* if $\nabla_{\gamma'}\gamma' = 0$. Here, ∇ is the Levi-Civita connection of (M,h).

A unit speed curve γ is said to be a *Frenet curve* if $h(\gamma'',\gamma'') \neq 0$. Like Euclidean geometry, every Frenet curve γ in (M,h) admits a Frenet frame field along γ. Here, a Frenet frame field $P = (\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)$ is an orthonormal frame field along γ such that $\mathbf{p}_1 = \gamma'(s)$ and P satisfies the following *Frenet-Serret formula* (cf. [2]; see also [4, 5]):

$$\nabla_{\gamma'} P = P \begin{pmatrix} 0 & -\varepsilon_1\kappa & 0 \\ \varepsilon_2\kappa & 0 & \varepsilon_2\tau \\ 0 & -\varepsilon_3\tau & 0 \end{pmatrix}. \tag{2.1}$$

The functions $\kappa \geq 0$ and τ are called the *curvature* and *torsion*, respectively. The vector fields $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are called *tangent vector field*, *principal normal vector field*, and *binormal vector field* of γ, respectively. The constants ε_2 and ε_3 defined by

$$\varepsilon_i = h(\mathbf{p}_i,\mathbf{p}_i), \quad i = 2,3 \tag{2.2}$$

are called *second causal character* and *third causal character* of γ, respectively. Note that $\varepsilon_3 = -\varepsilon_1 \cdot \varepsilon_2$.

As in the case of Riemannian geometry, a Frenet curve γ is a geodesic if and only if $\kappa = 0$.

A Frenet curve with constant curvature and zero torsion is called a *pseudo-circle*.

A helix is a Frenet curve whose curvature and torsion are constants. Pseudocircles are regarded as degenerate helices. Helices, which are not circles, are frequently called *proper helices*.

The mean curvature vector field $\mathbb{H}$ of a unit speed curve γ is $\mathbb{H} = \varepsilon_1 \nabla_{\gamma'}\gamma'$. If γ is a Frenet curve, then $\mathbb{H}$ is given by

$$\mathbb{H} = -\varepsilon_3 \kappa \mathbf{p}_2. \tag{2.3}$$

To close this section, we recall the notion of biharmonicity for unit speed curves.

Let $\gamma = \gamma(s)$ be a unit speed curve in a Lorentz 3-manifold (M,h) defined on an interval I. Denote by $\gamma^* TM$ the vector bundle over I obtained by pulling back the tangent bundle TM:

$$\gamma^* TM := \cup_{s\in I} T_{\gamma(s)} M. \tag{2.4}$$

The *Laplace operator* Δ acting on the space $\Gamma(\gamma^* TM)$ of all smooth sections of $\gamma^* TM$ is given explicitly by

$$\Delta = -\varepsilon_1 \nabla_{\gamma'} \nabla_{\gamma'}. \tag{2.5}$$

Definition 2.1. A unit speed curve $\gamma : I \to M$ in a Lorentz 3-manifold M is said to be *biharmonic* if $\Delta\mathbb{H} = 0$.

If M is the semi-Euclidean 3-space, then γ is biharmonic if and only if $\Delta\Delta\gamma = 0$.

3. Biharmonic curves.

Chen and Ishikawa classified biharmonic curves in semi-Euclidean 3-space. In particular, they showed that in Euclidean 3-space, there are no proper biharmonic curves (i.e., biharmonic curves which are not harmonic). On the other hand, in *indefinite* semi-Euclidean 3-space, there exist proper biharmonic curves. Here, we recall their classification theorem.

Theorem 3.1 (see [1]). *Let γ be a spacelike curve in indefinite semi-Euclidean 3-space $\mathbf{E}^3_\nu$. Then, γ is biharmonic if and only if γ is congruent to one of the following:*

(1) *a spacelike line;*

(2) *a spacelike curve $\gamma(s) = (as^3 + bs^2, as^3 + bs^2, s)$ in $\mathbf{E}^3_1$, where a and b are constants such that $a^2 + b^2 \neq 0$;*

(3) *a spacelike curve* $\gamma(s) = (a^2s^3/6, as^2/2, -a^2s^3/6+s)$ *in* $\mathbf{E}_1^3$*, where* a *is a nonzero constant;*

(4) *a spacelike curve* $\gamma(s) = (a^2s^3/6, as^2/2, a^2s^3/6+s)$ *in* $\mathbf{E}_2^3$*, where* a *is a nonzero constant.*

To give a differential geometric interpretation of the above result, we need to start with the following general result (cf. [2]).

Theorem 3.2. *Let* $\gamma : I \to M$ *be a Frenet curve in a Lorentz* 3*-manifold* (M,h)*. Denote by* Δ *the Laplace operator acting on* $\Gamma(\gamma^*TM)$*. Then,* γ *satisfies* $\Delta\mathbb{H} = \lambda\mathbb{H}$ *if and only if* γ *is a helix (including a geodesic). In this case, the eigenvalue* λ *is* $\lambda = -\varepsilon_3(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2)$*.*

Proof. Direct computation shows that

$$\Delta\mathbb{H} = -3\varepsilon_3\kappa\kappa'\mathbf{p}_1 - \varepsilon_2\{\kappa'' - \varepsilon_2\kappa(\varepsilon_1\kappa^2 - \varepsilon_3\tau^2)\}\mathbf{p}_2 - \varepsilon_1(2\kappa'\tau + \kappa\tau')\mathbf{p}_3. \tag{3.1}$$

Thus, $\Delta\mathbb{H} = \lambda\mathbb{H}$ if and only if

$$\kappa\kappa' = 0, \qquad 2\kappa'\tau + \kappa\tau = 0, \qquad \kappa'' - \varepsilon_2\kappa(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2) = -\varepsilon_1\lambda\kappa. \tag{3.2}$$

These formulae imply that γ is a spacelike or timelike helix whose curvature and torsion satisfy $\lambda = -\varepsilon_3(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2)$. □

Theorem 3.2 implies the following two results.

Corollary 3.3. *Let* γ *be a Frenet curve in a Lorentz* 3*-manifold* (M,h)*. Then,* γ *is a nongeodesic biharmonic curve if and only if it is one of the following:*

(1) γ *is a spacelike helix with a spacelike principal normal such that* $\kappa = \pm\tau$*;*

(2) γ *is a timelike helix such that* $\kappa = \pm\tau$*.*

Note that there exist no biharmonic spacelike curves in M with spacelike principal normals.

Corollary 3.4. *Let* γ *be a Frenet curve in* (M,h)*. Then,* γ *is a helix if and only if*

$$\nabla_{\gamma'}\nabla_{\gamma'}\nabla_{\gamma'}\gamma' - \mathcal{K}\nabla_{\gamma'}\gamma' = 0 \tag{3.3}$$

for some constant $\mathcal{K}$*. In this case, the constant* $\mathcal{K}$ *equals* $-\varepsilon_2(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2)$*.*

Note that Ikawa obtained Corollary 3.4 for timelike curves (see [3, Proposition 4.1]). Thus, we give here an analytic meaning of (3.3). Since we treat both spacelike and timelike curves in Corollary 3.4, we get a generalisation of [3, Proposition 4.1].

In the case where M is the Minkowski 3-space $\mathbf{E}_1^3$, it is known that helices with $\tau = \pm\kappa \neq 0$ are cubic curves, and one can explicitly give the formula of such helices (see, e.g., Kobayashi [6]). Moreover, it is easy to check that such spacelike helices are congruent to the curves given in Theorem 3.1.

Now, we rephrase the classification due to Chen and Ishikawa. Since case (4) in Theorem 3.1 is the image of a timelike helix satisfying $\kappa^2 = \tau^2 = a^2$ under the following anti-isometry from $\mathbb{E}^3_1$ onto $\mathbb{E}^3_2$:

$$\mathbb{E}^3_1 \ni (u, v, w) \longmapsto (w, v, u), \tag{3.4}$$

we may restrict our attention to curves in Minkowski 3-space $\mathbf{E}^3_1$.

Proposition 3.5. *Let γ be a unit speed curve in Minkowski 3-space $\mathbf{E}^3_1$. Then, γ is biharmonic if and only if γ is congruent to one of the following:*

(1) *a spacelike or timelike line;*

(2) *a spacelike curve such that $h(\gamma'', \gamma'') = 0$ is given by*

$$\gamma(s) = (as^3 + bs^2, as^3 + bs^2, s), \tag{3.5}$$

where a and b are constants such that $a^2 + b^2 \neq 0$;

(3) *a spacelike helix with a spacelike principal normal vector field satisfying $\kappa^2 = \tau^2 = a^2$;*

$$\gamma(s) = \left(\frac{a^2 s^3}{6}, \frac{as^2}{2}, -\frac{a^2 s^3}{6+s}\right); \tag{3.6}$$

(4) *a timelike helix satisfying $\kappa^2 = \tau^2 = a^2$;*

$$\gamma(s) = \left(\frac{a^2 s^3}{6+s}, \frac{as^2}{2}, \frac{a^2 s^3}{6}\right). \tag{3.7}$$

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